# Math 153001/14 Final exam solutions

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1. (a) Let $a_n = -1$.
   (b) One possible answer is $\sum_{n=0}^{\infty} \frac{1}{2^n} = 2$.
   (c) Let $f(x) = 1/x$. Then $f$ is a positive and decreasing function, and $\int_1^\infty f(x)\,dx = \infty$, so $\sum \frac{1}{n}$ diverges by the integral test.

2. $\|\vec{u}+\vec{v}\|^2 = (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) = \vec{u}\cdot\vec{u} + \vec{v}\cdot\vec{v} + 2\vec{u}\cdot\vec{v} = \|\vec{u}\|^2 + \|\vec{v}\|^2 + 2\vec{u}\cdot\vec{v}$ Since $\vec{u}, \vec{v}$ are unit vectors, we know $\|\vec{u}\| = \|\vec{v}\| = 1$. For the dot product, we have $\vec{u}\cdot\vec{v} = \|\vec{u}\|\|\vec{v}\|\cos(\pi/3) = 1\cdot 1\cdot \frac{1}{2}$.

   This gives $\|\vec{u}+\vec{v}\|^2 = 1 + 1 + 2\cdot\frac{1}{2} = 3$, so $\|\vec{u}+\vec{v}\| = \sqrt{3}$.

3. $\nabla f = (ovid, cvid, coid, covd, covi)$. (Sorry, this was a dumb question...)

4. (a) $\nabla f(x,y) = (2,-1)$. The direction, rescaled to be a unit vector is $\frac{1}{5}(3,-4)$. So

$$f'_{\vec{u}}(1,2) = \nabla f(1,2)\cdot\vec{u} = (2,-1)\cdot\frac{1}{5}(3,-4) = \frac{(2)(3)+(-1)(-4)}{5} = 2$$

   (b)

$$\begin{aligned} f'_{\vec{u}}(1,2) &= \lim_{h\to 0}\frac{f((1,2)+h\vec{u}) - f(1,2)}{h} \\ &= \lim_{h\to 0}\frac{f((1+\frac{3}{5}h, 2-\frac{4}{5}h)) - f(1,2)}{h} \\ &= \lim_{h\to 0}\frac{2(1+\frac{3}{5}h) - (2-\frac{4}{5}h) - 0}{h} \\ &= \lim_{h\to 0}\frac{2h}{h} \\ &= 2 \end{aligned}$$

5. (a) $\nabla f(x,y) = (y^2, 2xy)$. $\vec{r}'(t) = (3t^2, 2t)$. So

$$\frac{d}{dt}f(\vec{r}(t)) = \nabla f(\vec{r}(t))\cdot\vec{r}'(t) = \nabla f(t^3,t^2)\cdot(3t^2,2t) = (t^4, 2t^5)\cdot(3t^2,2t) = 3t^6 + 4t^6 = 7t^6$$

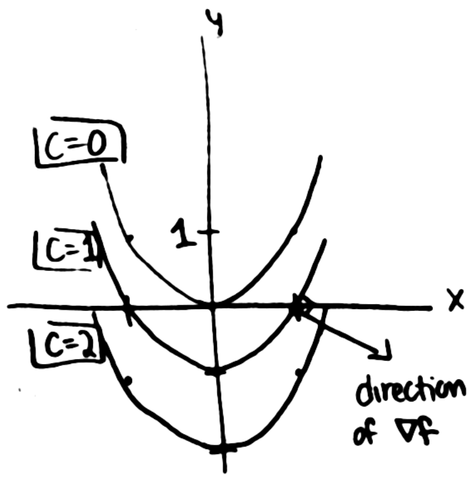
   (b) $\nabla f(1,2) = (4,4)$. So the unit vector is $\frac{1}{\sqrt{2}}(1,1)$. The rate of change in that direction is $\|\nabla f(1,2)\| = \|(4,4)\| = 4\sqrt{2}$.

6. (a) The graph of $f$ is given by $z = xy^2$ so it is a level curve of the function $g(x, y, z) = xy^2 - z$. $\nabla g = (y^2, 2xy, -1)$. $\nabla g(1, 2, 4) = (4, 4, -1)$.

   So the normal line goes through $(1, 2, 4)$ and has direction $(4, 4, -1)$. This means the scalar parametric equations for the line are $x(t) = 1 + 4t, y(t) = 2 + 4t, z(t) = 4 - t$

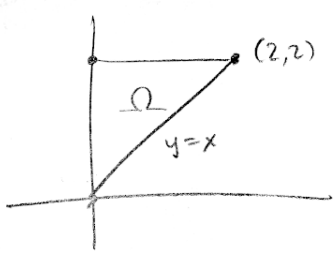
   (b) The tangent plane contains $(1, 2, 4)$ and has normal $(4, 4, -1)$. This means its equation is given by $4(x - 1) + 4(y - 2) - (z - 4) = 0$.

7. (a) Here is a sketch.




   (b) The point $(1, 0)$ is on the level curve $c = 1$, so $\nabla f$ at that point must be perpendicular to the curve. This narrows it down to two possible directions: up-left, or down-right. Since $\nabla f$ points in a direction in which $f$ is increasing, it must be down-right.

8. (a) Here is $\Omega$.




   $\iint_\Omega 1\, dx\, dy =$ area of $\Omega = 2$

   (b) Using horizontal lines:

$$\iint_\Omega xy\, dx\, dy = \int_0^2 \int_0^y xy\, dx\, dy = \int_0^2 \left[\frac{x^2 y}{2}\right]_0^y dy = \int_0^2 \frac{y^3}{2}\, dy = \left[\frac{y^4}{8}\right]_0^2 = 2$$

   (Alternatively, you could use vertical lines: $\iint_\Omega xy\, dx\, dy = \int_0^2 \int_x^2 xy\, dy\, dx$.)