Math 152

10/1/19

- Warmup problem: Determine $\lim_{x \to 0} f(x)$ for the following functions

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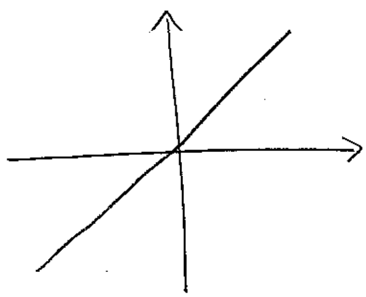

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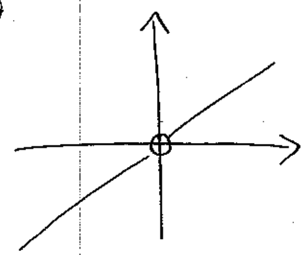

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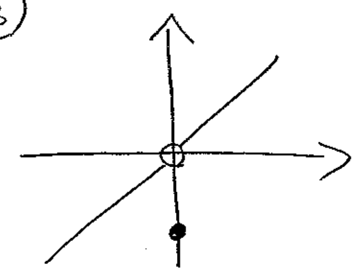

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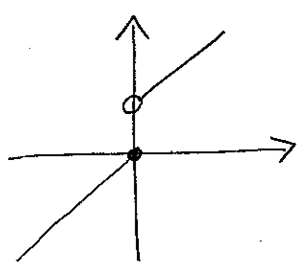

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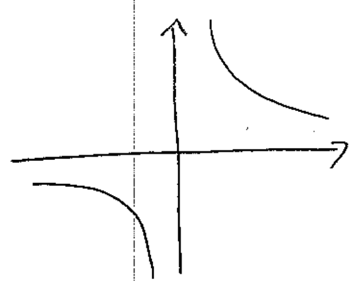

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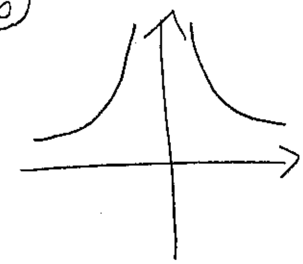

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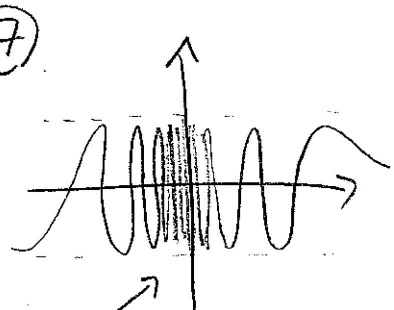

infinitely many oscillations

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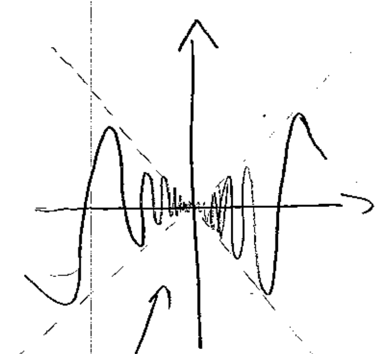


infinitely many oscillations

- Introductions: pair up and present.

  name, year, where you're from, intended major, interests, something fun you did during the summer.

- Go over syllabus, are there any questions?
- Back to warmup problem.

1. $f(x) = x$. $\quad \lim_{x\to 0} f(x) = 0$.

2. $f(x) = \begin{cases} x & \text{if } x \neq 0 \\ \text{undef.} & \text{if } x = 0 \end{cases} \quad \lim_{x\to 0} f(x) = 0$.

   Recall: When determining $\lim_{x\to c} f(x)$, the value of $f(c)$ does not matter

   You could also write this as $f(x) = \frac{x^2}{x}$ for example.

3. $f(x) = \begin{cases} x & \text{if } x \neq 0 \\ -1 & \text{if } x = 0 \end{cases} \quad \lim_{x\to 0} f(x) = 0$.

4. The limit does not exist. But, recall one-sided limits:

   From left: $\lim_{x\to 0^-} f(x) = 0$

   From right: $\lim_{x\to 0^+} f(x) = 1$.

   $\left(f(x) = \begin{cases} x & \text{if } x \leq 0 \\ x+1 & \text{if } x > 0 \end{cases} \right)$

5. $f(x) = \frac{1}{x}$. The limit does not exist.

We have: $\lim_{x\to 0^-} f(x) = -\infty$, $\lim_{x\to 0^+} f(x) = \infty$

But it's still correct to say that the left-sided and right-sided limits do not exist. (This is just some technicality... infinity is not a number)

6. $f(x) = \frac{1}{|x|}$. $\lim_{x\to 0} f(x) = \infty$.

(Again, technically, the limit does not exist.)

7. $f(x) = \sin\frac{1}{x}$. The limit does not exist.

As $x \to 0$, $f(x)$ keeps oscillating between -1 and 1, so it does not get close to any single number.

8. $f(x) = x \sin\frac{1}{x}$. $\lim_{x\to 0} f(x) = 0$.

(Infinitely many oscillations is not enough to prevent the limit from existing.)

