Lecture #1 (1/7/20)

Math 15300 section 14

Instructor: Alan Chang

Warmup:

(1) Sketch/name a function $f(x)$ which satisfies $\lim_{x \to \infty} f(x) = 0$

(2) ... which satisfies $\lim_{x \to \infty} f(x)$ does not exist.

Introductions: pair up and introduce
- name
- hobby
- something fun you did over winter break.

Quick overview of syllabus:

- $\left.\begin{array}{l}\text{office hours} \\ \text{problem sessions} \\ \text{TA's office hours}\end{array}\right\}$ T.B.D.

- Piazza: course website.

- Homework:
  - assigned after every lecture (posted by noon)
  - due every Th at 5pm
  - check your work/answers!
- Exams:

  Midterm 1: Tu 1/28 (week 4)

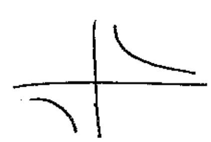
  Midterm 2: Th 2/20 (week 7)

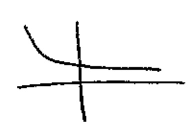
  Final: 3/19 Th. (week 11)
- Topics:
  - sequences and series
  - "elements of multivariable calculus"

Back to warmup problem.

(1) $\lim_{x \to \infty} f(x) = 0$:

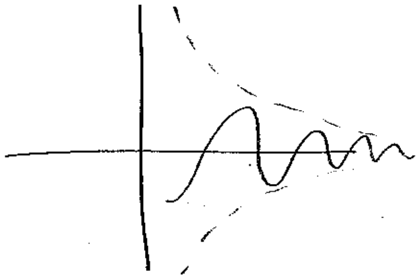
$f(x) = 0$

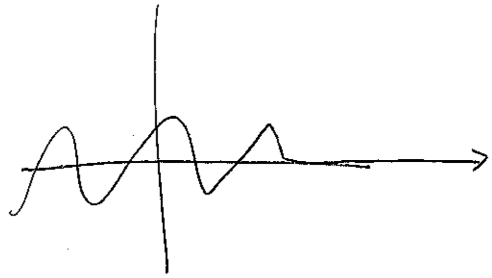
$f(x) = \frac{1}{x} \rightarrow$ 

$f(x) = \frac{1}{e^x} \rightarrow$ 

$f(x) = \frac{1}{x^2} \rightarrow$ 

You could also draw things like:

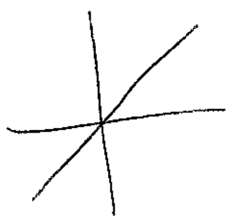


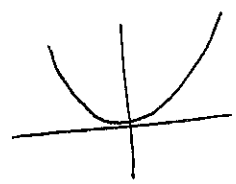


2. $\lim_{x\to\infty} f(x) =$ does not exist

$f(x) = x$ $\left(\lim_{x\to\infty} f(x) = \infty\right)$

$f(x) = x^2$

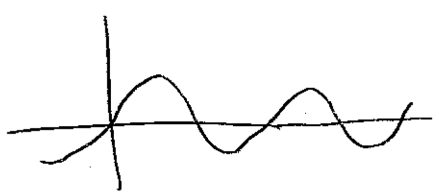


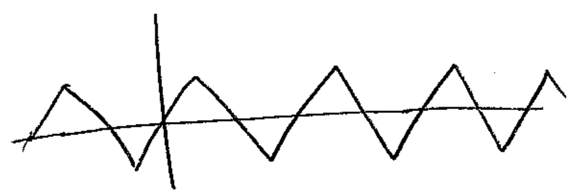


something bounded?

$f(x) = \sin x$

$f(x) =$ triangle wave





Let's try to come up with a definition of

$$\lim_{x\to\infty} f(x) = L.$$

Recall from last quarter: If c, L are real numbers,

"$\lim_{x \to c} f(x) = L$" means:

for all $\varepsilon > 0$, there exists a $\delta > 0$ such that

if $0 < |x-c| < \delta$, then $|f(x) - L| < \varepsilon$.

As a game: Two players: G and B.

The game starts with 3 things:

f (a function)

c, L (two real numbers)

G wants to show $\lim_{x \to c} f(x) = L$

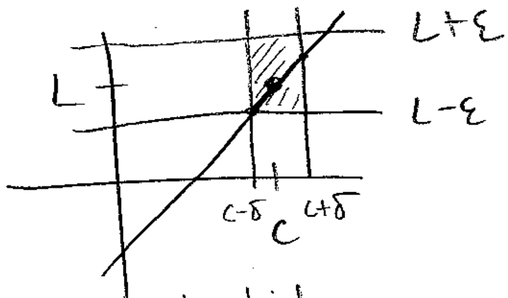
B wants to show $\lim_{x \to c} f(x) \neq L$

- B chooses a number $\varepsilon > 0$, and draws 2 horizontal lines $y = L + \varepsilon$ $y = L - \varepsilon$.

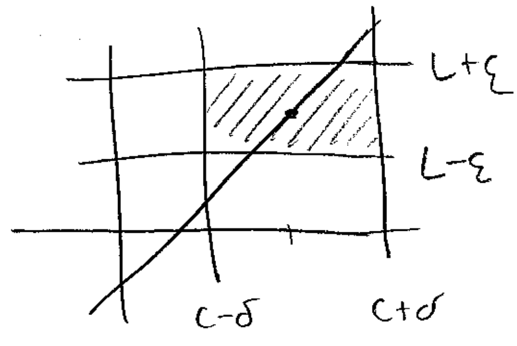
- G responds by choosing a number $\delta > 0$ and drawing 2 vertical lines $x = c + \delta$, $x = c - \delta$.

★ G's move is valid if

Every point of the graph between G's vertical lines (ignore the point $(c, f(c))$) lies between B's horizontal lines.



valid

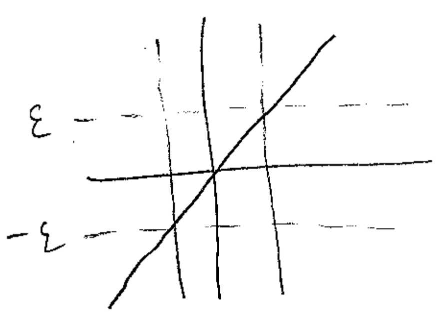


invalid

- If B can choose $\varepsilon > 0$ such that G has no valid response, then B wins. $(\lim_{x\to c} f(x) \neq L)$.
- Otherwise (i.e. if G always has a response, no matter what B does), G wins. $(\lim_{x\to c} f(x) = L)$.

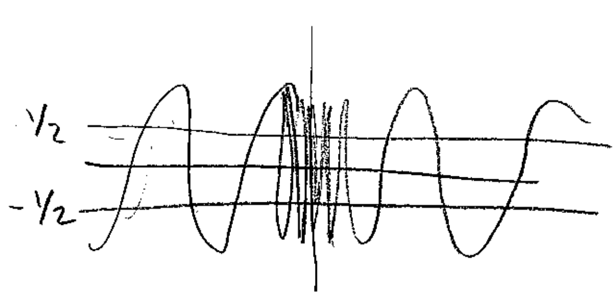
Recall examples:

$\lim_{x\to 0} x = 0$.



Take $\delta = \varepsilon$

$\lim_{x\to 0} \sin\frac{1}{x} \neq 0$.



Take $\varepsilon = 1/2$.

" $\lim_{x \to c} f(x) = L$ " means

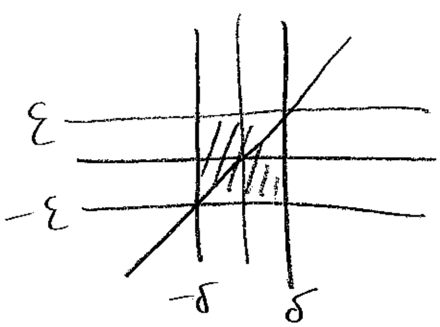
$\underbrace{\text{for all } \varepsilon > 0}_{1}$, $\underbrace{\text{there exists } \delta > 0}_{2}$ such that

$\underbrace{\text{if } 0 < |x-c| < \delta}_{3}$, $\underbrace{\text{then } |f(x) - L| < \varepsilon}_{4}$

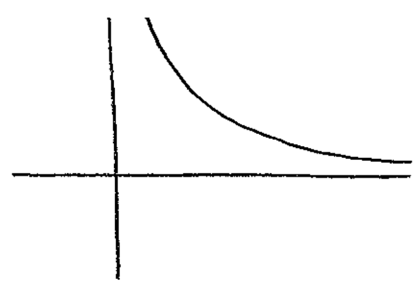
Question: How should we define $\lim_{x \to \infty} f(x) = L$?

(It doesn't make sense to plug in $c = \infty$ into the definition above.)

$\lim_{x \to 0} x = 0$

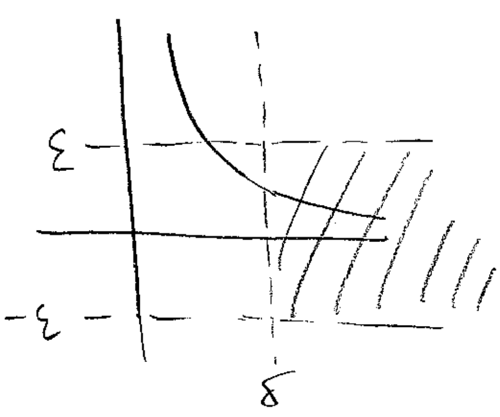


$\lim_{x \to \infty} \frac{1}{x} = 0$



[Ask students to discuss.].

Something is wrong with 3; let's change it.



Change 3 to "if $x > \delta$."

Geometrically, instead of looking at a normal box, we consider a box extending infinitely to the right.

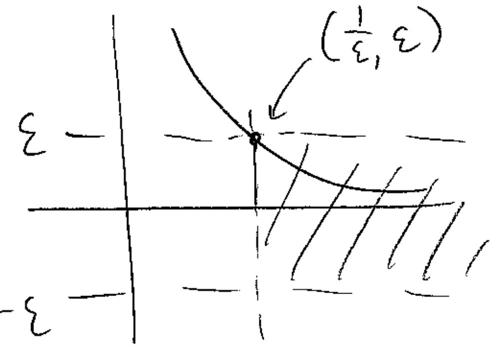
Also, mathematicians like using ε, δ for small numbers, so let's use a different letter instead:

<u>Definition</u>: "$\lim_{x\to\infty} f(x) = L$" means

For all $\varepsilon>0$, there exists $K>0$ such that if $x>K$ then $|f(x)-L|<\varepsilon$

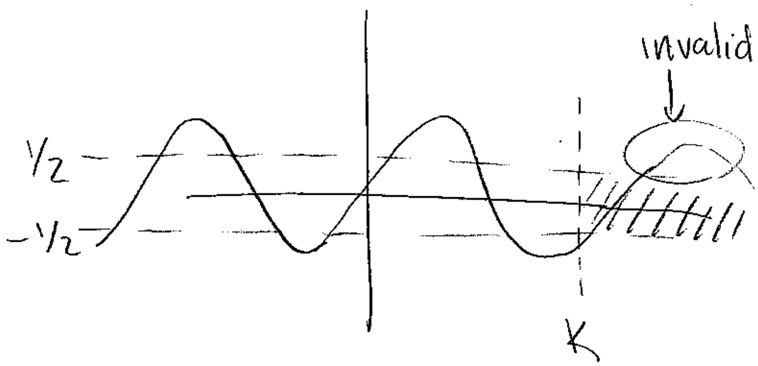
<u>Examples</u>:

1. $\lim_{x\to\infty} \frac{1}{x} = 0$.



No matter which ε B choose, G can respond with $K = 1/\varepsilon$.

② $\lim_{x\to\infty} \sin x \neq 0$:



If B choose $\varepsilon = 1/2$ then G has no valid response.

The reason we introduce this definition is that we need it when talking about sequences and series.

Lecture #2 (1/9/20).

Next topic: sequences.

Def: A sequence is a function whose domain is the positive integers ($\{1,2,3,\ldots\}$)

In theory we could write

" $f(x) = \frac{1}{x}, \quad x \in \{1,2,3,\ldots\}$ "

to represent the sequence $1, \frac{1}{2}, \frac{1}{3}, \ldots$,

but usually people write something like

" $a_n = \frac{1}{n}$ "

instead.

Q: How can we represent the sequence $1, 4, 9, 16, \ldots$?
(perfect squares)

A: $a_n = n^2$.

Q: What about $0, 1, 4, 9, 16, \ldots$?

A: $a_n = (n-1)^2$.

(The convention in this textbook is to start with $n=1$.)

Some ways to describe sequences:

Example: $a_n = n$ $(1, 2, 3, \ldots)$

This sequence is

- bounded below by 1
  (i.e., $a_n \geq 1$ for all n).
- bounded below by 0
  (i.e., $a_n \geq 0$ for all n).
- not bounded above.
- increasing
  (i.e. $a_{n+1} > a_n$ for all n)

Example:

$$\begin{array}{l} a_1 \;\; a_2 \\ \downarrow \;\; \downarrow \\ 1, 1, 2, 2, 3, 3, 4, 4, \ldots \end{array}$$

This sequence is not increasing.

However, it is nondecreasing.

In general, we have the following words:

- bounded above (by ...)
- bounded below (by ...)
- increasing
- nondecreasing
- decreasing
- nonincreasing

If a sequence satisfies one of these, it is monotonic.

A non-monotonic function?

$1, 0, 1, 0, 1, 0, \ldots$

Ex: $a_n = \dfrac{n}{n+1}$. What properties does this sequence have?

$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$

- bounded below by $\frac{1}{2}$.

$$\left.\begin{aligned} \frac{n}{n+1} &\geq \frac{1}{2} \\ 2n &\geq n+1 \\ n &\geq 1 \end{aligned}\right\}$$ This shows that $a_n \geq \frac{1}{2}$ for all n.

- bounded above by 1.

$$\frac{n}{n+1} \leq 1$$

$$n \leq n+1$$

$$0 \leq 1.$$

- increasing. several ways to see this:

1. $$\frac{a_{n+1}}{a_n} = \frac{\frac{n+1}{n+2}}{\frac{n}{n+1}} = \frac{(n+1)^2}{n(n+2)} = \frac{n^2+2n+1}{n^2+2n} > 1.$$

so $a_{n+1} > a_n$.

2. $$a_{n+1} - a_n = \frac{n+1}{n+2} - \frac{n}{n+1} = \frac{n^2+2n+1-(n^2+2n)}{(n+2)(n+1)}$$

$$= \frac{1}{(n+2)(n+1)} > 0$$

so $a_{n+1} > a_n$.

3. Consider the function $f(x) = \frac{x}{x+1}$ defined on all real numbers x except -1.

Then $a_n = f(n)$ for all n

$$f'(x) = \frac{x+1-x}{(x+1)^2} = \frac{1}{(x+1)^2} > 0.$$

so f is increasing.

so $f(1) < f(2) < f(3) < \dots$

so $a_1 < a_2 < a_3 < \dots$.

<u>Example</u>: $a_n = \frac{2^n}{n!}$

$a_1 = \frac{2}{1} = 2$

$a_2 = \frac{4}{2} = 2$

$a_3 = \frac{8}{6} = \frac{4}{3} = 1.33\dots$

$a_4 = \frac{16}{24} = \frac{2}{3} = 0.66\dots$

Maybe it starts decreasing at $n=2$?
How can we check this?

- $a_{n+1} - a_n$?
- $\frac{a_{n+1}}{a_n}$?
- $a_n = f(n)$, $f'(x)$?

In this case $\frac{a_{n+1}}{a_n}$ is the easiest to look at.

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}/(n+1)!}{2^n/n!} = \frac{2^{n+1}}{2^n} \frac{n!}{(n+1)!} = \frac{2}{n+1} \underbrace{> 1}_{\text{for all } n \geq 2}$$

so:
- a_n is nonincreasing
- a_n is decreasing for $n \geq 2$.
- a_n is bounded above by $a_1 = 2$.
- a_n is bounded below by 0.

Next topic: limit of a sequence.

We want to define $\lim_{n\to\infty} a_n$, but there's no need to do any extra work.

