# Take-home final exam

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You have two hours. Start the timer before looking at the next page. When time is up, please stop writing.

If a problem has a final answer, please draw a box around it.

Please present your solutions clearly and in an organized way. The use of a calculator is not allowed. Good luck!! ⌣̈

| Question | Points | Score |
|---|---|---|
| 1 | 100 | |
| 2 | 100 | |
| 3 | 50 | |
| 4 | 200 | |
| 5 | 200 | |
| 6 | 100 | |
| 7 | 100 | |
| 8 | 150 | |
| Total: | 1000 | |

This exam has 8 questions, for a total of 1000 points. The maximum possible score for each problem is given on the right side of the problem.


1. Here are some questions about sequences and series. [100]

   Each of (a), (b), and (c) is worth 50 points. The lowest score of the three will be dropped. (In other words, you only need to do two of the three parts, but you can do all three if you prefer.)

   (a) Write down a series $\sum_{n=0}^{\infty} a_n$ which satisfies the following two properties. You do not need to justify.

   - The series $\sum_{n=0}^{\infty} a_n$ diverges
   - The series $\sum_{n=0}^{\infty}(1+a_n)$ converges

   (b) Think of a convergent infinite series $\sum_{n=0}^{\infty} a_n$ with the following two properties:

   - The terms are positive (i.e., $a_n > 0$ for all $n$)
   - You know what value the series converges to.

   Write down the series itself as well as the value it converges to. You do not need to justify.

   (c) Show that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by using one of the convergence tests from class. Please explain why the test you chose can be applied to this series.

   *Note*: The $p$-series test is **not** one of the convergence tests from class.

2. Suppose $\vec{u}$ and $\vec{v}$ are unit vectors, and the angle between them is $\pi/3$ radians (60 degrees). Find the length of the vector $\vec{u}+\vec{v}$. [100]

   *Hint*: Start by expanding the expression $||\vec{u}+\vec{v}||^2$. Also, there's a formula involving $||\vec{u}||||\vec{v}||\cos\theta$ that may be helpful.

3. Let $f(c,o,v,i,d) = covid - 19$. (That is, multiply the five variables together, then subtract 19.) Calculate $\nabla f$. [50]

4. Let $f(x,y) = 2x - y$. Find the directional derivative of $f$ at the point (1,2) in the direction of the vector (3,−4) in the following two ways.

   (a) By the formula for the directional derivative involving the gradient. [100]

   (b) By the limit definition of the directional derivative. [100]

   *Caution*: Make sure you calculate the directional derivative correctly. Sometimes people forget a particular step when calculating the directional derivative.

5. Let $f(x,y) = xy^2$.

   (a) Using the multivariable chain rule, find the rate of change of $f$ with respect to $t$ along the curve $\vec{r}(t) = (t^3, t^2)$. [100]

   *Hint*: Your answer should be a function in $t$.

   (b) Find the unit vector in the direction in which $f$ increases most rapidly at the point $(1,2)$ and give the rate of change of $f$ in that direction. [100]

6. Let $f(x,y) = xy^2$ (same function as the previous problem).

   Consider the graph of $f$, i.e., $z = f(x,y)$. At the point $(1,2,4)$, find the following:

   (a) Scalar parametric equations for the normal line. [50]

   (b) An equation for the tangent plane. [50]


7. Let $f(x,y) = x^2 - y$.

   (a) Sketch the level curves $f = c$ with $c = 0, 1, 2$. Label each one with the value of $c$. 50

   (b) On your sketch, indicate the direction of $\nabla f$ at the point $(1,0)$ by drawing an arrow. Explain how you can determine the direction without having to actually calculate $\nabla f(1,0)$. 50

8. Let $\Omega$ be the triangle in the plane with vertices $(0,0), (0,2), (2,2)$.

   (a) Evaluate 50

$$\iint_\Omega 1\, dx\, dy$$

   (b) Evaluate 100

$$\iint_\Omega xy\, dx\, dy$$