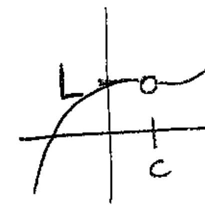
Recall: The pinching theorem.

$-|x| \le x \sin\frac{1}{x} \le |x|$ for all $x \neq 0$.

Since $\lim_{x\to 0}(-|x|) = 0$ and $\lim_{x\to 0}|x| = 0$, it follows that $\lim_{x\to 0} x \sin\frac{1}{x} = 0$.

Intuitively, $\lim_{x \to c} f(x) = L$ means:

"$f(x)$ is close to L for all $x \neq c$ which are close to c."

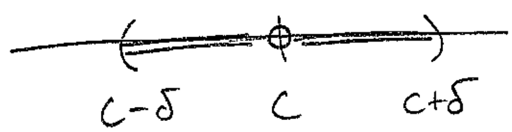


The examples from the warmup problem give you a good idea of how limits work.

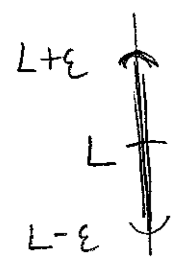
However, for modern mathematical standards, the definition of a limit given above is too imprecise.

Here is the rigorous definition: $\lim_{x \to c} f(x) = L$ means:

"For each $\varepsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x - c| < \delta$, then $|f(x) - L| < \varepsilon$."



x lies in this set
(note: c is not in the set)



f(x) lies in this set.

This can take some time/practice before it makes sense.

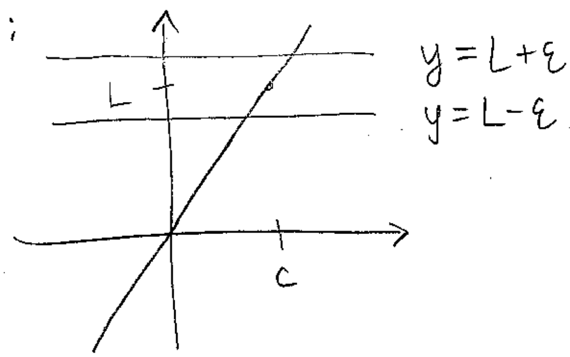
One way to think of this definition is to imagine it as a game between a "good guy" (G) and a "bad guy" (B).

The game starts with 3 things: f (function), c, L

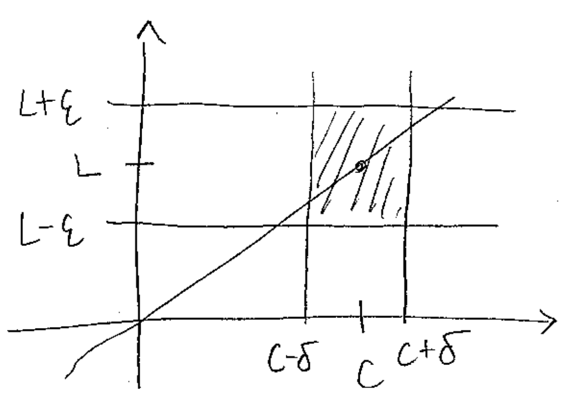
G wants to show $\lim_{x \to c} f(x) = L$.

B wants to show $\lim_{x \to c} f(x) \neq L$.

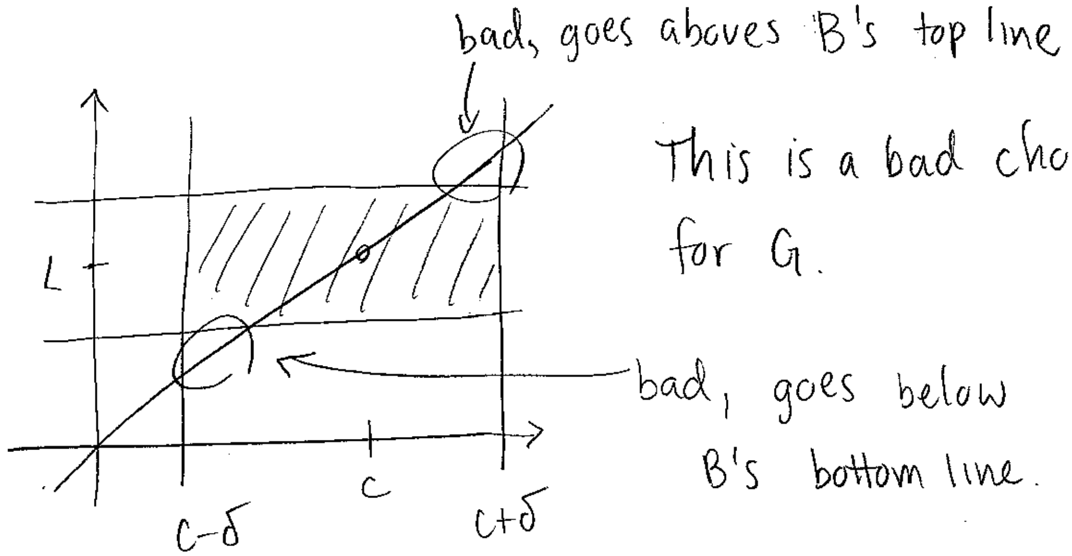
- B starts by choosing a number $\varepsilon > 0$ and drawing 2 horizontal lines on the graph:



- Then G has to respond by drawing 2 vertical lines so that when we're between these two lines, the graph stays with B's horizontal lines (with the exception of the point $(x, f(x))$)



← This is a good choice for G.