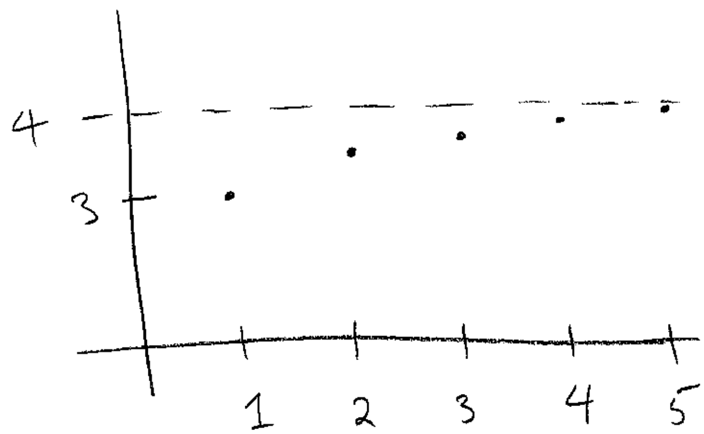
"$\lim_{n\to\infty} a_n = L$" means:

"For all $\varepsilon > 0$, there exists $K > 0$ such that if $n \geq K$, then $|a_n - L| < \varepsilon$."

(The book says K must be a positive integer, but that's not important.)

This is exactly the same definition as $\lim_{x\to\infty} f(x) = L$, except now, n must be a positive integer. The geometric picture stays the same, except the graph of $f(x)$ is replaced by a sequence of points.

Example: $a_n = \frac{4n-1}{n} = 4 - \frac{1}{n}$.



We want to show $a_n \to 4$.

If B chooses $\varepsilon > 0$, then G chooses a $K > 1/\varepsilon$.

If $n \geq K$ then $|a_n - 4| = \frac{1}{n} \leq \frac{1}{K} < \varepsilon$ as desired.

Lecture 3 (1/14/20)

Example (making sense of decimal expansions)

Let $x = 0.1234\,1234\,1234\ldots$

what does it mean to have a decimal expansion that is infinitely long?

By definition, x is the limit of the sequence

$a_1 = 0.1$
$a_2 = 0.12$
$a_3 = 0.123$
$a_4 = 0.1234$
$a_5 = 0.12341$
$\vdots$

In general, if $x = 0.b_1 b_2 b_3 \ldots$ (digits) then by definition
x is the limit of

$$a_1 = 0.b_1$$
$$a_2 = 0.b_1 b_2$$
$$\vdots$$
$$a_n = 0.b_1 b_2 \ldots b_n$$
$$\vdots$$

Let's consider $x = 0.999\ldots$

By definition, $x = \lim_{n\to\infty} a_n$,

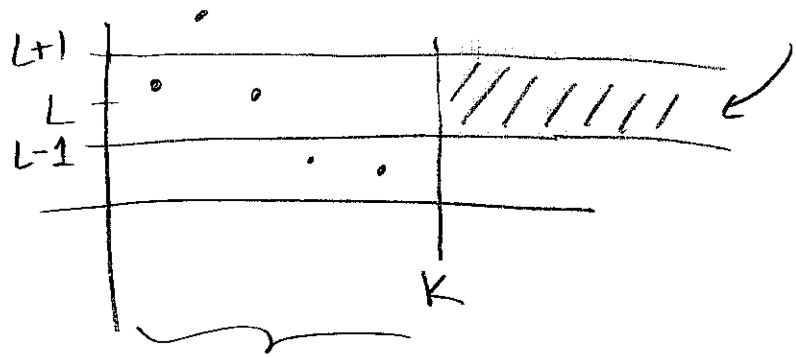
where
$$a_1 = 0.9 = \frac{9}{10} = 1 - \frac{1}{10}$$
$$a_2 = 0.99 = \frac{99}{100} = 1 - \frac{1}{100}$$
$$a_3 = 0.999 = \ldots = 1 - \frac{1}{1000}$$
$$\vdots$$
$$a_n = 1 - \frac{1}{10^n}.$$

From this we see $x = \lim_{n\to\infty} a_n = 1$.

Definition. A sequence that has a limit is said to be convergent. A sequence that has no limit is said to be divergent.

<u>Fact</u>: Every convergent seq. is bounded.

Suppose $a_n \to L$.



For $n \geq K$, the sequence lies in this box

For $n < K$, there are only finitely many terms.

The statement above is the same as:

Every unbounded sequence is divergent.

So: whenever you see an unbounded seq., you know right away it is divergent.

Some basic properties of limits of sequences (Note these are the same as for functions)

If $a_n \to L$ and $b_n \to M$ then

(i) $a_n + b_n \to L + M$

(ii) $\alpha a_n \to \alpha L$ (α is any real number)

(iii) $a_n b_n \to LM$

(iv) if $M \neq 0$ and each $b_n \neq 0$ then

$$\frac{1}{b_n} \to \frac{1}{M}, \quad \frac{a_n}{b_n} \to \frac{L}{M}$$

Example: $a_n = \dfrac{3n^4 - 2n^2 + 1}{n^5 - 3n^3} = \dfrac{3/n - 2/n^3 + 1/n^5}{1 - 3/n^2}$

divide by n^5

$$\lim_{n\to\infty} a_n = \frac{0+0+0}{1+0} = 0$$

The squeeze/sandwich/pinching theorem also works for sequences.

if $\left.\begin{array}{l} a_n \le b_n \le c_n \\ a_n \to L \\ c_n \to L \end{array}\right\}$ then $b_n \to L$

Monotone convergence thm: Suppose a_n is nondecreasing and bounded above. Then a_n converges to its least upper bound

Example:

$$a_n = \frac{\sin n}{n}.$$

$-\frac{1}{n} \le a_n \le \frac{1}{n}$, so $\lim_{n\to\infty} a_n = 0$.

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continuous functions:

If $c_n \to c$ and f is continuous

then $f(c_n) \to f(c)$. $\left[\lim_{n\to\infty} f(c_n) = f\left(\lim_{n\to\infty} c_n\right) = f(c)\right]$.

Example: $a_n = \sin\frac{1}{n}$. let $c_n = \frac{1}{n}$, $f(x) = \sin x$

since $\frac{1}{n} \to 0$, $a_n \to f(0) = 0$.

Lecture 4 (1/16/20)

Example: Fix $x > 0$. $\lim_{n\to\infty} x^{1/n} = ?$

Let $a_n = x^{1/n}$

Let $b_n = \ln(x^{1/n}) = \frac{1}{n}\ln x. \Rightarrow \lim_{n\to\infty} b_n = 0.$

$a_n = e^{b_n}$ so $\lim_{n\to\infty} a_n = \lim_{n\to\infty} e^{b_n} = e^{(\lim_{n\to\infty} b_n)} = e^0$
$= 1.$

Example: Fix x (any real number). $\lim_{n\to\infty} \frac{x^n}{n!} = ?$

There are many ways to show the limit is zero.
Here is one way:

$\boxed{\text{Maybe do this concretely with } x = 100?}$

$$a_n = \frac{x^n}{n!}$$

$$\frac{a_{n+1}}{a_n} = \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} = \frac{x}{n+1}.$$

Let N be an integer s.t. $N \geq 2x$.
Then if $n \geq N$, we have $n+1 \geq 2x$

So $\frac{a_{n+1}}{a_n} = \frac{x}{n+1} \leq \frac{1}{2}.$

So: $a_{N+1} \leq \frac{1}{2} a_N$
$a_{N+2} \leq \frac{1}{2} a_{N+1}$
$\vdots$

so $\quad a_n \le \frac{1}{2^{n-N}} a_N \quad$ if $\quad n \ge N.$

$$= \frac{1}{2^n} \cdot 2^N a_N$$

so $\quad \lim_{n\to\infty} a_n = 0. \quad$ (since $\frac{1}{2^n} \to 0$ and $2^N a_N$ is a constant).

For many other limits we can use L'Hôpital's rule:

Back to functions defined on real numbers.

Recall: $\quad$ If $\quad \lim_{x\to c} f(x) = L$

$\lim_{x\to c} g(x) = M \quad$ and $M \neq 0$

then $\quad \lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{M}.$

(Here, c can be ∞ also).

But: $\quad \lim_{x\to 0} \frac{\sin x}{x} \overset{??}{=} \frac{0}{0}$??

$\frac{0}{0}$ is called an indeterminate form.

so is $\frac{\infty}{\infty}$.

L'Hôpital's rule : (for calculating $\lim_{x\to c} \frac{f(x)}{g(x)}$)

If you get either $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}.$$

(Here c can be ∞.)

Example:

$\lim_{x\to 0} \frac{\sin x}{x}$ Taking limits of num/denom separately gives you $\frac{0}{0}$.

$$= \lim_{x\to 0} \frac{\cos x}{1} = 1.$$

Example: Let α be any positive number.

$\lim_{x\to\infty} \frac{\ln x}{x^\alpha}$ $\leftarrow$ get $\frac{\infty}{\infty}$

$$= \lim_{x\to\infty} \frac{\left(\frac{1}{x}\right)}{\alpha x^{\alpha-1}} = \lim_{x\to\infty} \frac{1}{\alpha}\frac{1}{x^\alpha} = 0 \quad \text{since } \alpha > 0.$$

Example

$$\lim_{x\to\infty} x^{1/x} = L$$

$\uparrow$ see prev example

$$\ln L = \lim_{x\to\infty} \ln(x^{1/x}) = \lim_{x\to\infty} \frac{\ln x}{x} = 0$$

$$\Rightarrow L = e^0 = 1 \Rightarrow \lim_{x\to\infty} x^{1/x} = 1.$$

Example:
$$\lim_{t\to\infty}\left(1+\frac{x}{t}\right)^t = L.$$

$$\ln L = \lim_{t\to\infty} \ln\left(1+\frac{x}{t}\right)^t = \lim_{t\to\infty} \underset{\substack{\uparrow\\ \infty}}{t}\ \underset{\substack{\uparrow\\ 0}}{\ln\left(1+\frac{x}{t}\right)}.$$

$$= \lim_{t\to\infty} \frac{\ln(1+x/t)}{1/t} \qquad \left(\frac{0}{0}\right).$$

$$= \lim_{t\to\infty} \frac{\frac{1}{1+\frac{x}{t}}\cdot x\,\frac{d}{dt}\left(\frac{1}{t}\right)}{\frac{d}{dt}\left(\frac{1}{t}\right)}$$

$$= \lim_{t\to\infty} \frac{x}{1+\frac{x}{t}} = x$$

$$\Longrightarrow \boxed{\lim_{t\to\infty}\left(1+\frac{x}{t}\right)^t = e^x}$$

From these calculations, we also get the following:

$$\begin{cases} \dfrac{\ln n}{n^\alpha} \to 0 & (\alpha>0). \\ n^{1/n} \to 0 \\ \left(1+\dfrac{x}{n}\right)^n \to e^x \end{cases}$$

(Pollev.com/alanchang)

Exercise: consider the following sequences:

$$n, \quad n^2, \quad n^{1/2}, \quad n^n, \quad n!, \quad \ln n, \quad e^n$$

Can you order them by how "fast" they grow?

$\left(\lim_{n\to\infty} \frac{a_n}{b_n} = 0 \leftarrow \text{when is this true?} \right)$.

Answer:

$$\ln n, \quad n^{1/2}, \quad n, \quad n^2, \quad e^n, \quad n!, \quad n^n$$

$\longleftarrow$ slowest $\qquad\qquad$ $\longrightarrow$ fastest.

(useful in C.S. analysis of algorithms).

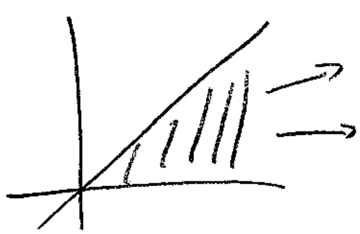
Next topic: improper integrals

$\int_a^b f(x)\,dx$ $\leftarrow$ bounded interval
$\leftarrow$ bounded function

① What if the interval is not bounded?

e.g. $\int_0^\infty f(x)\,dx$?

$$\int_0^\infty x\,dx$$



this has infinite area.

To see this algebraically,

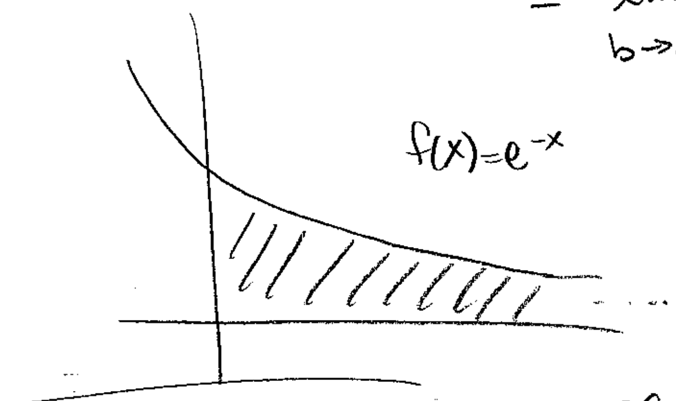
$$\int_0^\infty x\,dx = \lim_{b\to\infty}\int_0^b x\,dx = \lim_{b\to\infty}\frac{b^2}{2} = \infty.$$

What's an example when $\int_0^\infty f(x)\,dx$ is finite?

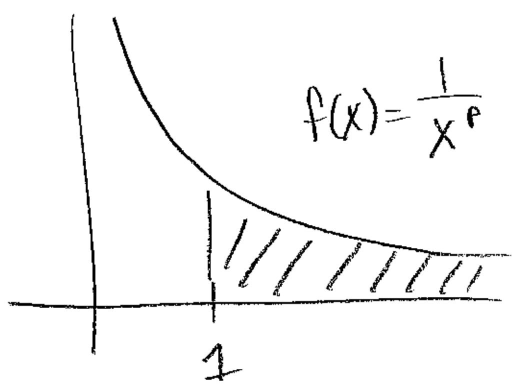
(Lecture 5 1/21/20)

$f(x) = e^{-x}$. $\displaystyle\int_0^\infty f(x)\,dx = \lim_{b\to\infty}\int_0^b e^{-x}\,dx = \lim_{b\to\infty}\left[\frac{e^{-x}}{-1}\right]_0^b$

$$= \lim_{b\to\infty}\left(\frac{e^{-b}}{-1} - \frac{1}{-1}\right) = 1.$$



Ex: Let's consider $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ $\quad p>0$.



For $p\neq 1$

$$\int_1^b \frac{1}{x^p}\,dx = \int_1^b x^{-p}\,dx$$

$$= \left[\frac{x^{-p+1}}{-p+1}\right]_1^b$$

$$= \frac{1}{1-p}\left(b^{1-p}-1\right).$$

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^p}\,dx = \lim_{b\to\infty}\frac{1}{1-p}\left(b^{1-p}-1\right) = \begin{cases}\frac{1}{p-1} & p>1\\ \infty & p<1\end{cases}$$

For $p=1$, $\displaystyle\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty}\ln b = \infty.$

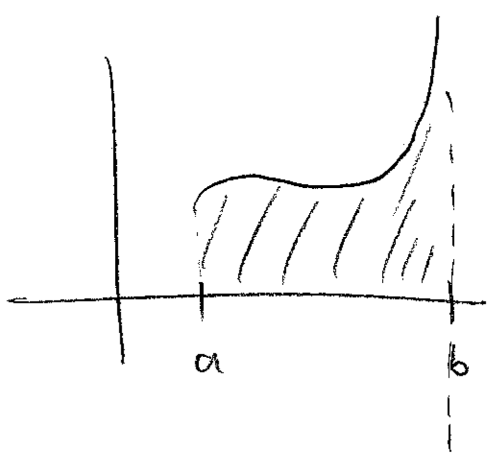
So: $\int_1^\infty \frac{dx}{x^p}$: converges if $p>1$
diverges if $p\le 1$.

For $\int_{-\infty}^{\infty} f(x)\,dx$, we define it as

$$\int_{-\infty}^{0} f(x)\,dx + \int_0^\infty f(x)\,dx.$$

This is how we handle unbounded intervals.

Unbounded functions?



Suppose f is unbounded at b. Can we make sense of

$$\int_a^b f(x)\,dx \,?$$

$$\int_a^b f(x)\,dx = \lim_{c\to b^-} \int_a^c f(x)\,dx.$$

Example:

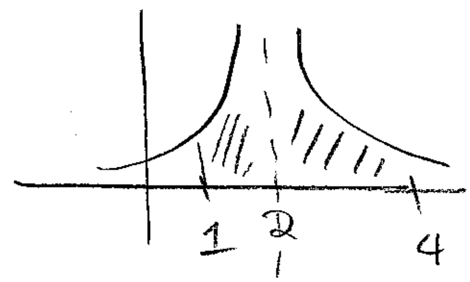
(a) $$\int_0^1 \underbrace{(1-x)^{-2/3}}_{\substack{\text{unbounded}\\ \text{at } x=1}}\,dx = \lim_{c\to 1^-} \underbrace{\int_0^c (1-x)^{-2/3}\,dx}_{-3(1-c)^{1/3}+3}$$

$$= \lim_{c \to 1^-} \left[-3(1-c)^{1/3} + 3 \right] = 3.$$

(b) $$\int_0^2 \frac{dx}{x} = \lim_{c \to 0^+} \int_c^2 \frac{dx}{x} = \lim_{c \to 0^+} \left[\ln x \right]_c^2$$

$$= \lim_{c \to 0^+} \left[\ln 2 - \ln c \right] = \infty.$$

Example: $$\int_1^4 \frac{dx}{(x-2)^2}$$



$$= \underbrace{\int_1^2 \frac{dx}{(x-2)^2}}_{} + \int_2^4 \frac{dx}{(x-2)^2}$$

$$\lim_{c \to 2^-} \int_1^c \frac{dx}{(x-2)^2} = \lim_{c \to 2^-} \left[-\frac{1}{c-2} - 1 \right] = \infty.$$

So the integral diverges.

