MATH 15200/14

Midterm #1

OCTOBER 22, 2019
ALAN CHANG

Please present your solutions clearly and in an organized way. Simplify all your final answers. If an answer box is given, write your final answer in the box. If you run out of room, continue on the extra pages provided at the end. **The use of a calculator is not allowed.** Good luck!! ◡̈

Full Name: Sample solutions

Student ID:

Question	Points	Score
1	40	
2	24	
3	15	
4	15	
5	20	
6	10	
7	20	
Total:	144	

This exam has 7 questions, for a total of 144 points. The maximum possible score for each problem is given on the right side of the problem.

Please do not open this exam packet unless you are instructed to do so.

1. Evaluate the following expressions. Show your work.

(a) $\int \left(\frac{4x^3+1}{x^2} + 3\cos x \right) dx = \boxed{2x^2 - \frac{1}{x} + 3\sin x + C}$ 10

$= \int (4x + x^{-2} + 3\cos x)\, dx$

$= 4 \cdot \frac{1}{2} x^2 + \frac{x^{-1}}{-1} + 3\sin x + C$

$= 2x^2 - \frac{1}{x} + 3\sin x + C$

(b) $\int_0^4 \frac{4x}{\sqrt{x^2+9}}\, dx = \boxed{8}$ 10

$u = x^2 + 9$ upper $= 4^2 + 9 = 16 + 9 = 25$

$du = 2x\, dx$ lower $= 0^2 + 9 = 9$

$= \int_9^{25} \frac{2}{\sqrt{u}}\, du = 2 \int_9^{25} u^{-1/2}\, du = 2 \left[\frac{u^{1/2}}{1/2} \right]_9^{25}$

$= 4 \left[25^{1/2} - 9^{1/2} \right] = 4(5-3) = 4 \cdot 2 = 8$

(c) $\dfrac{d}{dt}\displaystyle\int_t^1 (y^2+1)^3\,dy = \boxed{-(t^2+1)^3}$ 10

$$= -\frac{d}{dt}\int_1^t (y^2+1)^3\,dy$$

$$= -(t^2+1)^3 \leftarrow$$ (by fundamental theorem of calculus)

(d) $\displaystyle\int_0^1 \frac{d}{dt}\left[(t^2+1)^3\right] dt = \boxed{7}$ 10

$$= \left[t^2+1\right]_0^1 \leftarrow$$ (by fundamental theorem of calculus)

$$= (1^2+1)^3-(0^2+1)^3$$

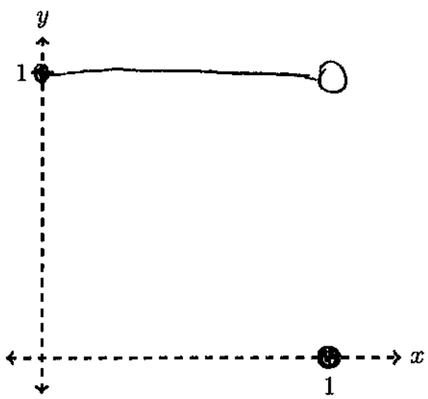
$$= 2^3-1^3$$

$$= 8-1$$

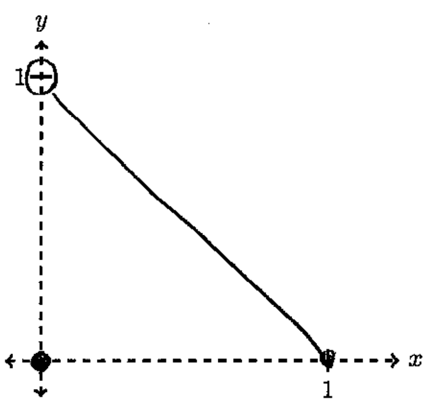
$$= 7$$

2. Sketch the graph of a function f that is **defined on** $[0,1]$ and meets the given conditions. If this is not possible, give a reason why.

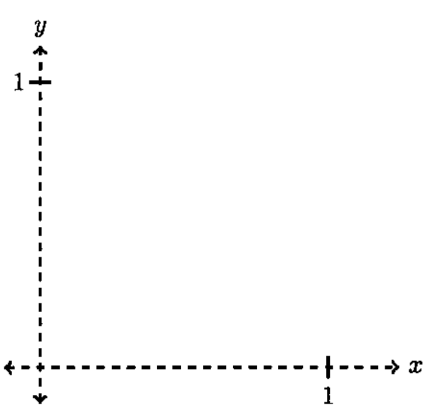
   (a) f is continuous on $(0,1)$ $\boxed{6}$
   f takes on the values 0 and 1
   f does **NOT** take on the value $\frac{1}{2}$



   (b) f is continuous on $(0,1]$ $\boxed{6}$
   f does **NOT** take on a maximum value