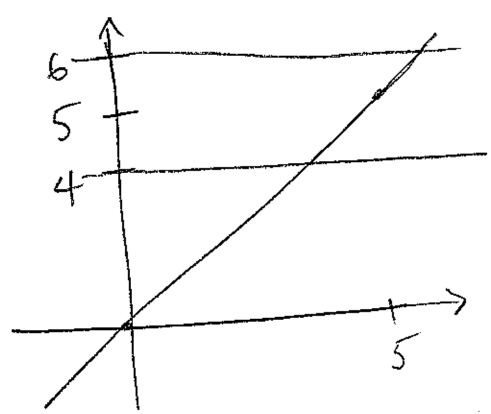
This is a bad choice for G.

And that's it.

B can choose any 2 horizontal lines he wants.
If he finds a choice for which G cannot respond, then B wins. ($\lim_{x \to c} f(x) \neq L$)
Otherwise (ie. if G always has a response, no matter what B does), G wins. ($\lim_{x \to c} f(x) = L$)

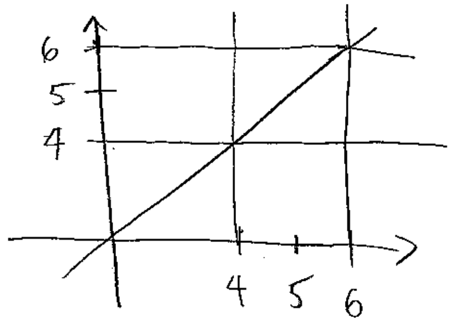
Example 1: $f(x) = x$, $c = 5$, $L = 5$. ($\lim_{x \to 5} x \stackrel{?}{=} 5$)

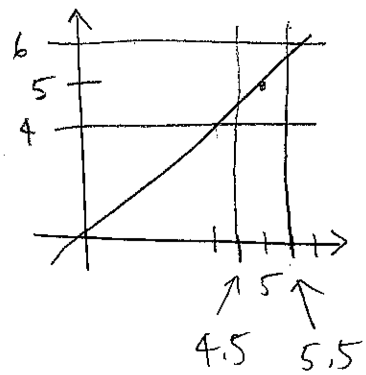
- Suppose B choose $\varepsilon = 1$.



Then G could respond with

$\delta = 1$: or $\delta = \frac{1}{2}$

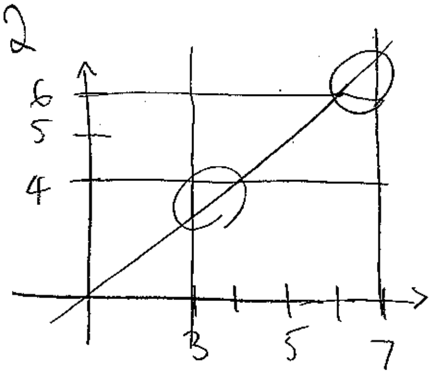




or any other $\delta \in (0, 1]$.

But G could <u>not</u> respond with

$\delta = 2$ or any other $\delta > 1$.



- Now suppose B chooses $\varepsilon = 0.5$?

  G can respond with $\delta = 0.5$.

- Observe. If B choose any $\varepsilon > 0$,
  G can respond with the same thing ($\delta = \varepsilon$).
  Why does this work? It's clear from the picture. In math language:

If $0<|x-5|<\delta$, then

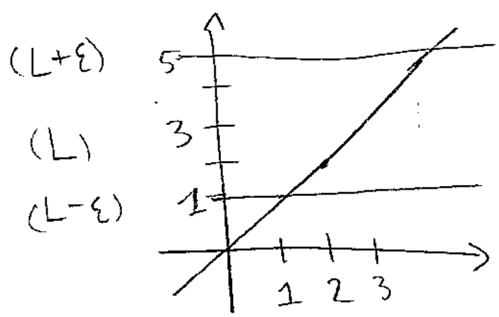
$$|f(x)-L| = |x-5| < \delta = \varepsilon.$$

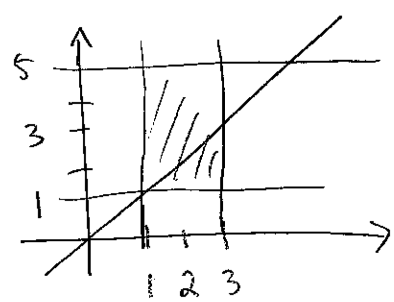
So: $\lim_{x\to 5} x = 5$.