WARNING: We have to identify the discontinuity ourselves.

We can't just apply FTC blindly:

$$\int_1^4 \frac{dx}{(x-2)^2} = \left[-\frac{1}{x-2} \right]_1^4 = -\frac{3}{2}. \quad \text{WRONG!}$$

Another example of improper integral:

Let n be a positive integer:

$$\int_0^\infty x^n e^{-x}\,dx = \lim_{b\to\infty} \int_0^b x^n e^{-x}\,dx$$

need some calculations.

$$= n \int_0^\infty x^{n-1} e^{-x}\,dx$$

For $n=0$:

$$\int_0^\infty x^0 e^{-x}\,dx = \int_0^\infty e^{-x}\,dx = 1.$$

Put these two facts together to conclude

$$\int_0^\infty x^n e^{-x}\,dx = n!$$

Next chapter: series

First of all, do not confuse the words "sequence" and "series"

sequence: $a_0, a_1, a_2, \dots$

series: $a_0 + a_1 + a_2 + \dots$

<u>Example</u>: $\underbrace{a_n = \frac{1}{2^n}}_{\text{geometric sequence}}$ (starting at $n=0$).

Geometric series: $\frac{1}{2^0} + \frac{1}{2^1} + \frac{1}{2^2} + \cdots$

$$= a_0 + a_1 + a_2 + \cdots$$

$$= \sum_{k=0}^{\infty} a_k$$

$$= \sum_{j=0}^{\infty} a_j$$

sigma notation:

$$a_m + a_{m+1} + \cdots + a_{n-1} + a_n = \sum_{k=m}^{n} a_k.$$

k is a "dummy variable"

$\int_a^b f(x)\,dx$ x is a "dummy variable"

What is the value of an infinite series $\sum_{k=0}^{\infty} a_k$?

Let's use "finite things" to study "infinite things" start with a sequence $a_0, a_1, a_2, \ldots$

To make sense of $a_0 + a_1 + a_2 + \cdots$ $\left(= \sum_{k=0}^{\infty} a_k \right)$

consider the partial sums:

$$S_0 = a_0$$

$$S_1 = a_0 + a_1$$

$$S_2 = a_0 + a_1 + a_2$$

$$\vdots$$

$$S_n = a_0 + a_1 + \cdots + a_n = \sum_{k=0}^{n} a_k$$

Definition: $$\sum_{k=0}^{\infty} a_k = \lim_{n \to \infty} \underbrace{\sum_{k=0}^{n} a_k}_{\text{partial sum}}$$

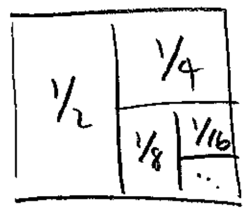
If the limit exists, we say the series converges. Otherwise we say it diverges.

Back to the geometric series example.

$$\sum_{k=0}^{\infty} \frac{1}{2^k} = \frac{1}{2^0} + \frac{1}{2^1} + \frac{1}{2^2} + \cdots$$

Let's just consider.

$$\underbrace{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots}$$



whole square has area 1

It seems like this should be 1.

So $\displaystyle\sum_{k=0}^{\infty}\frac{1}{2^k}$ should $=2$.

How do we see this?

$$S_n = \frac{1}{2^0}+\frac{1}{2^1}+\cdots+\frac{1}{2^n}.$$

very clever step: Multiply by $\frac{1}{2}$.

$$\frac{1}{2}S_n = \frac{1}{2^1}+\frac{1}{2^2}+\cdots+\frac{1}{2^n}+\frac{1}{2^{n+1}}.$$

Now subtract:

$$S_n-\frac{1}{2}S_n = \frac{1}{2^0}-\frac{1}{2^{n+1}}$$

$$\frac{1}{2}S_n = 1-\frac{1}{2^{n+1}}$$

$$S_n = 2-\frac{1}{2^n}.$$

$$\lim_{n\to\infty} S_n = 2 \implies \sum_{k=0}^{\infty}\frac{1}{2^k}=2.$$

For a geometric series with common ratio x:

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots$$

same argument.

$$S_n = 1 + x + \cdots + x^n.$$

$$x S_n = x + \cdots + x^n + x^{n+1}$$

$$(1-x) S_n = 1 - x^{n+1} \qquad \leftarrow \text{so clever!!}$$

$$S_n = \frac{1-x^{n+1}}{1-x}.$$

$$\sum_{k=0}^{\infty} x^k = \lim_{n\to\infty} \frac{1-x^{n+1}}{1-x} = \begin{cases} \frac{1}{1-x} & \text{if } |x|<1 \\ \infty & \text{if } |x|>1. \end{cases}$$

So:

(i) if $|x|<1$ then $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$

(ii) if $|x| \geq 1$ then $\sum_{k=0}^{\infty} x^k$ diverges.

Another example:

$$\sum_{k=0}^{\infty} \frac{1}{(k+1)(k+2)} \qquad a_k = \frac{1}{(k+1)(k+2)}$$

$S_n = a_0 + a_1 + \cdots + a_n$.

observe $\dfrac{1}{(k+1)(k+2)} = \dfrac{1}{k+1} - \dfrac{1}{k+2}$ (partial fraction decomp.)

so $S_n = a_0 + a_1 + a_2 + \cdots + a_n$

$$= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n+1} - \frac{1}{n+2}\right)$$

$$= \frac{1}{1} - \frac{1}{n+2}$$

telescoping sum!

so $\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 - \frac{1}{n+2}\right) = 1$.

So $\displaystyle\sum_{k=0}^{\infty} \frac{1}{(k+1)(k+2)} = \underline{1}$.

We see how to evaluate geometric series and telescoping series. In general, it is hard to tell what value a series converges to. We're happy enough with knowing whether it converges or diverges.

Example: Let $b_1, b_2, b_3 \ldots$ be integers between 0 and 9.

Consider $\displaystyle\sum_{k=1}^{\infty} \frac{b_k}{10^k}$. Does this converge?

partial sum: $\displaystyle t_n = \sum_{k=1}^{n} \frac{b_k}{10^k}$.

The sequence t_n is a nondecreasing sequence since $t_{n+1} - t_n = \dfrac{b_{n+1}}{10^{n+1}} \geq 0$.

Furthermore,

$$t_n = \sum_{k=1}^{n} \frac{b_k}{10^k} \leq \sum_{k=1}^{n} \frac{9}{10^k} = 9\left(\frac{1}{10^1} + \frac{1}{10^2} + \cdots + \frac{1}{10^n}\right)$$

$$\leq 9\left(\frac{1}{10^1} + \frac{1}{10^2} + \cdots\right) \quad \text{infinite sum.}$$

$$= 1$$

So the sequence t_n is bounded above by 1.

Therefore the sequence t_n converges.

so the series $\displaystyle\sum_{k=1}^{\infty} \frac{b_k}{10^k}$ converges.

This is how we define $0.b_1b_2b_3\ldots$

Properties of series:

(1) $$\sum_{k=0}^{n}(a_k+b_k) = (a_0+b_0)+\cdots+(a_n+b_n)$$
$$= (a_0+\cdots+a_n)+(b_0+\cdots+b_n)$$
$$= \sum_{k=0}^{n} a_k + \sum_{k=0}^{n} b_k.$$

(2) $$\sum_{k=0}^{n} \alpha a_k = \alpha \sum_{k=0}^{n} a_k$$

(just like properties of the definite integral)

Here are the infinite series version:

(1) If $\sum_{k=0}^{\infty} a_k$ converges and $\sum_{k=0}^{\infty} b_k$ converges,

then $$\sum_{k=0}^{\infty}(a_k+b_k) = \sum_{k=0}^{\infty} a_k + \sum_{k=0}^{\infty} b_k$$

(2) If $\sum_{k=0}^{\infty} a_k$ converges, then $\sum_{k=0}^{\infty} \alpha a_k = \alpha \sum_{k=0}^{\infty} a_k$.

Lecture 8 (1/30/20)

Next property.

If $\sum_{k=0}^{\infty} a_k$ converges then $a_k \to 0$.

Why is this true?

Suppose $\sum_{k=0}^{\infty} a_k$ converges. Let $L=\sum_{k=0}^{\infty} a_k$

Let $S_n = \sum_{k=0}^{n} a_k$. Then:

$$a_n = S_n - S_{n-1}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (S_n - S_{n-1})$$

$$\lim_{n\to\infty} a_n = L - L. = 0.$$

<u>Contrapositive</u>: (ask class).

If $a_k \not\to 0$ then $\sum_{k=0}^{\infty} a_k$ diverges

<u>Q</u>: Consider the statement

"If $a_k \to 0$ then $\sum_{k=0}^{\infty} a_k$ converges."

<u>Is this true?</u>

No! $a_k = \frac{1}{k}$. $a_k \to 0$

but $\sum_{k=0}^{\infty} \frac{1}{k} = \infty$ (by integral test, which we'll see soon).

Suppose we have

a Sequence $a_0, a_1, a_2, \ldots$ with nonneg. terms.

The sequence of partial sums $S_n = \sum_{k=0}^{n} a_k$.

is a nondecreasing sequence since

$$S_{n+1} - S_n = a_{n+1} \geq 0.$$

If the sequence of partial sums is bounded above, then by the monotone convergence theorem, S_n converges.

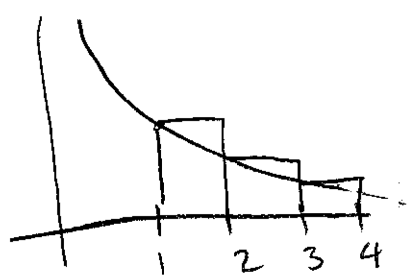
This means that the series $\sum_{k=0}^{\infty} a_k$ converges.

So: A series with nonnegative terms converges if and only if the sequence of partial sums is bounded

Here are 3 different tests that use this fact.

① Integral test.

Example: $a_n = \frac{1}{n}$. $f(x) = \frac{1}{x}$. we know $\int_1^\infty f(x)\,dx = \infty$



upper Riemann sum

From the picture:

$$a_1 + a_2 + a_3 \geq \int_1^4 f(x)\,dx.$$

So $S_n = a_1 + a_2 + \cdots + a_n \geq \underbrace{\int_1^{n+1} f(x)\,dx}_{\text{This goes to } \infty \text{ as } n \to \infty.}$

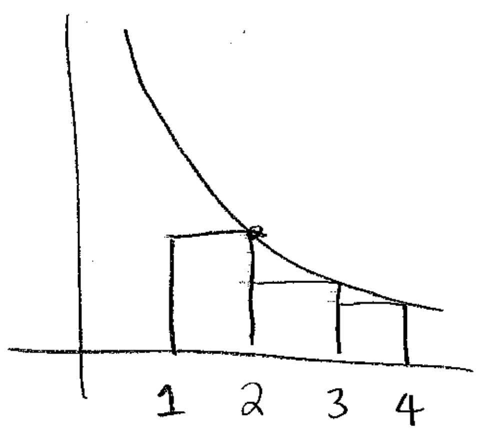
This goes to ∞ as $n \to \infty$.

So S_n is unbounded.

So $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges.

Example: $a_n = \frac{1}{n^2}$ $f(x) = \frac{1}{x^2}$.

We know $\int_1^{\infty} f(x)\,dx$ converges.



$\underbrace{a_2 + a_3 + a_4}_{S_4 - a_1} \leq \int_1^4 f(x)\,dx$

$S_n = a_1 + \int_1^n f(x)\,dx$

So $S_n \leq \underbrace{a_1 + \int_1^{\infty} f(x)\,dx}_{\text{this is just a constant.}}$ for all n.

this is just a constant.

So the seq S_n is bounded above. and increasing.
Therefore it converges.

In general, if f is continuous, positive, and decreasing, then

$$\underset{\text{(lower sum)}}{f(2)+\cdots+f(n+1)} \leq \int_1^{n+1} f(x)\,dx \leq \underset{\text{(upper sum)}}{f(1)+f(2)+\cdots+f(n)}$$

Integral test: If f is continuous, positive, and decreasing on $[1,\infty)$ then

$\sum_{k=1}^{\infty} f(k)$ converges if and only if $\int_1^{\infty} f(x)\,dx$ converges.

Example: Recall

$$\int_1^{\infty} \frac{1}{x^p}\,dx \begin{cases} \text{converges if } p>1 \\ \text{diverges if } p\leq 1. \end{cases}$$

So: $\sum_{k=1}^{\infty} \frac{1}{k^p}$ → converges if $p>1$
→ diverges if $p \leq 1$.

(What does it actually converge to? That is a difficult problem.)

$a_n = \sin^2(n\pi) + \frac{1}{n^2}$ $\quad \sum a_n = \infty$

$\int_1^{\infty} \left(\sin^2(x\pi) + \frac{1}{x^2}\right) dx < \infty$

(2) Basic comparison test.

Suppose a_k, b_k are nonnegative sequences and $a_k \le b_k$ for all k.

(i) If $\sum b_k$ converges then $\sum a_k$ converges

(2) if $\sum a_k$ diverges then $\sum b_k$ diverges

[(2) is just the contrapositive of (1).]

Q: If $\sum a_k$ conv. does $\sum b_k$ conv? Ask for counterex.

Let S_n be the partial sums of a_k

$$S_n = \sum_{k=0}^{n} a_k \le \sum_{k=0}^{n} b_k \le \sum_{k=0}^{\infty} b_k$$

since $a_k \le b_k$ ... since $b_k \ge 0$.

So if $\sum_{k=0}^{\infty} b_k$ is a finite number, then it is an upper bound for the sequence S_n.

This explains why (1) is true.

Note: we don't really need $a_k \le b_k$ for all k. It's enough to have this inequality for k sufficiently large.

Example : $\sum_{k=2}^{\infty} \frac{1}{\ln k}$ $\quad \ln k \leq k.$

$$\Rightarrow \frac{1}{\ln k} \geq \frac{1}{k}.$$

$\sum_{k=2}^{\infty} \frac{1}{k}$ diverges so $\sum_{k=2}^{\infty} \frac{1}{\ln k}$ diverges.

(3) Limit comparison test.

Let a_k, b_k be seq with positive terms.

If $a_k/b_k \to L$ and L is positive then

$\sum a_k$ converges if and only if $\sum b_k$ converges.

Why is this true?

Suppose $\frac{a_k}{b_k} \to L$ and L is positive.

Use the ε, K definition.

We can find $K > 0$ s.t. if $k \geq K$ then

$$\underbrace{\left| \frac{a_k}{b_k} - L \right| < \frac{L}{2}}$$

$$\frac{1}{2} L < \frac{a_k}{b_k} < \frac{3}{2} L$$

So $\left(\frac{1}{2}L\right) b_k \leq a_k \leq \left(\frac{3}{2}L\right) b_k$ for sufficiently large k.

Now we can use the basic comparison test!

<u>Example</u>: $a_k = \dfrac{3k^2+2k+1}{k^3+1}$ $\quad \sum a_k$??

Take the highest powers. Let $b_k = \dfrac{k^2}{k^3} = \dfrac{1}{k}$.

$$\frac{a_k}{b_k} = \frac{k(3k^2+2k+1)}{k^3+1} \longrightarrow 3.$$

We know $\sum b_k$ diverges.

So $\sum a_k$ diverges too.

<u>Next</u>: convergence tests of series $\sum a_k$ where a_k can have positive and negative terms.

Here is one way to test if $\sum a_k$ converges.

Fact:

If $\sum |a_k|$ converges then $\sum a_k$ converges.

Def: If $\sum |a_k|$ converges, we say the series $\sum a_k$ converges absolutely.

So the Fact can be restated as:

Absolutely convergent series are convergent.