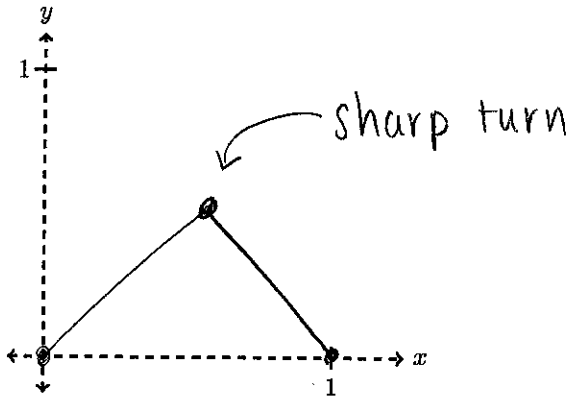
   (c) f is continuous on $[0,1]$ $\boxed{6}$
   f does **NOT** take on a maximum value



This is not possible, because of the extreme value theorem

6

(d) f is continuous on $[0,1]$
$f(0)=f(1)$
There does **NOT** exist a point $c \in (0,1)$ such that $f'(c)=0$



3. Show that $\lim_{x\to 2}\left(5-\frac{1}{2}x\right)=4$ using the (ϵ,δ) definition of the limit.

15

Let $\varepsilon > 0$.

Let $\delta = 2\varepsilon$.

Suppose $0 < |x-2| < \delta$.

Then $\left|\left(5-\frac{1}{2}x\right)-4\right|$

$= \left|1-\frac{1}{2}x\right|$

$= \frac{1}{2}|2-x|$

$= \frac{1}{2}|x-2|$

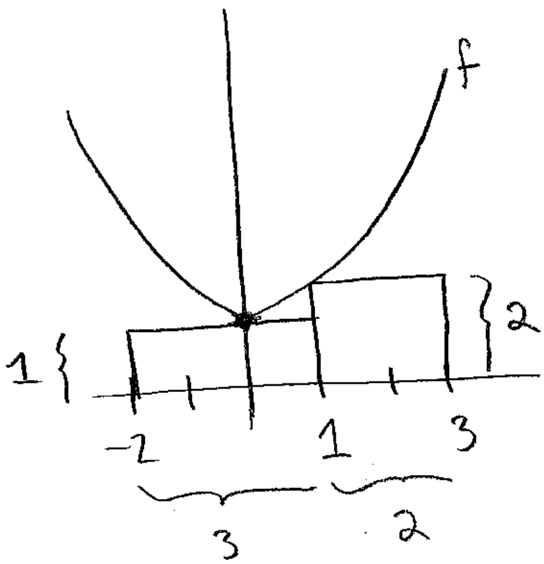
$< \frac{1}{2}\delta$

$= \varepsilon$.

4. Let $f(x) = x^2 + 1$. Consider the partition $P = \{-2, 1, 3\}$.

(a) Calculate the lower Riemann sum $L_f(P)$. Include a sketch of the function and the relevant rectangles. $\boxed{10}$

$L_f(P) = \boxed{7}$



$$L_f(P) = 1\cdot 3 + 2\cdot 2$$
$$= 3+4$$
$$= 7$$

(b) Write < (less than) or = (equals to) or > (greater than) into the box below to indicate the relation between the two quantities. You do not need to justify. $\boxed{5}$

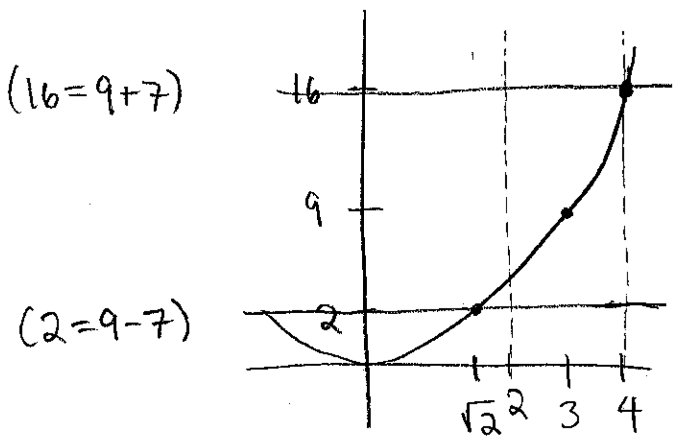
$$L_f(P) \ \boxed{<} \ \int_{-2}^{3} (x^2+1)\,dx$$

5. Consider the following statement:

"If $0 < |x-3| < \delta$, then $|x^2-9| < 7$."