(* lecture 2 begins here)

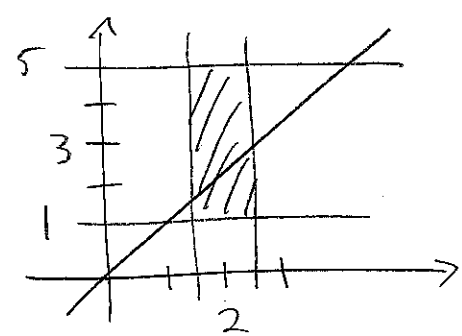
<u>Example 2</u>: $\lim_{x\to 2} x \stackrel{?}{=} 3$ $\qquad f(x)=x,\ c=2,\ L=3$

Suppose B chooses $\varepsilon=2$. Can G respond with a δ? Yes! Take $\delta=1$ for example. (Any $\delta\in(0,1]$ works)



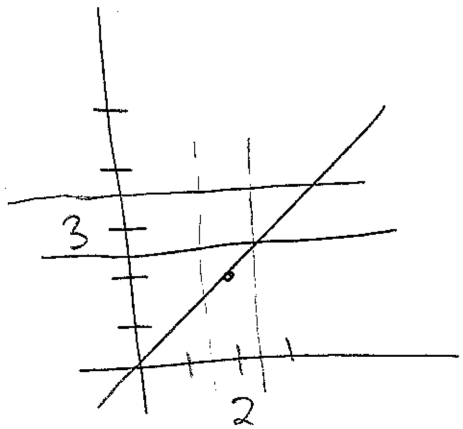


($\delta=1$)



($\delta=0.5$)

Now suppose B choose $\varepsilon=0.5$.



No matter how G responds there will always be a point between the 2 vertical lines (not counting (2,2)) which lies below B's bottom line.

For example if $x_0 = 2 - \frac{\delta}{2}$, then x_0 lies between the 2 vertical lines

(i.e. $0 < |x_0 - 2| < \delta$)

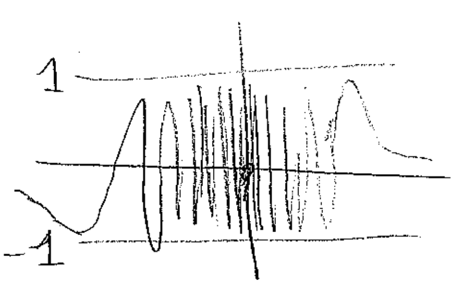
but: $f(x_0) = x_0 = 2 - \frac{\delta}{2} < 2 < 2.5 = L - \varepsilon$.

so $f(x_0) - L < -\varepsilon$

so $|f(x_0) - L| > \varepsilon$.

so B wins. $\lim_{x \to 2} x \neq 3$.

<u>Example 3</u>: $\lim_{x \to 0} \sin\frac{1}{x} \stackrel{?}{=} 0$ $f(x) = \sin\frac{1}{x}$, $c = 0$, $L = 0$



Who wins? Observe that

$$1 = \sin\frac{\pi}{2} = \sin\left(\frac{\pi}{2} + 2\pi\right) = \sin\left(\frac{\pi}{2} + 4\pi\right)$$
$$= \sin\left(\frac{\pi}{2} + 6\pi\right) = \cdots$$

$\sin\left(\frac{\pi}{2}+2\pi k\right)=1$ for all integers k.

so: $f\left(\frac{1}{\frac{\pi}{2}+2\pi k}\right)=1$ for all integers k.

So if B picks $\varepsilon = 1/2$, there is no way for G to pick δ, since for any $\delta>0$ we can always find a large enough k so that $0<\frac{1}{\frac{\pi}{2}+2\pi k}<\delta$.

So B wins. In fact $\lim_{x\to 0} \sin\frac{1}{x} \neq L$ for any value of L.
(B always wins, no matter what L is.)

(Ask: How to change the game for one-sided limits?)

For more practice: you can read the examples in section 2.2 of the textbook. We'll also do more examples in the first problem session.

(Lecture 3 begins here)

Now that we have defined limits, we can define continuity:

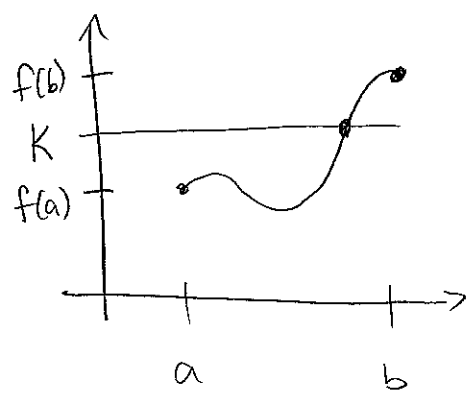
Definition: We say that f is continuous at c if

$$\lim_{x\to c} f(x) = f(c).$$

f is continuous from the left at c if $\lim_{x \to c^-} f(x) = f(c)$.

f is - - - - - - - - right - - - $\lim_{x \to c^+} f(x) = f(c)$.

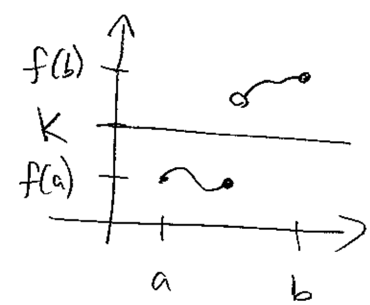
Recall two basic theorems about continuous functions.