Example:

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$$

Q: Is this absolutely convergent?

$$\sum_{k=1}^{\infty} \left| \frac{(-1)^{k+1}}{k^2} \right| = \sum_{k=1}^{\infty} \frac{1}{k^2} \leftarrow \text{we know this converges.}$$

So the series is absolutely convergent.

Therefore, the series itself converges.

Example:

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

If we take absolute values.

$$\sum_{k=1}^{\infty} \left|\frac{(-1)^{k+1}}{k}\right| = \underbrace{\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots}_{\text{This diverges.}}$$

So: $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ is <u>not</u> absolutely convergent.

But does it converge?

<u>Alternating series test</u>.

Let $a_0, a_1, a_2, \ldots$ be a <u>decreasing</u> seq of <u>positive</u> numbers. Then

$$\underbrace{\sum_{k=0}^{\infty} (-1)^k a_k}_{a_0 - a_1 + a_2 - a_3 + \cdots} \text{ converges } \underline{\text{if and only if}} \quad a_k \to 0.$$

Example:
Since $a_k = \frac{1}{k}$ is a decreasing seq of positive numbers and $a_k \to 0$, by the alternating series test:

$$\sum_{k=1}^{\infty} (-1)^{k+1} \cdot \frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots \quad \text{converges.}$$

Def: A series that is convergent but not absolutely convergent is called conditionally convergent.

Conditionally convergent series are weird!

Facts: $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \dots = \ln 2.$

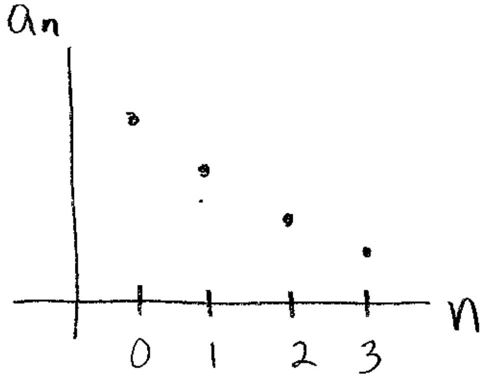
negative terms

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \dots = \frac{1}{2} \ln 2.$$

positive terms

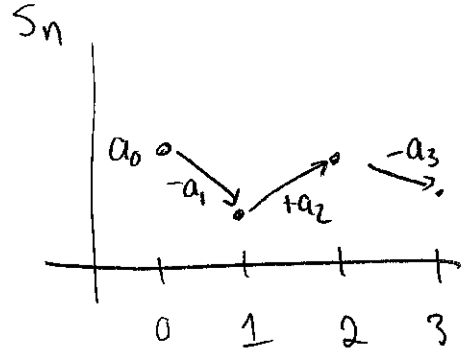
You cannot rearrange the order of terms in a conditionally convergent series. If you do, you might change the value of the infinite series.

Picture for alternating series test.



We're interested in the series $\sum (-1)^k a_k$ (not the series $\sum a_k$).

Let $S_n = \sum_{k=0}^{n} (-1)^k a_k$



Because the step sizes are getting smaller:

- $S_0 > S_2 > S_4 > S_6 > \cdots$
- $S_1 < S_3 < S_5 < S_7 < \cdots$

If the step size a_k converge to zero, then the two sequences ($S_0, S_2, S_4, \ldots$ and $S_1, S_3, S_5, \ldots$) will converge to the same value.

Furthermore, if $L = \sum_{k=0}^{\infty} (-1)^k a_k$, then:

$$\begin{cases} L < S_n & \text{for all even } n \\ L > S_n & \text{for all odd } n \end{cases}$$

(This is not worth memorizing. Just remember the picture.)

Fun applications of seqs and series:

(1) Taylor series:

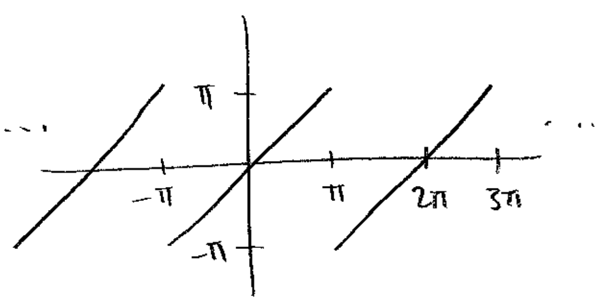
$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin x = x - \frac{x^3}{6} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

- useful for numerical analysis
- can be used to show $e^{ix} = \cos x + i \sin x$.

(2) Fourier series



$$f(x) = 2\sin x - \frac{2}{2}\sin 2x + \frac{2}{3}\sin 3x - \dots$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n} \sin nx.$$

useful for: signal processing, time series, quantum mechanics, ...

(3) Chaos theory.

The logistic map. Fix a number r and define the sequence $x_{n+1} = r x_n (1 - x_n)$.

The behavior of this sequence changes dramatically with r.

Lecture 10
2/6/20

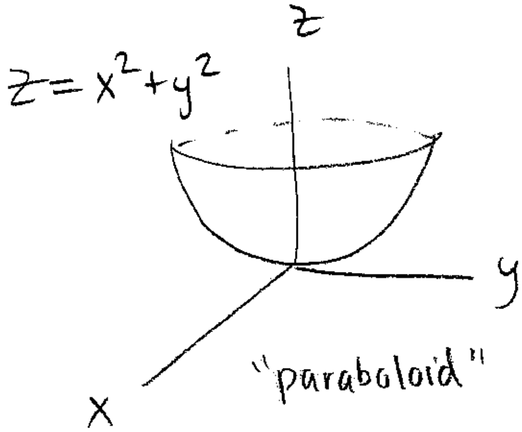
Second half of the course

Multivariable calculus.

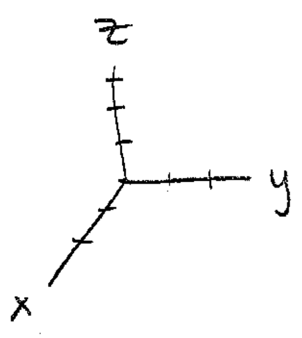
Single variable: $y=f(x) \longrightarrow$ can be plotted in the plane

multivar: $z=f(x,y)$

For functions of 2 variables, we can plot them in 3D space.

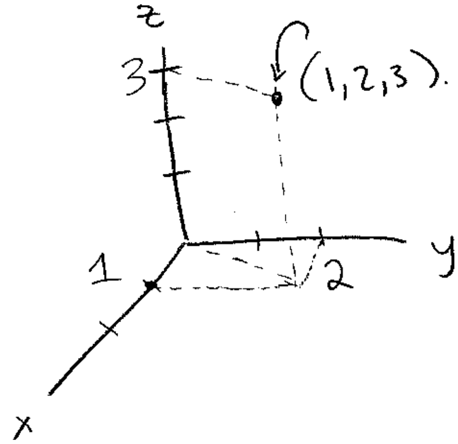


Rectangular coordinates in 3D.



By convention we use "right-handed" coordinate systems.

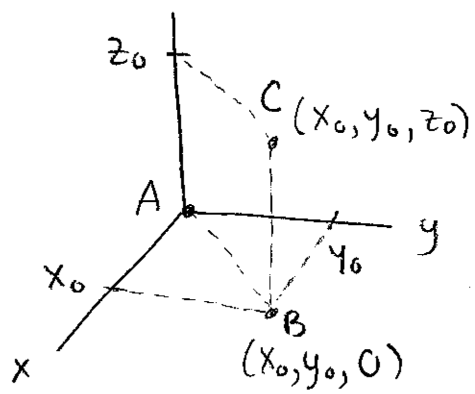
consider the point $(1,2,3)$.



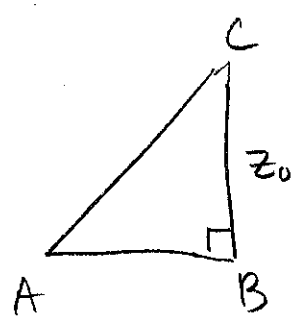
Q: what is the equation of the xy-plane?

A: $z = 0$.

Q: What is the distance between (x_0, y_0, z_0) and the origin?



$$(\text{length of } AB)^2 = x_0^2 + y_0^2$$



so $\underbrace{(\text{length } AB)^2 + z_0^2}_{x_0^2 + y_0^2 + z_0^2} = (\text{length } AC)^2$.

so distance between (x_0, y_0, z_0) and $(0,0,0)$

is $\sqrt{x_0^2 + y_0^2 + z_0^2}$

Distance between (x_1, y_1, z_1) and (x_2, y_2, z_2)

is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Example: what is the equation for the sphere of radius r centered at (a,b,c)?

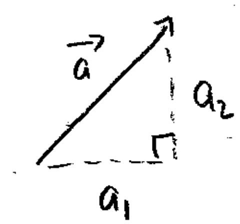
A: $(x-a)^2+(y-b)^2+(z-c)^2=r^2$.

(Just like for a circle in 2D).

Next topic: vectors:

In 2D first. A vector is something that has a magnitude and a direction.

$\vec{a}=(a_1, a_2)$ $\rightsquigarrow$



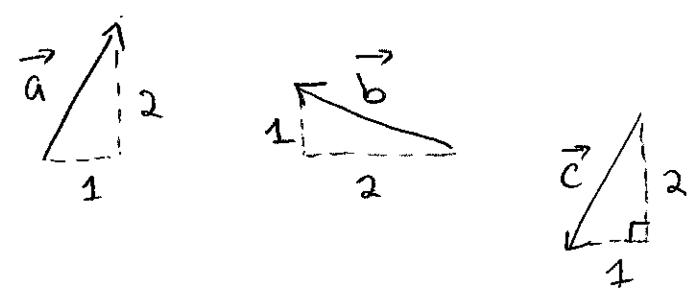
(can represent: displacement velocity force)

example:

$\vec{a}=(1,2)$

$\vec{b}=(-2,1)$

$\vec{c}=(-2,-1)$.



Basic vector operations.

1. scalar multiplication.

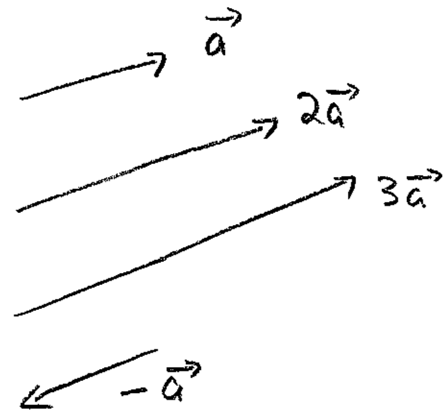
$\vec{a}=(a_1,a_2)$. $\alpha\vec{a}=(\alpha a_1, \alpha a_2)$.

Example: $\vec{a} = (2,1)$.

$\Rightarrow$ $2\vec{a} = (4,2)$

$3\vec{a} = (6,3)$

$-\vec{a} = (-2,-1)$.



So scalar multiplication can rescale and reflect vectors.

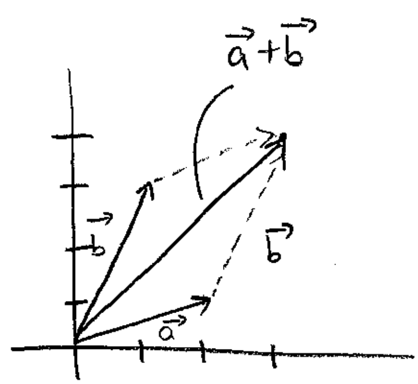
② vector addition:

$$\left.\begin{aligned}\vec{a} &= (a_1, a_2) \\ \vec{b} &= (b_1, b_2)\end{aligned}\right] \longrightarrow \vec{a}+\vec{b} = (a_1+b_1, a_2+b_2).$$

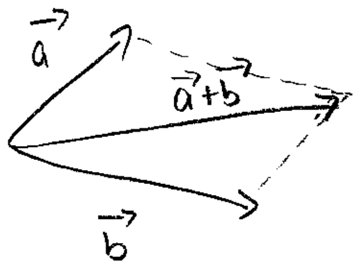
Example. $\vec{a} = (2,1)$.

$\vec{b} = (1,3)$.

$\vec{a}+\vec{b} = (3,4)$.



If you think of vectors as displacement, then $\vec{a}+\vec{b}$ is the result when you "move along $\vec{a}$" then "move along $\vec{b}$"



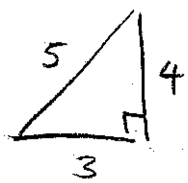
③ Norm: (or magnitude).

$\vec{a} = (a_1, a_2)$.

we define $\|\vec{a}\| = \sqrt{a_1^2 + a_2^2}$ (= length of the vector)

Ex: $\vec{a} = (3,4)$.

$\|\vec{a}\| = \sqrt{3^2+4^2} = 5$



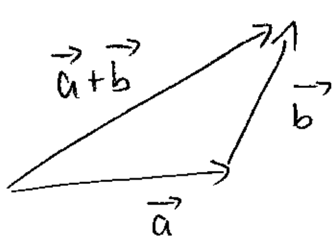
Basic properties of norms.

(1) $\|\vec{a}\| \geq 0$

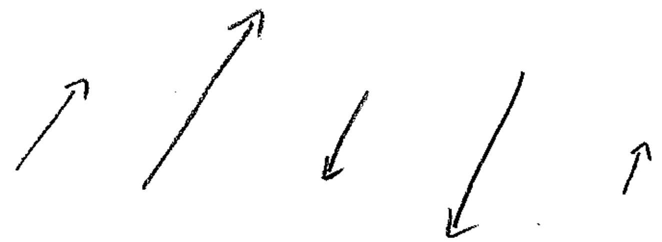
$\|\vec{a}\| = 0$ if and only if $\vec{a} = \vec{0}$

(2). $\|\alpha \vec{a}\| = |\alpha| \, \|\vec{a}\|$.

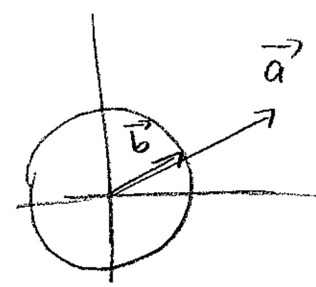
(3). $\|\vec{a}+\vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|$ (triangle inequality).



Def: Two vectors are <u>parallel</u> if one is a scalar multiple of the other.



Question: If $\vec{a}=(2,1)$ what is $\vec{b}$?



<u>Unit vectors</u>: If $\vec{a} \neq \vec{0}$.

we can define $\vec{u}_{\vec{a}} = \dfrac{1}{\|\vec{a}\|}\,\vec{a}$.

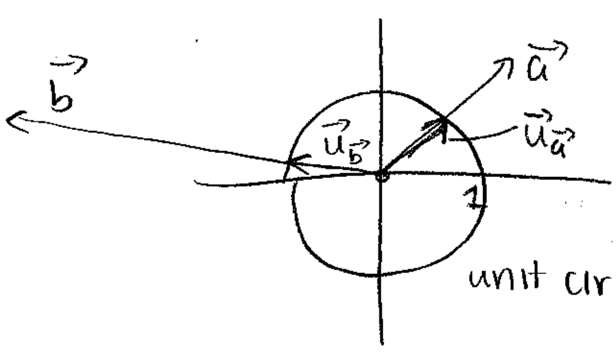
This rescales $\vec{a}$, so $\vec{u}_{\vec{a}}$ has norm 1.

It is called a <u>unit vector</u>.

e.g. $\vec{a} = (1,1)$.

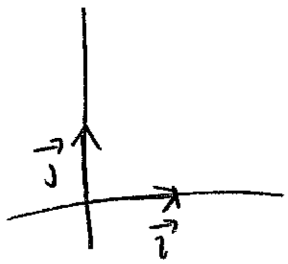
$\|\vec{a}\| = \sqrt{1^2+1^2} = \sqrt{2}$.

$\vec{u}_{\vec{a}} = \dfrac{1}{\sqrt{2}}(1,1) = \left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)$.



unit circle.

consider the unit vectors $\vec{i} = (1,0)$
$\vec{j} = (0,1)$.



We can use $\vec{i}$ and $\vec{j}$ as building blocks for all other vectors (in 2D).

$$(3,4) = (3,0) + (0,4) = 3(1,0) + 4(0,1)$$
$$= 3\vec{i} + 4\vec{j}.$$

Everything above was for 2D.

For 3D, you only need to make one change: add an extra component. Everything else is the same (but may be harder to visualize on paper).

$\vec{a} = (a_1, a_2, a_3)$. $\quad \|\vec{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

$\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ where $\begin{cases} \vec{i} = (1,0,0) \\ \vec{j} = (0,1,0) \\ \vec{k} = (0,0,1) \end{cases}$

Dot product of 2 vectors:

(Back to 2D although everything works in 3D.)