(a) Find a $\delta > 0$ so that the statement above is **TRUE**. Please justify with a sketch and a brief explanation. $\boxed{10}$

$\delta =$ $\boxed{1}$

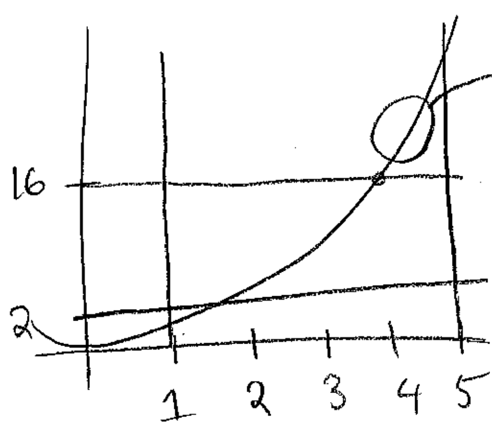


The two vertical lines should be between $\sqrt{2}$ and 4.

So $\delta = 1$ works.

(b) Find a $\delta > 0$ so that the statement above is **FALSE**. Please justify with a sketch and a brief explanation. $\boxed{10}$

$\delta =$ $\boxed{2}$



this part of the graph is not within the two horizontal lines.

10

6. Consider the functions $f(x) = x^2 - 2$ and $g(x) = -2x^2 + 1$.
Sketch the region bounded by the two curves, and find the area of that region.

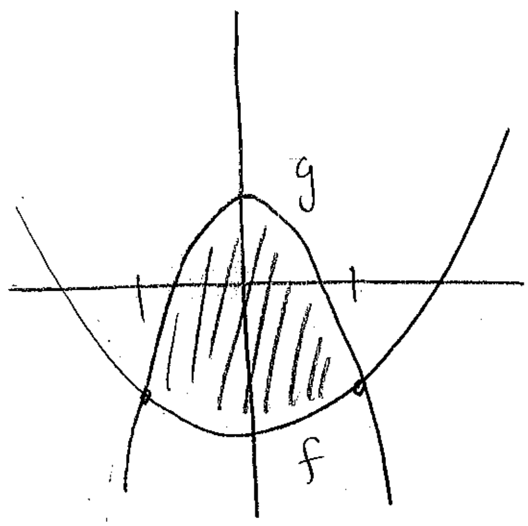
area = $\boxed{4}$

Intersection points:

$$x^2 - 2 = -2x^2 + 1$$
$$3x^2 = 3$$
$$x^2 = 1$$
$$x = \pm 1$$

$$f(1) = 1^2 - 2 = -1 \qquad f(-1) = (-1)^2 - 2 = -1$$

$$(1, -1) \qquad (-1, -1)$$



$$\text{area} = \int_{-1}^{1} (g(x) - f(x))\, dx$$
$$= \int_{-1}^{1} ((-2x^2+1) - (x^2-2))\, dx$$
$$= \int_{-1}^{1} (-3x^2 + 3)\, dx$$
$$= \left[-x^3 + 3x\right]_{-1}^{1}$$
$$= (-1^3 + 3\cdot 1) - (-(-1)^3 + 3(-1))$$
$$= 2 - (-2)$$
$$= 4$$

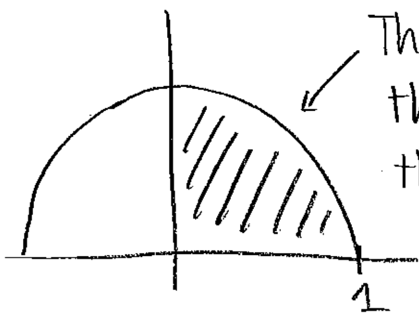
7. Consider the integral $\int_0^1 \sqrt{1-x^2}\,dx$.

In this problem, we will evaluate this integral in two different ways.

(a) Give a geometric interpretation of the integral as the area of some region in the xy-plane. Please include a sketch. Use basic (i.e., high school level) geometry to evaluate the integral. 10

$\int_0^1 \sqrt{1-x^2}\,dx = \boxed{\pi/4}$

$y=\sqrt{1-x^2}$ is a semicircle



The integral is the area of this region (quarter circle)

area of quarter circle

$= \frac{1}{4} \cdot \pi 1^2$

$= \frac{\pi}{4}$

(b) Fact: $\frac{d}{dx}\left[x\sqrt{1-x^2}+\arcsin x\right] = 2\sqrt{1-x^2}$. 10

Use this fact (which you do not need to prove) and the fundamental theorem of calculus to evaluate the integral.

$\int_0^1 \sqrt{1-x^2}\,dx = \boxed{\pi/4}$

$= \frac{1}{2}\left[x\sqrt{1-x^2} + \arcsin x\right]_0^1$

$= \frac{1}{2}\left[1\sqrt{1-1^2} + \arcsin 1 - \left(0\sqrt{1-0^2} + \arcsin 0\right)\right]$

$= \frac{1}{2}\left(0 + \frac{\pi}{2} - (0+0)\right)$

$= \frac{\pi}{4}$

This is blank space.

This is more blank space.

This is even more blank space.

If you are bored, you can draw something (e.g. your favorite Pokémon).