(1) Intermediate value theorem: (Ask students to state)

If f is continuous on $[a,b]$ and K is any number between f(a) and f(b), then there exists a $c \in (a,b)$ such that $f(c) = K$.

If f is not continuous, then this is no longer true:

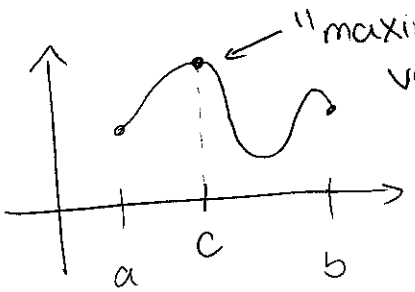


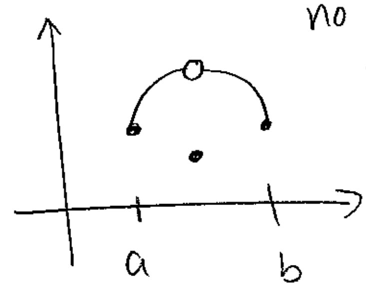
(2) Extreme value theorem: If f is continuous on a bounded and closed interval $[a,b]$, then f takes on both a maximum value and a minimum value on $[a,b]$. (Ask: what are the hypotheses?)

"takes on a maximum value" means:

there exists a $c \in [a,b]$ s.t. $f(x) \leq f(c)$

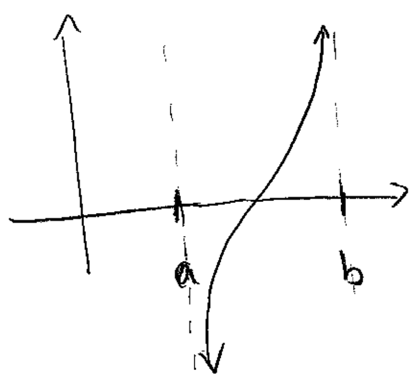
for all $x \in [a,b]$.



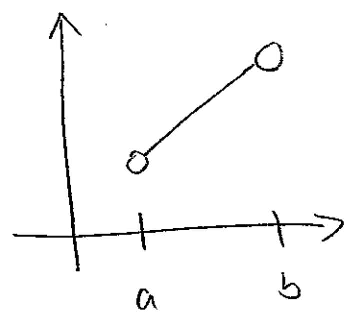


Need bounded in hypothesis. $f(x)=x$ has no maximum value on $[0,\infty)$.

Need closed in hypothesis. Consider



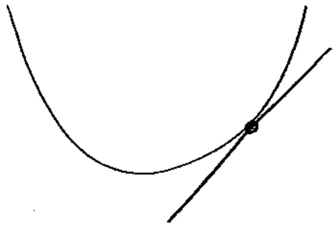
or



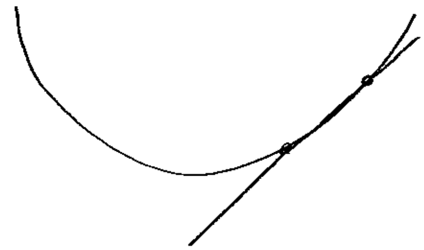
In both cases f is continuous on the bounded <u>open</u> interval (a,b), but it has no max or min.

Next topic: derivatives.

Start with a function f. We want to find the line tangent to f at the point $(x, f(x))$. (if it exists).

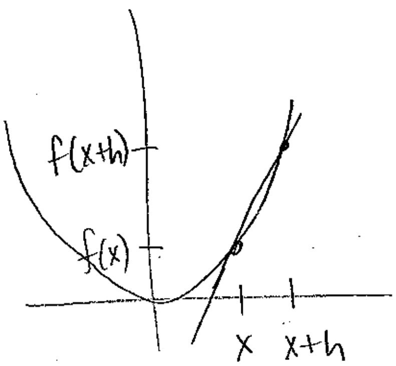


tangent line



secant line.

Idea: use secant line to approximate tangent line.



slope of secant line

$$= \frac{\text{rise}}{\text{run}} = \frac{f(x+h)-f(x)}{h}.$$

$$\text{slope of tangent line} = \underbrace{\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}}_{\text{by definition this is } f'(x)}.$$

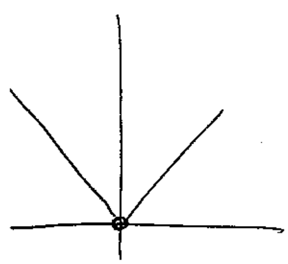
Example 1: $f(x)=x^2$. $x=1$.

$$f'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{(1+h)^2-1}{h}$$

$$= \lim_{h\to 0} \frac{1+2h+h^2-1}{h} = \lim_{h\to 0} \frac{2h+h^2}{h} = \lim_{h\to 0}(2+h) = 2.$$

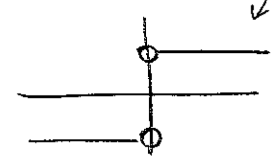
Q: What's an example of a function s.t. $f'(0)$ does not exist?

