MATH 15300/14

Midterm #1

JANUARY 28, 2020
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Please present your solutions clearly and in an organized way. Simplify all your final answers. If an answer box is given, write your final answer in the box. If you run out of room, continue on the extra pages provided at the end. **The use of a calculator is not allowed.** Good luck!! ◡̈

Full Name: sample solutions

Student ID:

Question	Points	Score
1	5	
2	21	
3	20	
4	21	
5	10	
6	20	
7	20	
Total:	117	

This exam has 7 questions, for a total of 117 points. The maximum possible score for each problem is given on the right side of the problem.

Please do not open this exam packet unless you are instructed to do so.

1. Consider the sequence $a_n = (-1)^n n$ (which starts with a_1).

   Also consider the sequence of partial sums $s_n = \sum_{k=1}^{n} a_k$.

   (a) Write down the first four terms of the sequence a_n. (You do not need to show any work.) $\boxed{1}$

   $a_1, a_2, a_3, a_4 = \boxed{-1,\ 2,\ -3,\ 4}$

   (b) Write down the first four terms of the sequence s_n. (You do not need to show any work.) $\boxed{4}$

   $s_1, s_2, s_3, s_4 = \boxed{-1,\ 1,\ -2,\ 2}$

   $S_1 = -1$

   $S_2 = -1 + 2 = 1$

   $S_3 = -1 + 2 - 3 = -2$

   $S_4 = -1 + 2 - 3 + 4 = 2$

2. Evaluate the following limits of sequences.

   (a) $\lim_{n\to\infty} \frac{\cos(n^5)}{4n^3+1} = \boxed{0}$ $\boxed{7}$

$$-\frac{1}{4n^3+1} \leq \frac{\cos(n^5)}{4n^3+1} \leq \frac{1}{4n^3+1}$$

$$\lim_{n\to\infty}\left(-\frac{1}{4n^3+1}\right) = 0 \qquad \lim_{n\to\infty}\left(\frac{1}{4n^3+1}\right) = 0$$

so by squeeze theorem,

$$\lim_{n\to\infty} \frac{\cos(n^5)}{4n^3+1} = 0$$

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(b) $\lim\limits_{n\to\infty} (e^n + 2)^{3/n} =$ $\boxed{e^3}$ 7

Let $L = \lim\limits_{n\to\infty} (e^n+2)^{3/n}$

$$\ln L = \lim_{n\to\infty} \ln(e^n+2)^{3/n}$$

$$= \lim_{n\to\infty} \frac{3\ln(e^n+2)}{n}$$

L'Hôpital

$$= \lim_{n\to\infty} \frac{3\cdot \frac{1}{e^n+2}e^n}{1}$$

$$= 3 \implies L = e^3$$

(c) $\lim\limits_{n\to\infty} n\sin(2\pi n) =$ $\boxed{0}$ 7

$$\lim_{n\to\infty} n\sin(2\pi n)$$

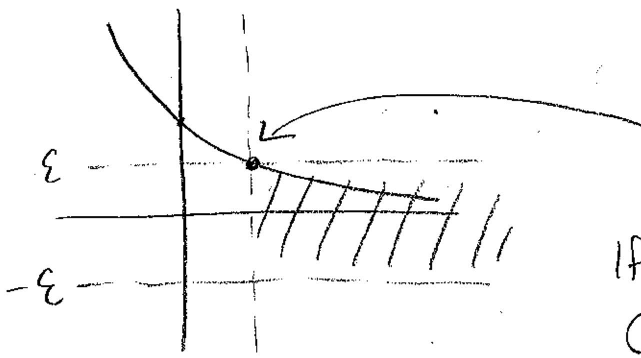
$$= \lim_{n\to\infty} n\cdot 0$$

$$= \lim_{n\to\infty} 0$$

$$= 0$$

3. For the following, you should use the ϵ, K definition of a limit, but you do not need to write a formal proof. It is enough to include a sketch along with a short explanation.

(a) Show that $\lim\limits_{x \to \infty} e^{-x} = 0$. $\boxed{10}$



$e^{-x} = \varepsilon$

$e^{x} = 1/\varepsilon$

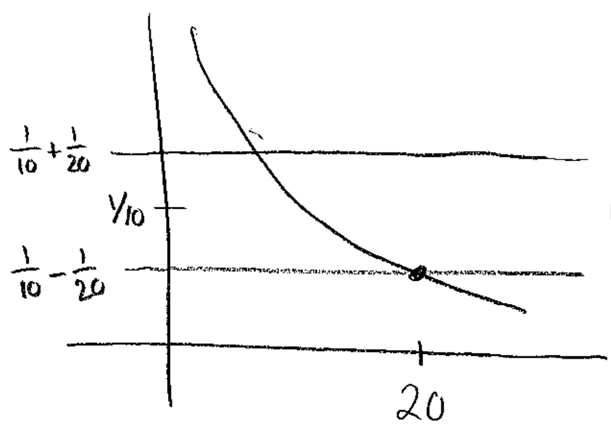
$x = \ln(1/\varepsilon)$

If B chooses $\varepsilon > 0$, then G can respond with $K = \ln(1/\varepsilon)$.

(From the picture we see that if $x > K$, then $|e^{-x} - 0| < \varepsilon$.)

(b) Show that $\lim\limits_{x \to \infty} \frac{1}{x} \neq \frac{1}{10}$. $\boxed{10}$

If B chooses $\varepsilon = \frac{1}{20}$, then G has no valid response.



(For $x > 20$, the entire graph is not between the two horizontal lines.)

4. For each part, write down a sequence a_n which satisfies the given properties. If it is not possible, give a reason why.

