"Remarkable property" of antiderivative of $1/x$.

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Define $L(x) = t\int_1^x \frac{1}{t}\,dt$.
The textbook says that the function L satisfies the following "remarkable property":

$$L(ab) = L(a) + L(b) \text{ for all positive numbers } a \text{ and } b.$$

Here is the proof I gave in class for this.

The left-hand side is $L(ab) = \int_1^{ab} \frac{1}{t}\,dt$.
The right-hand side is $L(a) + L(b) = \int_1^a \frac{1}{t}\,dt + \int_1^b \frac{1}{t}\,dt$.

Step 1: First, let's show that

$$\int_1^b \frac{1}{t}\,dt = \int_a^{ab} \frac{1}{t}\,dt \qquad (*)$$

Start with the left-hand side of (*) and make the change of variables $u(t) = at$. Then $du = a\,dt$, and the new upper and lower bounds are $u(b) = ab$ and $u(1) = a$. So:

$$\int_1^b \frac{1}{t}\,dt = \int_a^{ab} \frac{1}{\frac{1}{a}u} \cdot \frac{1}{a}\,du = \int_a^{ab} \frac{1}{u}\,du = \int_a^{ab} \frac{1}{t}\,dt$$

(The first equality is by the u-substitution. The second equality is because the two $1/a$ cancel each other. The third equality is because for a definite integral, the variable you integrate with respect to is just a dummy variable.)
Thus we have shown $(*)$.

Step 2: Using $(*)$, we have

$$L(a) + L(b) = \int_1^a \frac{1}{t}\,dt + \int_1^b \frac{1}{t}\,dt = \int_1^a \frac{1}{t}\,dt + \int_a^{ab} \frac{1}{t}\,dt = \int_1^{ab} \frac{1}{t}\,dt = L(ab)$$

QED

(If you find that there are too many variables, keep in mind that a and b are just constants. You can substitute in values, e.g., $a = 3$ and $b = 5$.)