<u>Example 2</u>: $f(x) = |x|$

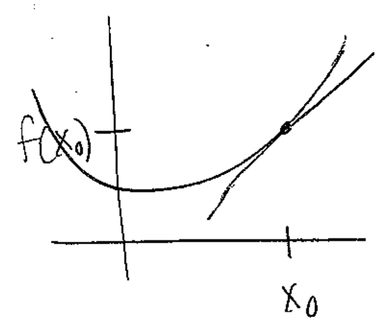


$$f'(0) = \lim_{h \to 0} \frac{|0+h| - 0}{h} = \lim_{h \to 0} \frac{|h|}{h}$$

limit does not exist:

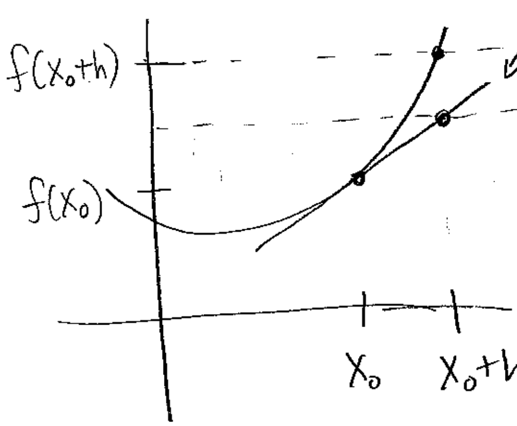


Why are we interested in tangent lines and derivatives?



The derivative $f'(x_0)$ tells you the best linear approximation to f near x_0.

slope $f'(x_0)$.



$f(x_0) + f'(x_0)h$.] error(h).

$$f(x_0+h) = \underbrace{f(x_0) + f'(x_0)h}_{\text{linear approximation}} + \underbrace{\text{error}(h)}_{\text{should be small}}$$

What do we mean by "error(h) is small"?

$$\underbrace{\frac{f(x_0+h)-f(x_0)}{h} - f'(x_0)} = \frac{\text{error}(h)}{h}.$$

$$\lim_{h\to 0}(\text{LHS}) = f'(x_0)-f'(x_0) = 0,$$

so $$\boxed{\lim_{h\to 0} \frac{\text{error}(h)}{h} = 0} \quad (*)$$

"goes to zero faster than a linear function"

Q: what's an example of a function with this property?

If we change the slope of our linear approximation

$$f(x_0+h) = f(x_0) + \underbrace{f'(x_0)}h + \text{error}(h)$$

i.e. if we change this number

the error term would not have property (*).

So this gives us another way to think about the derivative, and it lets us get a feeling of why some basic properties of derivatives are true.

<u>Basic properties</u>: Suppose $f'(x_0)$, $g'(x_0)$ both exist.

(1) Sum rule: $\underbrace{(f+g)'(x_0)} = ?$

This number should give the best linear approximation to $f+g$ near x_0.

$$f(x_0+h) = f(x_0) + f'(x_0)h + E_1(h)$$

$$g(x_0+h) = g(x_0) + g'(x_0)h + E_2(h)$$

$$\Longrightarrow (f+g)(x_0+h) = (f+g)(x_0) + \underbrace{[f'(x_0)+g'(x_0)]}\,h + \underbrace{[E_1(h)+E_2(h)]}_{\text{this is very small}}$$

This is the slope of the lin. approx!

So $\boxed{(f+g)'(x_0) = f'(x_0) + g'(x_0)}$

(2) scalar multiple rule: Let α be a real number.

$$f(x_0+h) = f(x_0) + f'(x_0)h + E(h).$$

$$\alpha f(x_0+h) = \alpha f(x_0) + \underbrace{\alpha f'(x_0)}_{\text{slope}}\,h + \alpha E(h).$$

So $\quad (\alpha f)'(x_0) = \alpha \cdot (f'(x_0))$

(3) Product rule: $(fg)'(x_0) = ?$

$$f(x_0+h) = f(x_0) + f'(x_0)h + E_1(h)$$
$$g(x_0+h) = g(x_0) + g'(x_0)h + E_2(h).$$

Multiply together:

	$f(x_0)$	$f'(x_0)h$	$E_1(h)$
$g(x_0)$	$f(x_0)g(x_0)$	$g(x_0)f'(x_0)h$	small
$g'(x_0)h$	$f(x_0)g'(x_0)h$	$f'(x_0)g'(x_0)h^2$	small
$E_2(h)$	small	small	small.

Note $\lim_{h\to 0} \frac{f'(x_0)g'(x_0)h^2}{h} = 0$, so that term is small too.

$$\Longrightarrow f(x_0+h)g(x_0+h) = f(x_0)g(x_0) + [f(x_0)g'(x_0) + g(x_0)f'(x_0)]h + \underbrace{E_3(h)}_{\text{small}}.$$

$$\Longrightarrow \boxed{(fg)'(x_0) = f(x_0)g'(x_0) + g(x_0)f'(x_0)}$$

Remember: The product rule is just the distributive property of multiplication!