$\vec{a} = (a_1, a_2) \qquad \vec{b} = (b_1, b_2).$

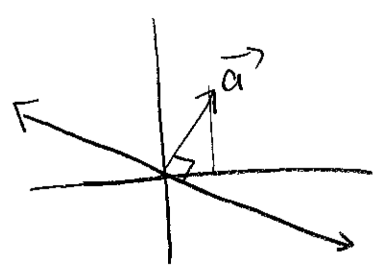
$$\vec{a} \cdot \vec{b} = \underbrace{a_1 b_1 + a_2 b_2}_{\text{the output is a scalar, not a vector.}}$$

(in 3D $(a_1, a_2, a_3) \cdot (b_1, b_2, b_3) = a_1 b_1 + a_2 b_2 + a_3 b_3$)

What is a geometric interpretation of the dot product?

<u>Q</u>: when is $\vec{a} \cdot \vec{b} = 0$? For example, when $\vec{a} = (1,2)$, which vectors $\vec{b}$ satisfy $\vec{a} \cdot \vec{b} = 0$?

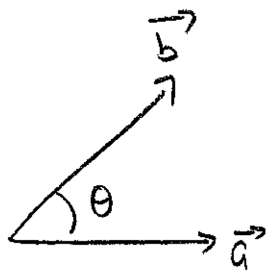
$$0 = \vec{a} \cdot \vec{b} = b_1 + 2b_2 \implies b_2 = -\tfrac{1}{2} b_1$$



$\vec{b}$ should be parallel to this line

In general: $\vec{a} \cdot \vec{b} = 0$ if and only if $\vec{a}$ and $\vec{b}$ are perpendicular to each other.

More general, $\vec{a} \cdot \vec{b}$ tells you the angle between the two vectors.



If θ = angle between $\vec{a}, \vec{b}$, then

$$\boxed{\vec{a} \cdot \vec{b} = \|\vec{a}\| \, \|\vec{b}\| \cos\theta}$$

(To prove this, use law of cosines). (I'll give a HW problem on this)

Note: if $\theta = \frac{\pi}{2}$ then $\cos\theta = 0$.

Properties of the dot product:

(1) $\vec{a} \cdot \vec{a} = (a_1, a_2) \cdot (a_1, a_2) = a_1 a_1 + a_2 a_2$

$$= a_1^2 + a_2^2 = \|\vec{a}\|^2$$

(2) $\vec{a} \cdot \vec{0} = 0$.

(3) $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$.

(4) $(\alpha \vec{a}) \cdot \vec{b} = \alpha (\vec{a} \cdot \vec{b})$.

(5) $\vec{a} \cdot (\vec{b} + \vec{c})$

$= \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$.

Example:

(1) Find the angle between

$$\vec{a} = 2\vec{i} + \vec{j} + \vec{k} = (2,1,1)$$

$$\vec{b} = \vec{i} + \vec{j} - 3\vec{k} = (1,1,-3)$$

$$\vec{a} \cdot \vec{b} = 2 \cdot 1 + 1 \cdot 1 + 1 \cdot (-3) = 0.$$

so $\vec{a}, \vec{b}$ are perpendicular. The angle between is $\pi/2$.

(2). Find angle between

$$\vec{a} = (2,3,2)$$

$$\vec{b} = (1,2,-1).$$

$$\vec{a} \cdot \vec{b} = 2 \cdot 1 + 3 \cdot 2 + 2 \cdot (-1) = 6.$$

$$\|\vec{a}\| = \sqrt{2^2+3^2+2^2} = \sqrt{17}$$

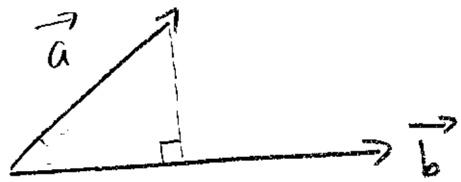
$$\|\vec{b}\| = \sqrt{1^2+2^2+(-1)^2} = \sqrt{6}.$$

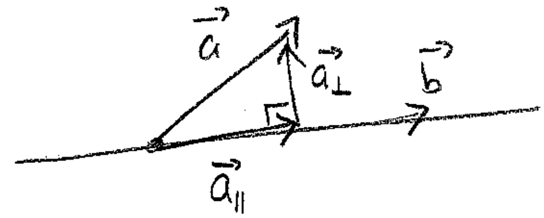
$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \, \|\vec{b}\|} = \frac{6}{\sqrt{17}\sqrt{6}}$$

use calculator: $\theta \approx 0.935$ radians

Projections:

We'll define the "projection of $\vec{a}$ onto $\vec{b}$".





We can split up $\vec{a} = \vec{a}_\parallel + \vec{a}_\perp$ where

$\vec{a}_\parallel$ is parallel to $\vec{b}$, $\vec{a}_\perp$ is perpendicular to $\vec{b}$.

We define $\text{proj}_{\vec{b}}\vec{a}$ to be $\vec{a}_\parallel$.

To calculate $\vec{a}_\parallel$: we know $\vec{a}_\parallel = \alpha \vec{b}$ for some scalar α.

We just need to solve for α:

$$\vec{a}\cdot\vec{b} = (\vec{a}_\parallel + \vec{a}_\perp)\cdot\vec{b}$$

$$= \underbrace{(\vec{a}_\parallel\cdot\vec{b})} + \underbrace{(\vec{a}_\perp\cdot\vec{b})}_{\text{zero}}$$

$$= \alpha\, \vec{b}\cdot\vec{b}$$

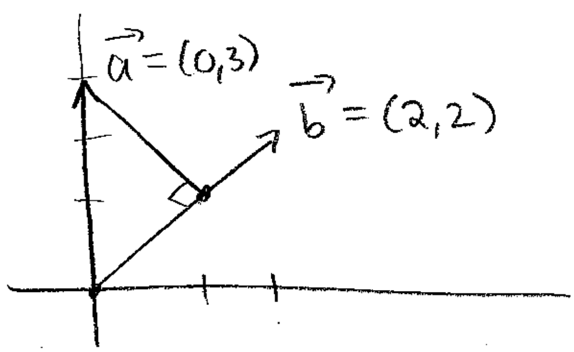
$$\Longrightarrow \quad \alpha = \frac{\vec{a}\cdot\vec{b}}{\vec{b}\cdot\vec{b}}$$

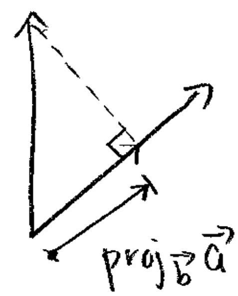
So $\text{proj}_{\vec{b}}\vec{a} = \vec{a}_\parallel = \dfrac{\vec{a}\cdot\vec{b}}{\vec{b}\cdot\vec{b}}\,\vec{b}$.

This formula is simpler if we consider $\vec{u_{\vec{b}}}$ instead of $\vec{b}$:

$$\text{proj}_{\vec{b}}\vec{a} = \text{proj}_{(\vec{u_{\vec{b}}})}\vec{a} = \frac{\vec{a}\cdot\vec{u_b}}{\vec{u_b}\cdot\vec{u_b}}\vec{u_b} = (\vec{a}\cdot\vec{u_b})\,\vec{u_b}.$$

Example:





$$\text{proj}_{\vec{b}}\vec{a} = \frac{(0,3)\cdot(2,2)}{(2,2)\cdot(2,2)}(2,2).$$

$$= \frac{6}{8}(2,2) = \frac{3}{4}(2,2) = \left(\frac{3}{2},\frac{3}{2}\right)$$

So: dot products are useful for calculating projections. (more on that in linear algebra)

(application: "projections can be used to compress high dimensional data")

(also: least squares regression)

Next topic: lines

In 2D, lines are given by a linear equation

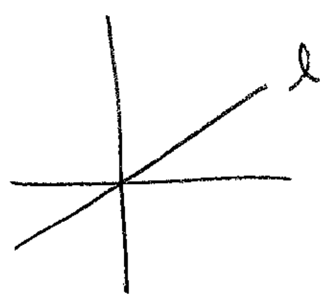
$$ax+by=c.$$

But in 3D, a linear equation gives you a plane.

So let's find a different way to represent lines that works in both 2D, 3D (and higher dim)

We'll use parametric equations.

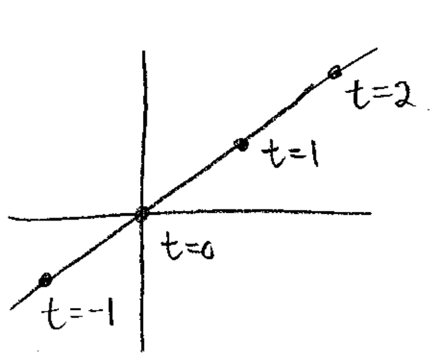
Example:



ℓ is the line through the origin with slope $\frac{1}{2}$.

$\vec{d} = (2,1)$ is a "direction vector" for the line

Consider $\vec{r}(t) = t\vec{d}$.



As we vary t, $\vec{r}(t)$, traces out the line ℓ.

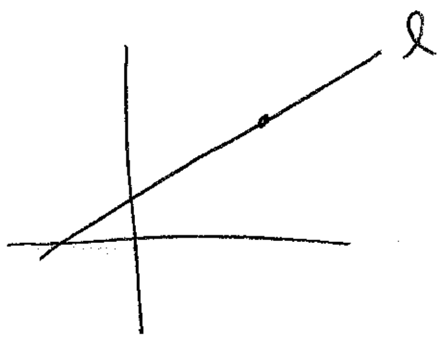
So the line is parametrized by $\vec{r}(t)$.

$$\vec{r}(t) = t\vec{d} = (2t, t).$$

Alternatively we can write $\begin{cases} x(t) = 2t \\ y(t) = t \end{cases}$

"scalar parametric equation"

Example :



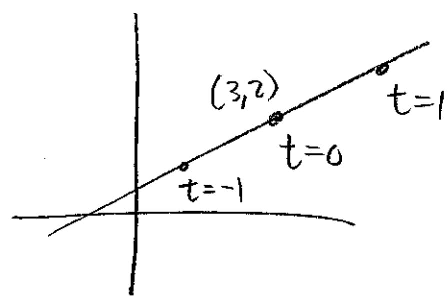
l is the line
- through $(3,2)$
- with direction $\vec{d} = (2,1)$.

What change can we make from the previous example?

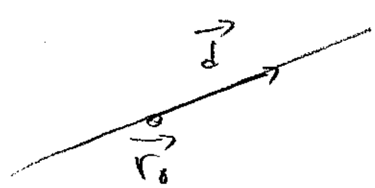
$$\vec{r}(t) = (3,2) + t(2,1)$$

or

$$\begin{cases} x(t) = 3 + 2t \\ y(t) = 2 + t. \end{cases}$$



In general: The line through the point $\vec{r_0}$ and with direction vector $\vec{d}$ can be given by $\boxed{\vec{r}(t) = \vec{r_0} + t\vec{d}}$



In 3D: (Lecture 12, 2/13/20)

Example: Line through $(1,-1,2)$ with direction vector $(2,-3,1)$.

$$\vec{r}(t) = (1,-1,2) + t(2,-3,1)$$
$$= (1+2t, -1-3t, 2+t).$$

$$\begin{cases} x(t) = 1+2t \\ y(t) = -1-3t \\ z(t) = 2+t. \end{cases}$$

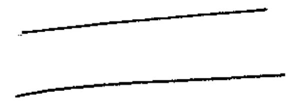
If we want the equations for a line without t, we can solve for t:

$$\begin{cases} t = \frac{x-1}{2} \\ t = \frac{y+1}{-3} \\ t = z-2 \end{cases} \Longrightarrow \underbrace{\frac{x-1}{2} = \frac{y+1}{-3} = z-2.}_{\text{this really 2 equations.}}$$

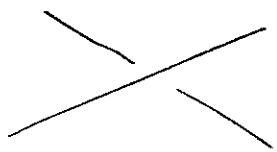
$$\begin{cases} \frac{x-1}{2} = \frac{y+1}{-3} \\ \frac{y+1}{-3} = z-2 \end{cases}$$

2 equations: each one defines a plane in 3D. The two planes intersect in our given line.

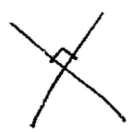
Def: • 2 lines are parallel if their direction vectors are parallel.



• 2 lines are skew if they do not intersect, but they are not parallel (in 3D, but not possible in 2D.).



• 2 lines are perpendicular if they intersect and their direction vectors are perpendicular



Next: planes

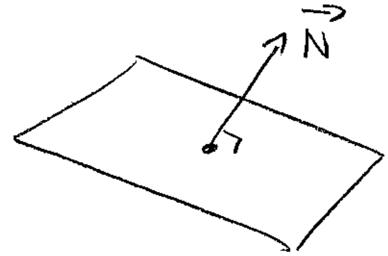
Example: $3x-2y+z=0$ this is a plane.

How does it look?

$(x,y,z)\cdot(3,-2,1)=0.$

$\Rightarrow$ Every point on the plane is perpendicular to $(3,-2,1)$.

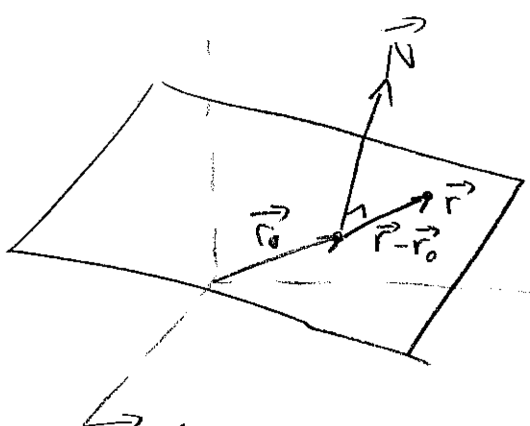
Let $\vec{N} = (3,-2,1)$. We call it the normal vector of the plane.



So $3x - 2y + 1 = 0$
is the plane through $(0,0,0)$ with normal vector $(3,-2,1)$.

Example:

Plane through $\vec{r_0} = (1,0,2)$ with normal vector $\vec{N} = (3,-2,1)$?



If $\vec{r}$ is on the plane, then $\vec{r}-\vec{r_0}$ is perpendicular to $\vec{N}$.

$\vec{r} = (x,y,z)$.

$$\vec{N} \cdot (\vec{r} - \vec{r_0}) = 0.$$

$$(3,-2,1)\cdot(x-1,\, y,\, z-2) = 0$$

$$3(x-1) - 2y + 1(z-2) = 0.$$

$$3x - 2y + z - 5 = 0$$

In general:

plane through (x_0, y_0, z_0)

normal vec (A, B, C).

$\Rightarrow$ equation is

$$A(x-x_0)+B(y-y_0)+C(z-z_0)=0.$$

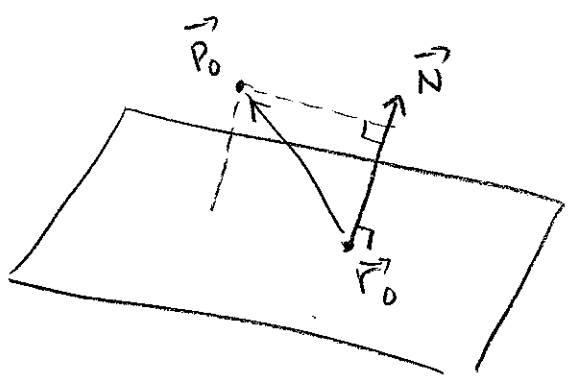
Distance between a point and a plane:

<u>Example</u>: distance between

$\vec{P_0}=(3,-5,2)$ (point) and $8x-2y+z=5$? (plane).

$\vec{N}=(8,-2,1)$

let's pick $\vec{r_0}=(0,0,5)$



Geometrically, we want to project $\vec{P_0}-\vec{r_0}$ onto $\vec{N}$.

$$\text{proj}_{\vec{N}}(\vec{P_0}-\vec{r_0})=\frac{(\vec{P_0}-\vec{r_0})\cdot\vec{N}}{\vec{N}\cdot\vec{N}}\vec{N}$$

$$=\frac{(3,-5,-3)\cdot(8,-2,1)}{69}(8,-2,1).=\frac{31}{69}(8,-2,1)$$

The length is $\frac{31}{69}\sqrt{69}$

Functions of several variables

Lecture 13
2/18/20.

Sometimes a quantity depends on 2 or more variables.