(a) the sequence a_n is bounded and it diverges **7**

$a_n = (-1)^n$

It is bounded above by 1 and below by -1.

It oscillates between 1 and -1 so it diverges.

(b) the sequence a_n diverges and the series $\sum_{k=1}^{\infty} a_k$ converges **7**

This is impossible.

If $\sum_{k=1}^{\infty} a_k$ converges, then a_n converges to 0.

(c) the sequence a_n converges and the series $\sum_{k=1}^{\infty} a_k$ diverges **7**

Consider the constant sequence $a_n = 1$.

- a_n converges to 1
- $S_n = \sum_{k=1}^{n} a_k = n \implies \lim_{n\to\infty} S_n = \infty$

$\implies \sum_{k=1}^{\infty} a_k$ diverges

10

5. For the four boxes in the table, write "Y" if the sequence satisfies the particular property, and write "N" if it does not. Please justify your answers in the space below.

	bounded above	bounded below	increasing	decreasing
$a_n = \frac{n^3}{n+2}$	N	Y	Y	N

① $\lim_{n\to\infty} \frac{n^3}{n+2} = \infty$ so a_n is not bounded above.

② $\frac{n^3}{n+2} \geq 0$ for all n, so a_n is bounded below

③ $f(x) = \frac{x^3}{x+2}$

$$f'(x) = \frac{(x+2)(3x^2) - x^3}{(x+2)^2} = \frac{2x^3 + 6x^2}{(x+2)^2} > 0$$

f is increasing when $x > 0$, so a_n is an increasing sequence.

(Therefore it is not decreasing.)

6. Evaluate the following series.

(Note that the starting value of k is different for different parts.)

(a) $\displaystyle\sum_{k=1}^{\infty}\left(\frac{1}{\sqrt{k}}-\frac{1}{\sqrt{k+1}}\right) = \boxed{1}$ 5

partial sums:

$$S_n = \sum_{k=1}^{n}\left(\frac{1}{\sqrt{k}}-\frac{1}{\sqrt{k+1}}\right) = \left(\frac{1}{\sqrt{1}}-\underbrace{\frac{1}{\sqrt{2}}\right)+\left(\frac{1}{\sqrt{2}}}_{\text{cancel}}-\underbrace{\frac{1}{\sqrt{3}}\right)+\ldots}_{\text{cancel}}+\left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\right)$$

$$= \frac{1}{\sqrt{1}} - \frac{1}{\sqrt{n+1}} = 1 - \frac{1}{\sqrt{n+1}}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1-\frac{1}{\sqrt{n+1}}\right) = 1-0 = 1$$

(b) $\displaystyle\sum_{k=1}^{\infty}\left(\sqrt{k}-\sqrt{k+1}\right) = \boxed{-\infty \quad \text{(diverges)}}$ 5

$$S_n = (\sqrt{1}-\sqrt{2})+(\sqrt{2}-\sqrt{3})+\ldots+(\sqrt{n}-\sqrt{n+1})$$

$$= 1-\sqrt{n+1}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1-\sqrt{n+1}\right) = -\infty$$

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(c) $\sum_{k=0}^{\infty} 2^k = \boxed{\infty \text{ (diverges)}}$ 5

Use the formula for a geometric series:

$$\sum_{k=0}^{\infty} x^k = \begin{cases} \frac{1}{1-x} & \text{if } -1<x<1 \\ \text{diverges} & \text{otherwise.} \end{cases}$$

In this case $x=2$ so the series diverges

(d) $\sum_{k=0}^{\infty} \frac{1+2^k}{4^k} = \boxed{10/3}$ 5

$$= \sum_{k=0}^{\infty} \left(\frac{1}{4}\right)^k + \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k$$

$$= \frac{1}{1-\frac{1}{4}} + \frac{1}{1-\frac{1}{2}}$$

$$= \frac{1}{\frac{3}{4}} + \frac{1}{\frac{1}{2}}$$

$$= \frac{4}{3} + 2$$

$$= \frac{10}{3}$$

7. Evaluate the following integrals.

(a) $\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \boxed{\pi}$ 10

$$= \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2}$$

$$= \lim_{a \to -\infty} \int_{a}^{0} \frac{dx}{1+x^2} + \lim_{b \to \infty} \int_{0}^{b} \frac{dx}{1+x^2}$$

$$= \lim_{a \to -\infty} \left[\arctan x\right]_{a}^{0} + \lim_{b \to \infty} \left[\arctan x\right]_{0}^{b}$$

$$= \lim_{a \to -\infty} \left(\arctan 0 - \arctan a\right) + \lim_{b \to \infty} \left(\arctan b - \arctan 0\right)$$

$$= 0 - \left(-\frac{\pi}{2}\right) + \frac{\pi}{2} - 0 = \pi$$

(b) $\int_{-1}^{1} \frac{dx}{x} = \boxed{\text{diverges}}$ 10

$$= \int_{-1}^{0} \frac{dx}{x} + \int_{0}^{1} \frac{dx}{x}$$

$$\hookrightarrow \lim_{b \to 0^+} \int_{b}^{1} \frac{dx}{x} = \lim_{b \to 0^+} \left[\ln|x|\right]_{b}^{1}$$

$$= \lim_{b \to 0^+} \left(\ln 1 - \ln b\right)$$

$$= \infty$$

$$\to \lim_{b \to 0^-} \int_{-1}^{b} \frac{dx}{x} = \lim_{b \to 0^-} \left[\ln|x|\right]_{-1}^{b}$$

$$= \lim_{b \to 0^-} \left(\ln|b| - \ln 1\right)$$

$$= -\infty$$

This is blank space.

Name a Pokémon (or something else if you prefer):