4 Quotient rule: Suppose $g(x_0) \neq 0$.
What is $\left(\frac{f}{g}\right)'(x_0)$.

Idea: Division is the inverse of multiplication. Let's find a clever way to use the product rule.

$$f = \frac{f}{g} \cdot g$$

$$\Rightarrow \quad f'(x_0) = \left(\frac{f}{g} \cdot g\right)'(x_0)$$

To save space, I'm going to drop the "(x_0)" from the calculations below:

$$f' = \left(\frac{f}{g} g\right)'$$

$$= \left(\frac{f}{g}\right)' g + \frac{f}{g}(g') \qquad \text{(product rule)}$$

Now solve for $\left(\frac{f}{g}\right)'$:

$$\left(\frac{f}{g}\right)' g = f' - \frac{f}{g} g' = \frac{f'g - g'f}{g}$$

$$\Rightarrow \boxed{\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}}$$

<u>Chain rule</u>: $(f\circ g)'(x_0) = ?$

(Recall $(f\circ g)(x) = f(g(x))$.).

Goal: $(f\circ g)(x_0+h) = (f\circ g)(x_0) + [\,??\,]h + (\text{small})$.

$$
\begin{aligned}
(f\circ g)(x_0+h) &= f(g(x_0+h)) \\
&= f(g(x_0) + \underbrace{g'(x_0)h + \text{small}}_{\text{this is the new "}h\text{"}}) \\
&= f(g(x_0)) + f'(g(x_0))[g'(x_0)h + \text{small}] + \text{small}. \\
&= f(g(x_0)) + \underbrace{f'(g(x_0))\,g'(x_0)}\,h + \text{small}.
\end{aligned}
$$

$$\Longrightarrow \boxed{(f\circ g)'(x_0) = f'(g(x_0))\,g'(x_0)}$$

picture:

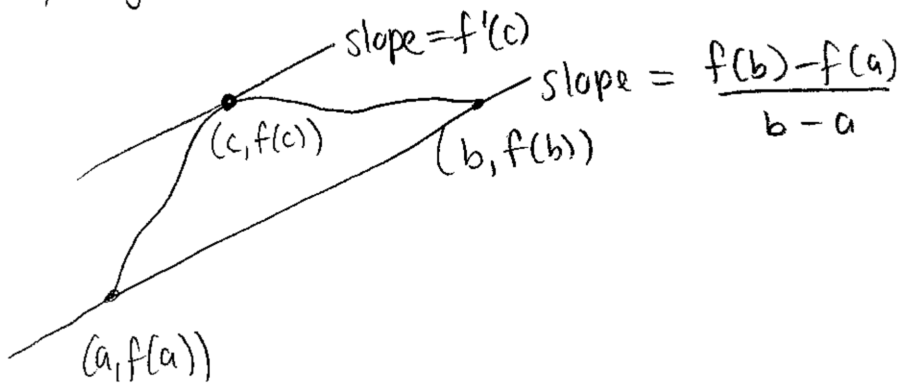
$$x_0 \longrightarrow \boxed{g} \longrightarrow g(x_0) \longrightarrow \boxed{f} \longrightarrow f(g(x_0))$$

$$x_0+h \longrightarrow \boxed{g} \longrightarrow \begin{matrix} g(x_0)+g'(x_0)h \\ (+\text{small}) \end{matrix} \longrightarrow \boxed{f} \longrightarrow \begin{matrix} f(g(x_0)) \\ + f'(g(x_0))g'(x_0)h \\ (+\text{small}) \end{matrix}$$

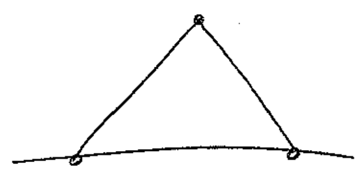
(* Lecture 4 starts here)

Mean value theorem: If f is differentiable on the open interval (a,b) and continuous on the closed interval $[a,b]$, then there is at least one number c in (a,b) for which $f'(c) = \dfrac{f(b)-f(a)}{b-a}$.

(It's like the IVT: If you try to "draw" a function f, you feel that this statement is "obvious")

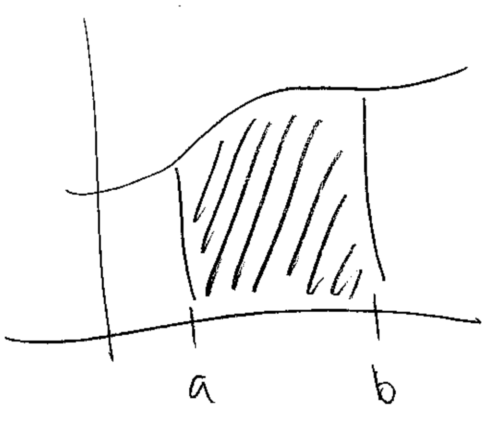


If f is not differentiable...



That's the end of the "Math 151 review" part.