Example: force = mass × acceleration

$$f(m,a) = ma$$

$$\text{Kinetic energy} = \frac{1}{2}(\text{mass})(\text{velocity})^2$$

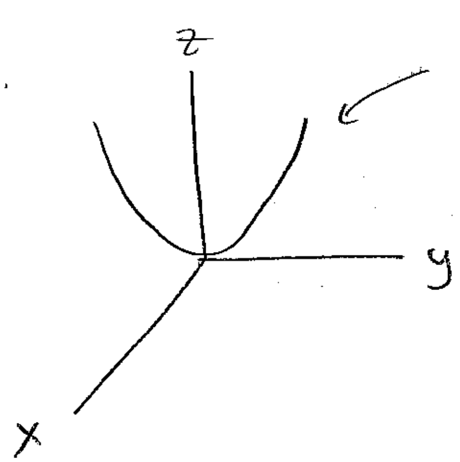
$$K(m,v) = \frac{1}{2}mv^2$$

How can we visualize functions of 2 variables?

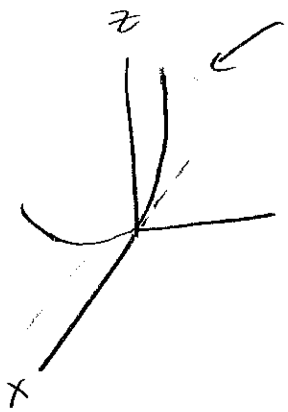
Example: $f(x) = x^2 + y^2$

(Domain: all real x, all real y.
Range: $[0,\infty)$.)

$z = x^2 + y^2$.

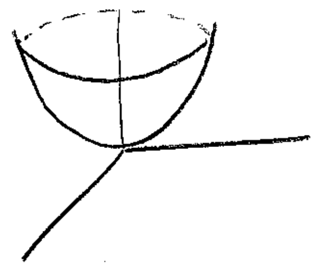


in the yz-plane $x=0$, so you get a parabola ($z=y^2$).



in the xz-plane, it's also a parabola.

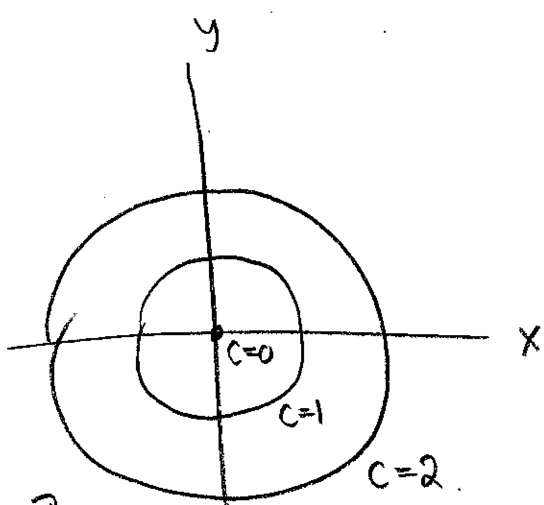
in general: you get the surface of revolution generated by the parabola.



Here's another way to "plot" this function.

Level curves. $f(x,y) = x^2 + y^2$.

Q: when does $f(x,y) = c$?



$c=1$: circle of radius 1

$c=4$: - - - - - 2

$c=9$: - - - - - 3.

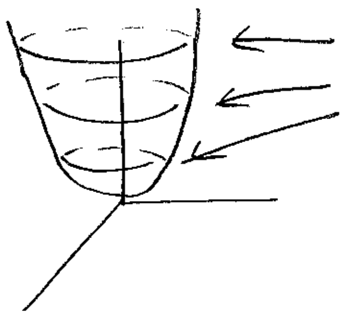
$c=0$: just the origin.

Each of these curves is called a "level curve" for f.

Very useful in mapping mountainous terrain.

→ "topographic map"

For $f(x) = x^2 + y^2$, the level curves are circles.



these circles correspond to the level curves.

<u>Example</u>:

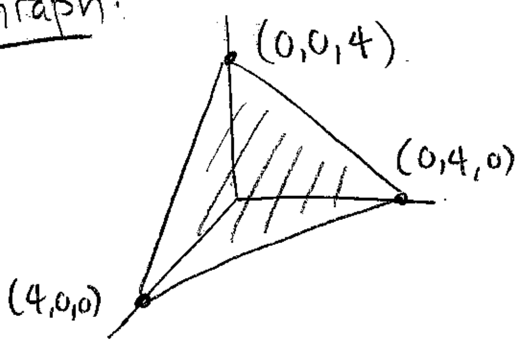
$$f(x,y) = 4 - x - y.$$

The graph is a plane. $z = 4 - x - y.$

$$x + y + z = 4.$$

$\Rightarrow$ normal vector $\vec{N} = (1,1,1)$.

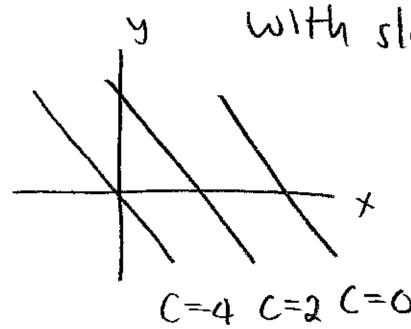
<u>Graph</u>:



Level curves:

$$4 - x - y = C.$$

$$\underbrace{y = -x - C + 4}_{\text{lines in } xy\text{-plane with slope } -1}.$$

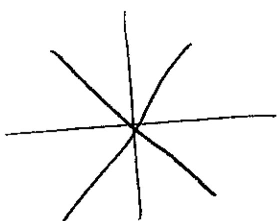


Example: $f(x) = x^2 - y^2$.

Level curves:

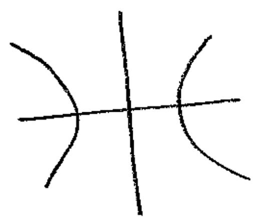
$x^2 - y^2 = c$.

$x^2 - y^2 = 0$



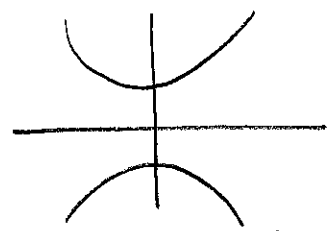
(two lines)

$x^2 - y^2 = 1$



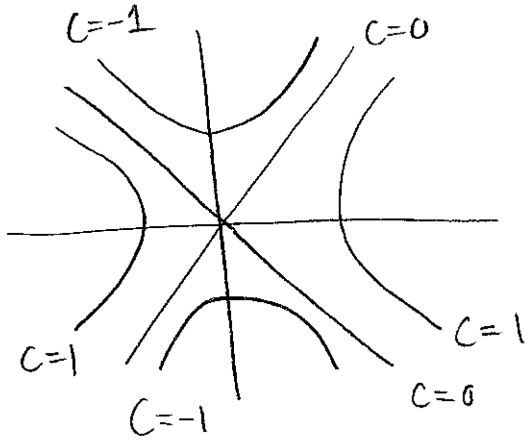
(hyperbola)

$x^2 - y^2 = -1$



(hyperbola).

so:



the 3D shape is hard to draw. It looks like a saddle. The origin is called a "saddle point."

(The shape of the graph is a "hyperbolic paraboloid.")

So: Functions of 2 variables:

- graphs are in 3D, might be hard to draw
- level curves are in 2D, easier to draw.

Functions of 3 variables? $f(x,y,z)$.

- Graph is in 4D...
- "Level surfaces" ($f(x,y,z)=c$) are in 3D.

<u>Example</u>: $f(x,y,z) = x^2+y^2+z^2$.

$f(x,y,z)=c$ is a sphere with radius $\sqrt{c}$.

Next topic: Partial derivatives.

<u>Example</u>: $f(x,y) = 3x^2y - 5x\cos\pi y$.

We can think of y as constant and differentiate with respect to x:

$$\frac{\partial f}{\partial x}(x,y)\left(= f_x(x,y)\right) = 6xy - 5\cos\pi y.$$

Or, we can think of x as constant and differentiate with respect to y.

$$\frac{\partial f}{\partial y}(x,y) = \left(f_y(x,y)\right) = 3x^2 + 5\pi x\sin\pi y$$

- f_x (or $\frac{\partial f}{\partial x}$) is called the <u>partial derivative</u> of <u>f with respect</u> to x.
- f_y (or $\frac{\partial f}{\partial y}$) - - - - - - - - wrt y.

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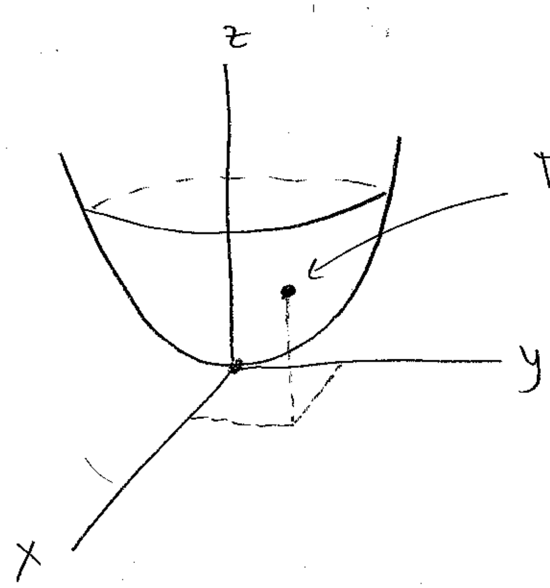
<u>Example</u>: $f(x,y) = x^2 + 2y^2$

$f_x(x,y) = 2x$

$f_y(x,y) = 4y$.

$\Longrightarrow \begin{cases} f(1,2) = 9 \\ f_x(1,2) = 2 \\ f_y(1,2) = 8 \end{cases}$

How can we interpret these geometrically?



This is the point (1,2,9).

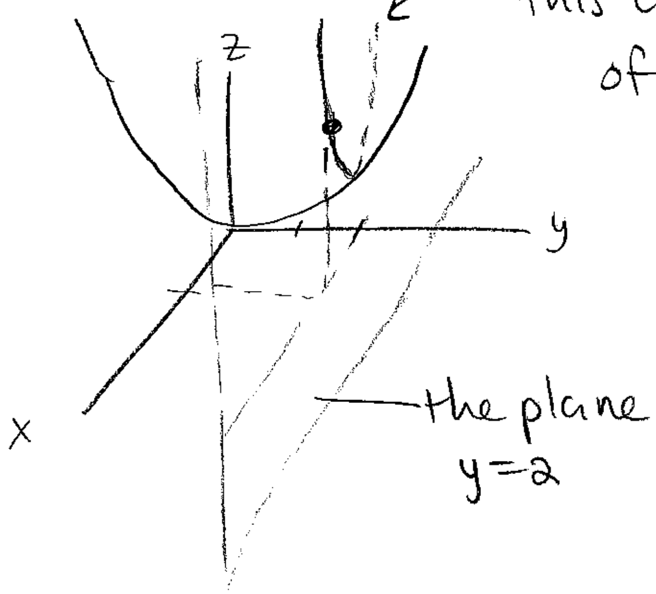
Can we interpret $f_x(1,2)$ and $f_y(1,2)$ as slopes of tangent lines?

Yes! In calculating $f_x(1,2)$ we keep y fixed (to the value 2).

$y=2$ corresponds to a plane parallel to xz plane.

In this plane ($y=2$), we have $z = x^2 + 2y^2 = x^2 + 8$.

this curve is the intersection of the graph with the plane



In the plane we can think of z as a function of x alone.

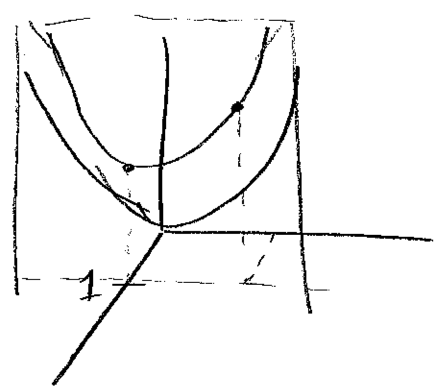
$$z = g(x) = x^2 + 8.$$

$$g'(x) = 2x + 0.$$

$$g'(1) = 2.$$

so: $f_x(1,2)$ is the slope of the curve inside this plane.

Similarly for $f_y(1,2)$. Fix $x=1$.



$f_y(1,2)$ is the slope of the curve at $(1,2,9)$ inside the plane $x=1$.

You can also define partial derivatives of functions of more variables.

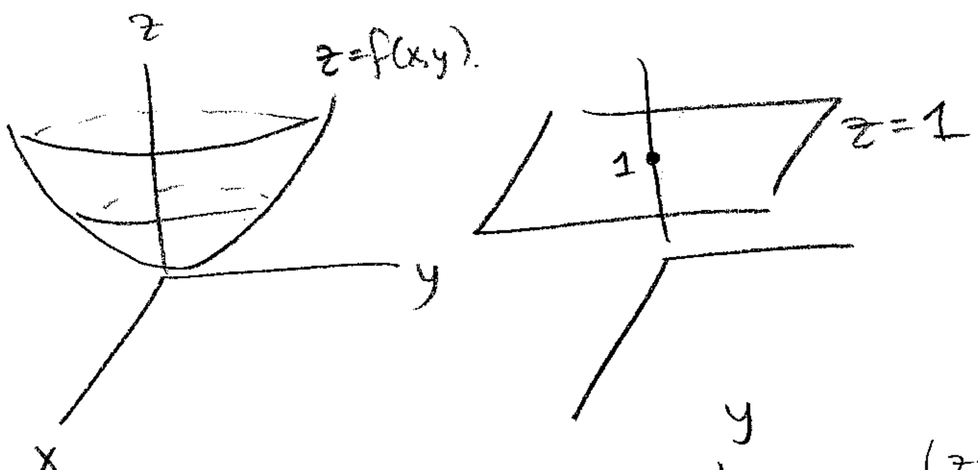
Ex: $f(x,y,z) = xy^2z^3$

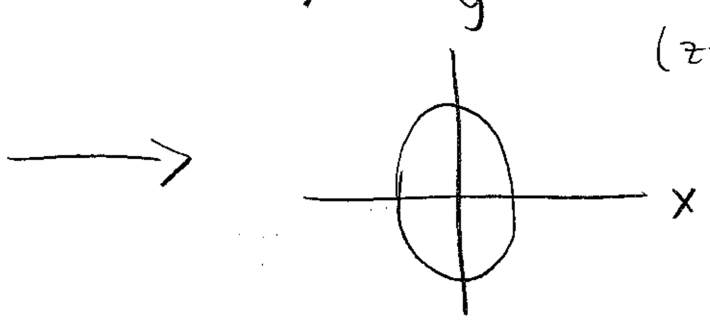
$$\Rightarrow \begin{cases} f_x(x,y,z) = y^2z^3 \\ f_y(x,y,z) = 2xyz^3 \\ f_z(x,y,z) = 3xyz^2. \end{cases}$$

(These have geometric interpretations too.)

Actually, more on $f(x) = x^2+2y^2$.

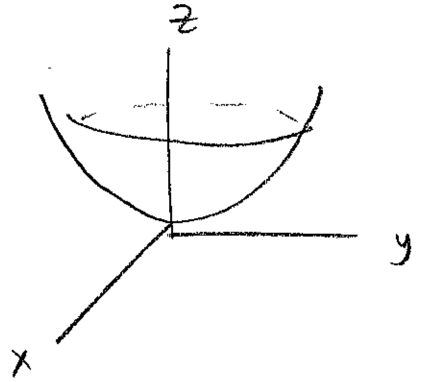
If we "slice" the graph of f by planes parallel to xy-plane (i.e. of the form $z=c$) we get level curves:

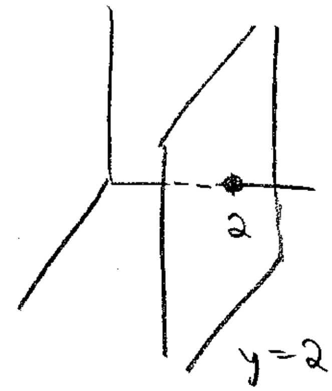




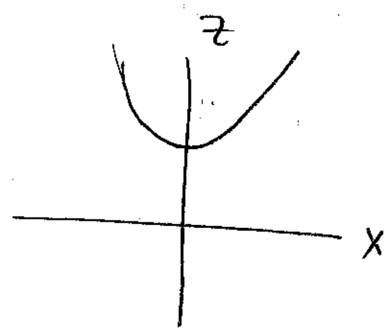
(z=1 plane).

We can also "slice" by planes of the form $y = c$.





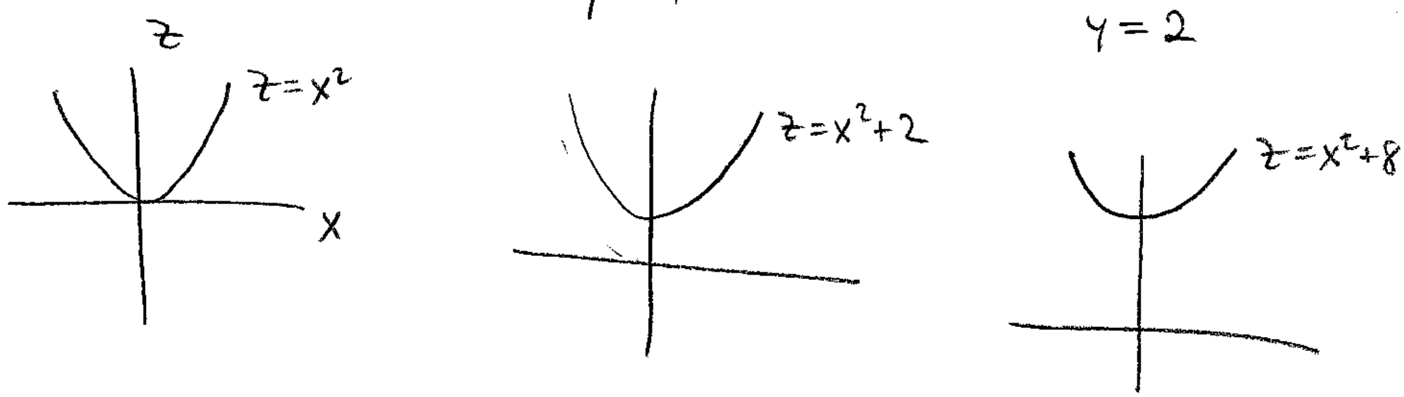
$\Longrightarrow$



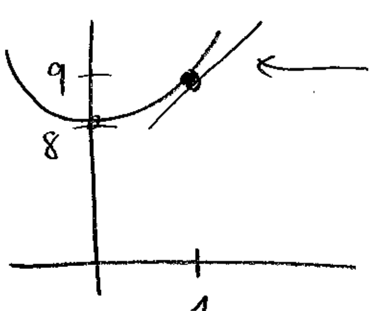
$$z = f(x,2) = x^2 + 8$$

In general:

$y = 0$ $\qquad$ $y = 1$ $\qquad$ $y = 2$



On the plane $y = 2$, consider $x = 1$ $(z = 9)$.



the slope of this tangent line is

$$f_x(1,2) = 2.$$

Higher order derivatives:

In single-variable calculus, you have $f^{(n)}(x)$.

(differentiate f wrt x n times).

You can do the same here:

Example: $f(x,y) = \sin x^2y$.

$$f_x(x,y) = 2xy\cos x^2y.$$

$$f_y(x,y) = x^2\cos x^2y.$$

$$f_{xx} = (f_x)_x = -4x^2y^2\sin x^2y + 2y\cos x^2y$$

$$\left.\begin{aligned} f_{xy} &= (f_x)_y = -2x^3y\sin x^2y + 2x\cos x^2y \\ f_{yx} &= (f_y)_x = -2x^3y\sin x^2y + 2x\cos x^2y \\ f_{yy} &= (f_y)_y = -x^4\sin x^2y \end{aligned}\right\} \text{ask class to calculate}$$

Notice that $f_{xy} = f_{yx}$.

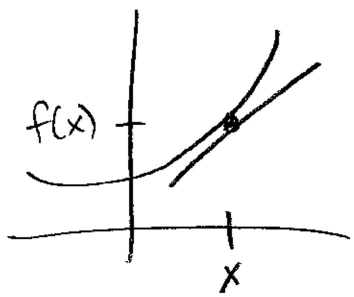
In fact:

If f_x f_y f_{xy} f_{yx} are all continuous,

then $f_{xy} = f_{yx}$. "equality of mixed partials"

Next topic: differentiability.

Recall: $f(x)$.



We say f is differentiable at x if "f is like a linear function near x":

$$f(x+h) - f(x) \approx \underbrace{ah}_{\substack{\text{This is a linear} \\ \text{function} \\ g(h)=ah}}$$

In this case, a is called the <u>derivative</u> of f at x, and it's denoted $f'(x)$.

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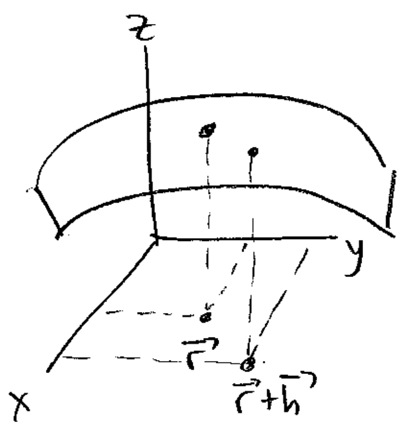
In the multivariable setting.

$f(x,y)$. Let $\vec{r} = (x,y)$. We say f is differentiate at $\vec{r}$ if "f is like a linear function near $\vec{r}$"



$$f(\vec{r}+\vec{h}) - f(\vec{r}) \approx \vec{a} \cdot \vec{h}$$

In this case, $\vec{a}$ is called the <u>gradient</u> of f at $\vec{r}$, and it's denoted $\nabla f(\vec{r})$

Note: ∇f: The input is a 2D vector $\vec{r}$.
The output is also a 2D vector.
(in order for $\nabla f \cdot \vec{h}$ to make sense).

How do we actually calculate ∇f? Let $\nabla f(\vec{r}) = (a,b)$.

Go back to: $f(\vec{r}+\vec{h}) - f(\vec{r}) \approx \nabla f(\vec{r}) \cdot \vec{h}$

and plug in some things for $\vec{h}$.

① Let $\vec{h} = (t,0)$.

$$\vec{r}+\vec{h} = (x+t, y).$$

So $$f(x+t,y) - f(x,y) \approx \underbrace{\nabla f(\vec{r}) \cdot (t,0)}_{at}.$$

So $$\frac{f(x+t,y)-f(x,y)}{t} \approx a.$$

But this is the difference quotient for $\frac{\partial f}{\partial x}(x,y)$!.

So $a = \frac{\partial f}{\partial x}(x,y)$. Similarly $b = \frac{\partial f}{\partial y}(x,y)$.

<u>So</u>: If f is <u>differentiable</u> at $\vec{r}$,

then $\nabla f(\vec{r}) = \left(\frac{\partial f}{\partial x}(\vec{r}), \frac{\partial f}{\partial y}(\vec{r}) \right)$.

<u>Caution</u>: unlike in single-variable calculus, it is possible for $\frac{\partial f}{\partial x}(\vec{r})$, $\frac{\partial f}{\partial y}(\vec{r})$ to be defined even though f is not differentiable at $\vec{r}$.

We won't worry too much about this.

<u>Example</u>: $f(x,y) = x^2 + 2y^2$.

$$\left.\begin{array}{l} f_x(x,y) = 2x \\ f_y(x,y) = 4y \end{array}\right\} \Longrightarrow \nabla f(x,y) = (2x, 4y).$$

<u>Example</u>: $f(x,y,z) = \sin xy^2z^3$.

$$\nabla f = (\underbrace{y^2z^3 \cos xy^2z^3}_{(f_x)}, \underbrace{2xyz^3 \cos xy^2z^3}_{(f_y)}, \underbrace{3xy^2z^2 \cos xy^2z^3}_{(f_z)})$$

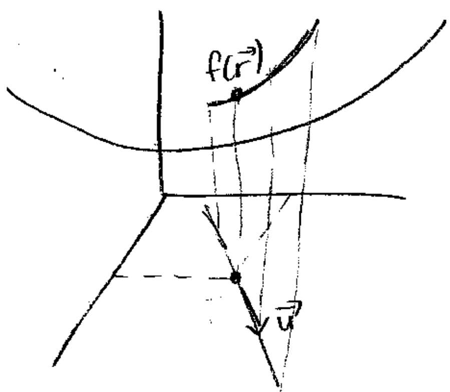
$$= yz^2 \cos xy^2z^3 \, (yz, 2xz, 3xy).$$

Next: directional derivatives

Suppose f is differentiable at $\vec{r}$.

Let $\vec{u}$ be a unit vector.

Can we make sense of $\lim_{t \to 0} \dfrac{f(\vec{r}+t\vec{u}) - f(\vec{r})}{t}$?



interpretation: slice the graph with a "vertical plane in direction $\vec{u}$"
look at slope of tangent line there.

This is called the directional derivative of f at $\vec{r}$ in direction $\vec{u}$.

Recall: $$f(\vec{r}+\vec{h}) - f(\vec{r}) \approx \nabla f(\vec{r}) \cdot \vec{h}$$

Let $\vec{h} = t\vec{u}$:

$$f(\vec{r}+t\vec{u}) - f(\vec{r}) \approx \nabla f(\vec{r}) \cdot (t\vec{u})$$

$$\Rightarrow \frac{f(\vec{r}+t\vec{u}) - f(\vec{r})}{t} \approx \nabla f(\vec{r}) \cdot \vec{u}.$$

So the directional derivative is given by.

$$f'_u(\vec{r}) = \boxed{\nabla f(\vec{r})\cdot\vec{u}.}$$

<u>Example</u>: $f(x,y) = x^2 + y^2$. Find dir'nal deriv.

point $(1,2)$. vector $(2,-3)$.

$$\vec{u} = \frac{1}{\sqrt{13}}(2,-3)$$ ($\leftarrow$ divide to get unit vector).

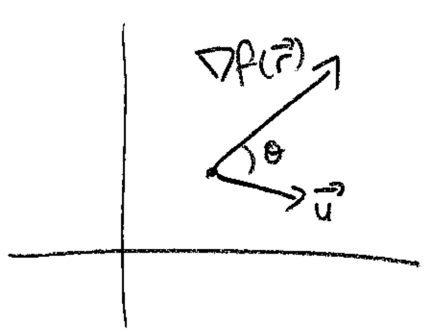
$$\nabla f(x,y) = (2x, 2y).$$

$$\nabla f(1,2) = (2,4)$$

$$f'_u(1,2) = \nabla f(1,2)\cdot\vec{u} = (2,4)\cdot\frac{1}{\sqrt{13}}(2,-3) = -\frac{8}{\sqrt{13}}$$

$$\approx -2.219.$$

Recall the dot product identity. Lecture 17 3/3/20



$$\nabla f(\vec{r})\cdot\vec{u} = \|\nabla f(\vec{r})\|\,\|\vec{u}\|\cos\theta$$

$$= \|\nabla f(\vec{r})\|\cos\theta$$

since $\vec{u}$ is a unit vector.

Q: which direction should we pick to maximize the directional derivative?

$$f'_{\vec{u}}(\vec{r}) = \nabla f(\vec{r}) \cdot \vec{u} = \underbrace{\|\nabla f(\vec{r})\| \cos\theta}.$$

To maximize this, we want $\cos\theta = 1 \implies \theta = 0$.

pick $\vec{u}$ to be the same direction as $\nabla f(\vec{r})$.

So: From the point $\vec{r}$ in the domain of f,

- the direction of $\nabla f(\vec{r})$ is the direction in which f increases most rapidly.
- the directional derivative in this direction is $\|\nabla f(\vec{r})\|$.

Q: If $\frac{\partial f}{\partial x}(\vec{r}) = 0$ for all $\vec{r}$, is f constant?

A: No! consider $f(x,y) = y$.

But: If $\nabla f(\vec{r}) = \vec{0}$ for all $\vec{r}$, then f is constant.

This means that two functions f and g satisfying $\nabla f = \nabla g$ differ by a constant.

Gradients and the chain rule.

- in single-variable calculus:

$f(x(t))$ is composition of $f(x)$, $x(t)$.

$$\frac{d}{dt}\left[f(x(t))\right] = \underbrace{\frac{df}{dx}}_{\text{how } f \text{ changes wrt } x.}\ \underbrace{\frac{dx}{dt}}_{\text{how } x \text{ changes wrt } t.}$$

- in multivariable?

$$f \xrightarrow[\text{on}]{\text{dep}} x \xrightarrow[\text{on}]{\text{dep}} t$$

$f(x,y)$. $x(t)$ $y(t)$

combine: $f(x(t), y(t))$.

$$\frac{d}{dt}\left[f(x(t), y(t))\right] = ?$$

chain rule:

$$\frac{d}{dt}\left[f(x(t),y(t))\right] = \underbrace{\frac{\partial f}{\partial x}\frac{dx}{dt}}_{} + \underbrace{\frac{\partial f}{\partial y}\frac{dy}{dt}}_{}.$$

each term looks like the single var chain rule. You just add them up.

$$\begin{tikzcd} & x \arrow[dr, "\text{dep on}"] & \\ f \arrow[ur, "\text{dep on}"] \arrow[dr, "\text{dep on}"'] & & t \\ & y \arrow[ur, "\text{dep on}"'] & \end{tikzcd}$$

shorthand notation for chain rule.

$f(\vec{r})$ $\qquad \vec{r}(t) = (x(t), y(t))$.

$$\nabla f(\vec{r}) = \left(\frac{\partial f}{\partial x}(\vec{r}), \frac{\partial f}{\partial y}(\vec{r})\right) \qquad \vec{r}\,'(t) = (x'(t), y'(t)) = \left(\frac{dx}{dt}(t), \frac{dy}{dt}(t)\right).$$

So the chain rule can also be written:

$$\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}) \cdot \vec{r}\,'(t)$$

Written this way, the multivariate chain rule more closely resembles the single-var chain rule.

Example: $f(x,y) = x^y$ $\qquad x(t)=t,\ y(t)=t$

$(\vec{r}(t) = (t,t))$.

$$f(\vec{r}(t)) = t^t.$$

Let's use the chain rule to calculate $\frac{d}{dt}(t^t)$. ((we also calculated this last quarter))

$$\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t).$$

$$\frac{\partial f}{\partial x} = y x^{y-1} \qquad \frac{\partial f}{\partial y} = (\ln x)\, x^y.$$

$$\Rightarrow \nabla f(x,y) = (y x^{y-1}, (\ln x) x^y).$$

$$\nabla f(t,t) = (t\, t^{t-1}, (\ln t) t^t)$$

$$= (t^t, (\ln t) t^t).$$

And $\vec{r}(t) = (t,t)$

$\vec{r}\,'(t) = (1,1)$.

So $\frac{d}{dt} f(\vec{r}(t)) = (t^t, (\ln t)t^t) \cdot (1,1)$

$$= t^t + (\ln t) t^t$$

$$= t^t (1 + \ln t).$$

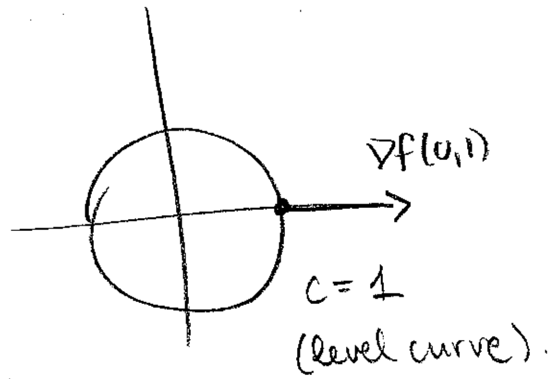
Next topic: gradients and level curves

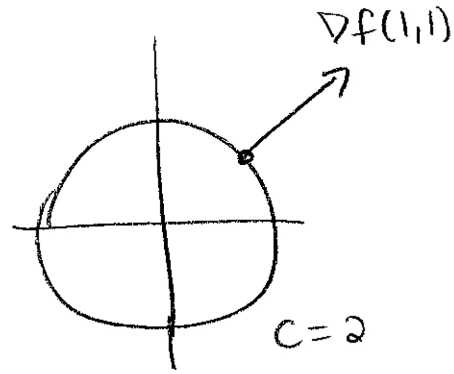
Ex: $f(x,y) = x^2 + y^2$

$\nabla f(x,y) = (2x, 2y)$.
$= 2(x,y)$

$\nabla f(0,1) = (0,2)$

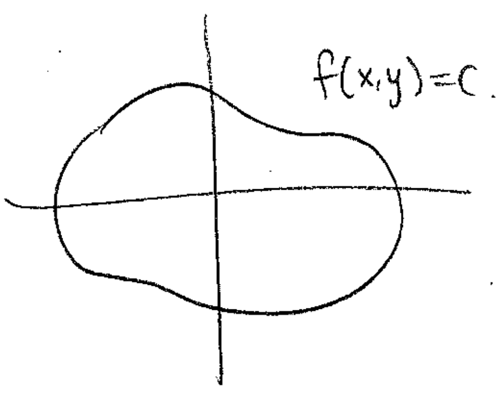
$\nabla f(1,1) = (2,2)$





Fact: $\nabla f(\vec{r})$ is perpendicular to the level curve of f which passes through $\vec{r}$.

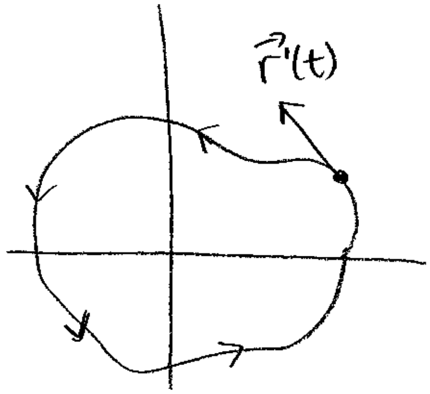
To prove this use chain rule:



Let $\vec{r}(t) = (x(t), y(t))$ be a parametrization of $f(x,y) = c$. (level curve).

So $f(\vec{r}(t)) = c$ for all t.

Differentiate wrt t: $\underbrace{\nabla f(\vec{r}(t))}_{\text{gradient}} \cdot \underbrace{\vec{r}\,'(t)}_{\text{velocity vector}} = 0$.



$\vec{r}'(t)$ is tangent to the curve. at $\vec{r}(t)$

so $\nabla f(\vec{r}(t))$ is perpendicular to the curve at $\vec{r}(t)$.

This also seems reasonable if you remember that $\nabla f(\vec{r})$ points in the direction of greatest increase of f.

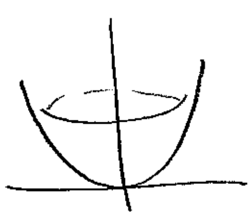
(You don't want to go in the direction of a level curve since that would lead to no increase.)

Functions of 3 variables $f(x,y,z)$.

consider level surface $f(x,y,z)=c$.

$\nabla f(x,y,z)$ is perpendicular to the level surface passing through (x,y,z).

Example: $z=x^2+y^2$ is a level surface of $f(x,y,z)=x^2+y^2-z$ $(f=0)$.



$\nabla f = (2x, 2y, -1)$.

This gives you the normal vector of the tangent plane.

In general, if $f(x,y)$ is a func of 2 variables, its graph $z=f(x,y)$ is a level surface of $g(x,y,z)=f(x,y)-z$.

$$\frac{\partial g}{\partial x}=\frac{\partial f}{\partial x} \qquad \frac{\partial g}{\partial y}=\frac{\partial f}{\partial y} \qquad \frac{\partial g}{\partial z}=-1$$

$$\Rightarrow \nabla g = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1\right)$$

so the tangent plane to $z=f(x,y)$ has normal vector $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1\right)$.

More generally.

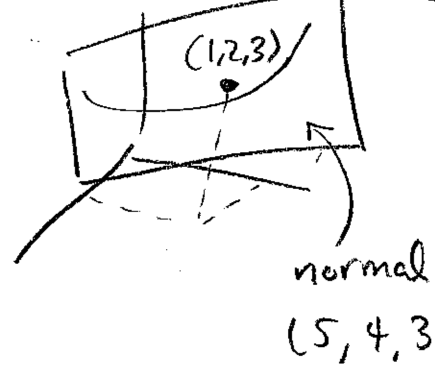
Example. Find an equation for the plane tangent to the surface $xy+yz+zx=11$ at the point $(1,2,3)$.

Let $f(x,y,z)=xy+yz+zx$.

$$\nabla f=(y+z,\ x+z,\ x+y)$$

$\nabla f(1,2,3)=(5,4,3)$. ← This gives the normal vector of the tangent plane.

So the equation is

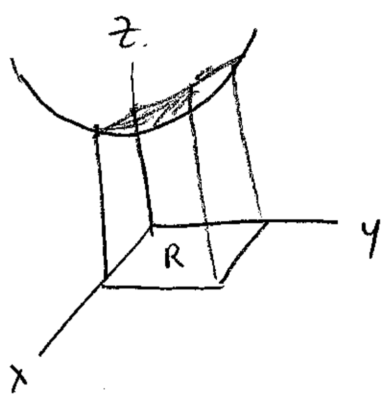


$$5(x-1)+4(y-2)+3(z-3)=0.$$

So gradients can be used for:

- tangent planes
- linear approximations
- directional derivatives
- chain rule
- normals to level curves/surfaces.

Next topic: Integration (Lecture 19. 3/10/20.)

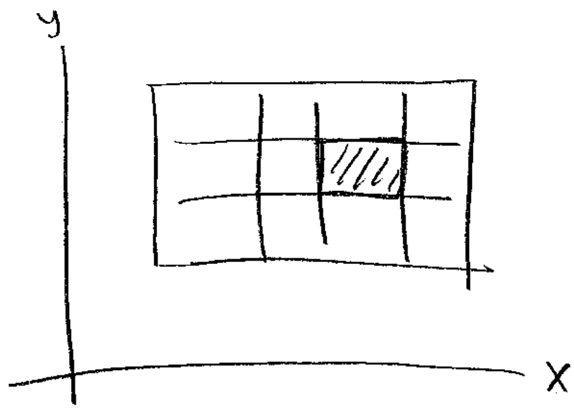


$f(x,y)$. R is a rectangle in the xy plane.

Q: what is the volume of the solid that is bounded below by R and above by the graph of f?

(This is denoted $\iint_R f(x,y)\,dx\,dy$

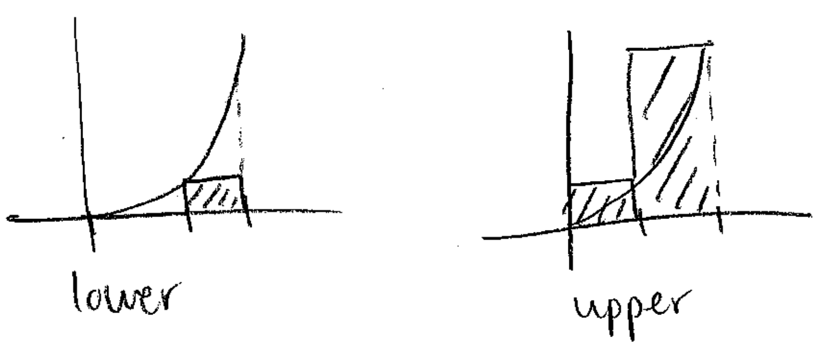
<u>Idea:</u> We can partition R into smaller rectangles.



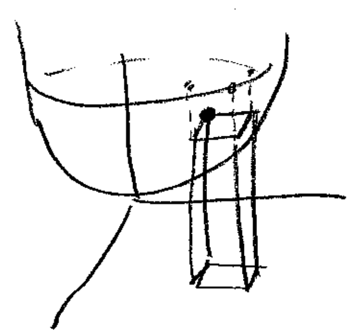
We can calculate the max and min on each of these rectangles.

$\longrightarrow$ upper and lower Riemann sums.

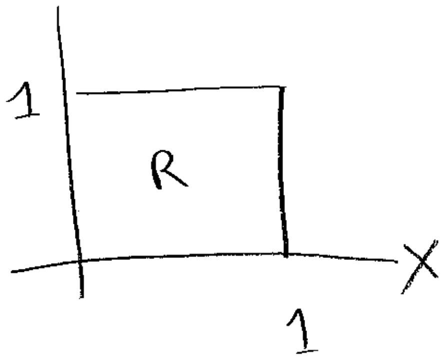
in single var:



same picture in multivar: the rectangles from 1-var become rectangular prisms.

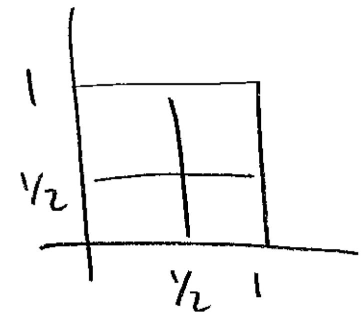


Example: $f(x,y) = x^2 + y^2$.

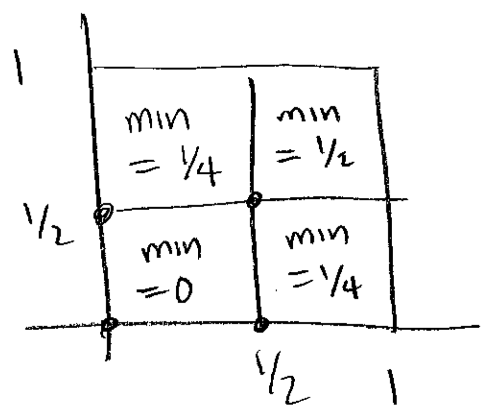


R is the square
$0 \le x \le 1$
$0 \le y \le 1$.

Consider the partition P:
(into 4 rectangles)

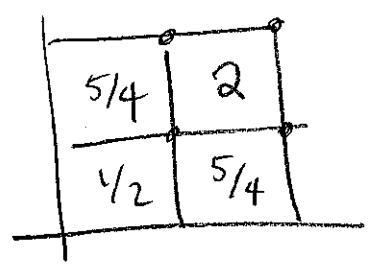


lower sum. calculate the min of f on each rectangle



$$L_f(P) = 0 \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{4}$$
$$= \frac{1}{4}$$

upper sum



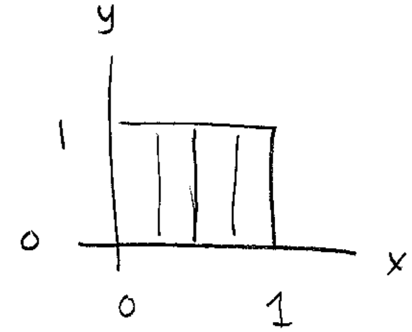
$$U_f(P) = \frac{1}{2} \cdot \frac{1}{4} + \frac{5}{4} \cdot \frac{1}{4} + \frac{5}{4} \cdot \frac{1}{4} + 2 \cdot \frac{1}{4}$$
$$= \frac{5}{4}.$$

so $$\frac{1}{4} \le \iint_R (x^2+y^2)\,dx\,dy \le \frac{5}{4}.$$

How do we actually calculate the volume

Repeated/ Iterated integrals.

$f(x,y) = x^2 + y^2$ $\quad$ $0 \le x \le 1$, $0 \le y \le 1$



Fix x, vary y.

$$\iint_R (x^2+y^2)\,dx\,dy =$$

$$= \int_0^1 \underbrace{\left[\int_0^1 (x^2+y^2)\,dy \right]}_{\text{for the inner integral } x \text{ is fixed.}} dx$$

$$\int_0^1 (x^2+y^2)\,dy = \left[x^2 y + \frac{y^3}{3} \right]_0^1$$

$$= \left[x^2(1) + \frac{1^3}{3} \right] - \left[x^2 0 + \frac{0^3}{3} \right]$$

$$= x^2 + \frac{1}{3}$$

$$= \int_0^1 \left(x^2 + \frac{1}{3} \right) dx$$

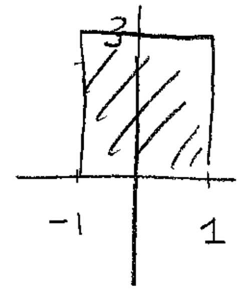
$$= \left[\frac{x^3}{3} + \frac{1}{3}x \right]_0^1 = \boxed{\frac{2}{3}}$$

Example 2 :

$f(x,y)=x^2+y^2$ R is the rectangle

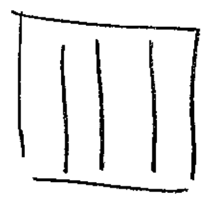
$-1 \leq x \leq 1$

$0 \leq y \leq 3$.



Then the volume is

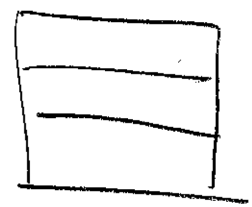
$$\int_{-1}^{1}\left[\int_{0}^{3}(x^2+y^2)\,dy\right]dx$$



Fix x, vary y.

It can also be written

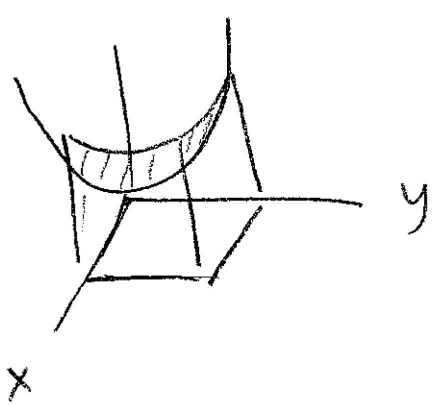
$$\int_{0}^{3}\left[\int_{-1}^{1}(x^2+y^2)\,dx\right]dy$$



Fix y, vary x.

ask class to calculate.

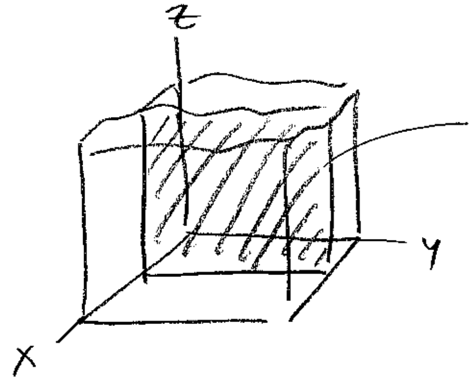
Geometric interpretation.



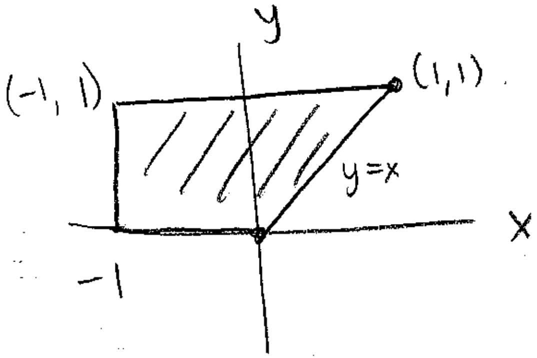
Fix x: (take slice parallel to yz plane)

$\int_0^3 (x^2+y^2)\,dx$

is the area of a crosssection of the solid



What if the region we're integrating over is not a rectangle?

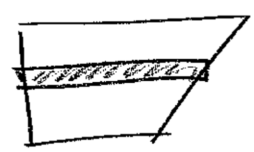


Call the region Ω.

If we fix $y \in [0,1]$ then x lies between -1 and y.

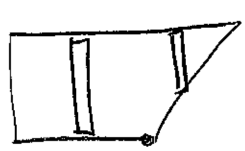
Ω consists of all pts with

$0 \le y \le 1 \qquad -1 \le x \le y$.



so $$\iint_{\Omega} f(x,y)\,dx\,dy = \int_0^1 \left[\int_{-1}^{y} f(x,y)\,dx \right] dy$$

We could also start by fixing $x \in [-1,1]$.



but then the range of y is more complicated

If $-1 \le x \le 0$: then $0 \le y \le 1$.

If $0 \le x \le 1$ then $x \le y \le 1$.

So $\iint_{\Omega} f(x,y)\,dx\,dy$

$$= \int_{-1}^{0}\left[\int_{0}^{1} f(x,y)\,dy\right]dx + \int_{0}^{1}\left[\int_{x}^{1} f(x,y)\,dy\right]dx$$

let's try $f(x,y) = 2x+4y$.

Ask class to calculate.

$$\int_0^1 \int_{-1}^{y} 2x+4y \,dx\,dy = \int_0^1 \left[x^2+4yx\right]_{-1}^{y} dy$$

$$= \int_0^1 \left[y^2+4y^2\right] - \left[+1-4y\right] dy.$$

$$= \int_0^1 5y^2+4y-1\,dy$$

$$= \frac{5}{3}+2-1 = \frac{8}{3}.$$

$$\int_{-1}^{0}\int_{0}^{1}(2x+4y)\,dy\,dx$$

$$= \int_{-1}^{0}\left[2xy+2y^2\right]_0^1 dx$$

$$= \int_{-1}^{0} 2x+2\,dx$$

$$= \left[x^2+2x\right]_{-1}^{0}$$

$$= 0-((-1)^2+2(-1))$$

$$= 1.$$

$$\int_{0}^{1}\int_{x}^{1}(2x+4y)\,dy\,dx$$

$$= \int_{0}^{1}\left[2xy+2y^2\right]_x^1 dx$$

$$= \int_{0}^{1}\left[2x+2-\left[2x^2+2x^2\right]\right] dx$$

$$= \int_{0}^{1} -4x^2+2x+2\,dx$$

$$= -\frac{4}{3}+1+2.$$

So final answer $1-\frac{4}{3}+1+2 = \frac{8}{3}$ ✓