Next topic. Integration:

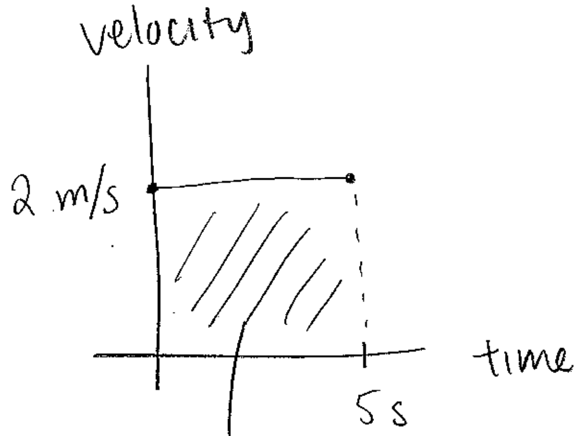


Some questions.

① How do you find the area under a curve?

② What does this have to do with derivatives?

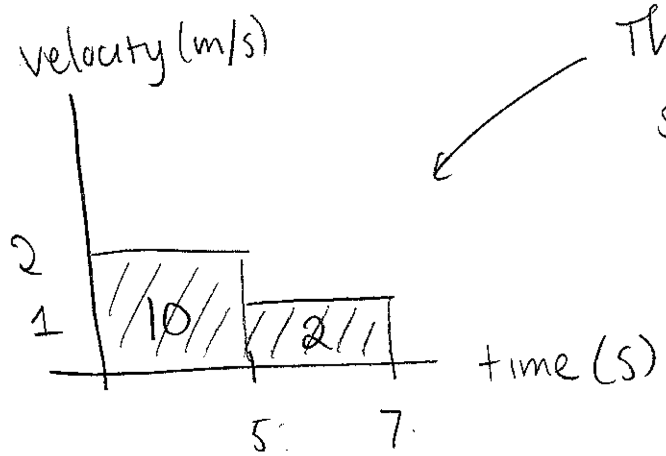
Recall: For an object moving at a constant speed, distance = speed × time.



If an object moves at speed 2 m/s for 5 s, then distance covered is (2 m/s)(5 s) = 10 m.

distance covered is area of this rectangle.

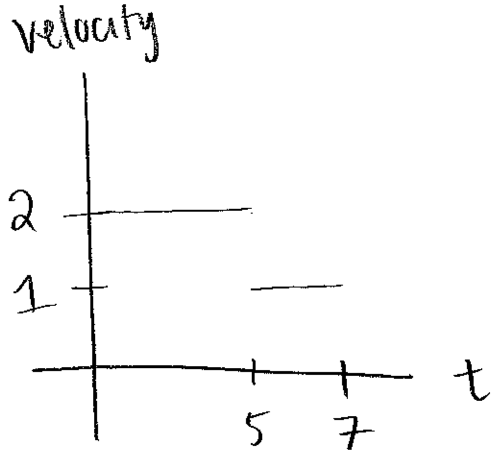
What if the velocity changes?

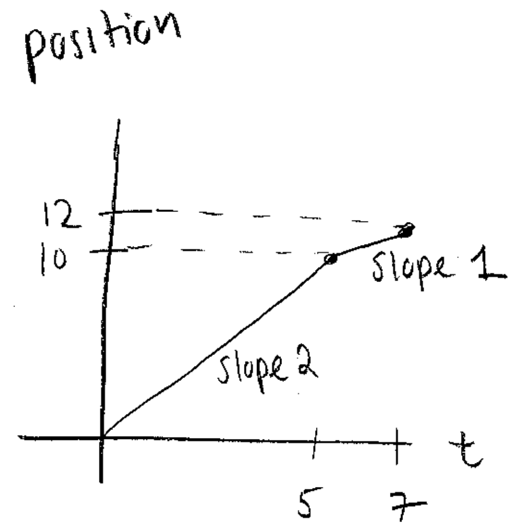


The distance covered is still the area under the graph.

Q: If the object starts at position 0 where is the object at time $t = 3$?

at time $t = 6$?



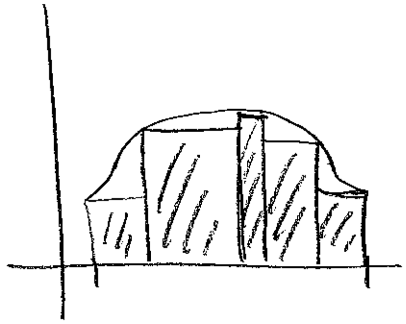


Velocity function $\xrightarrow{\text{area under graph}}$ position function

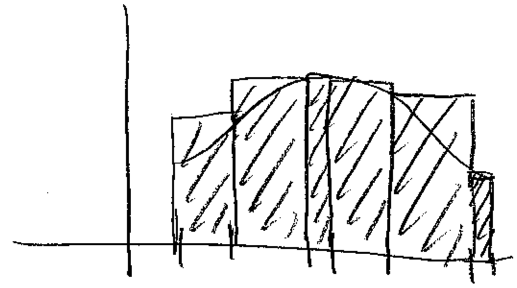
Velocity function $\xleftarrow[\text{slope of tangent line.}]{}$ position function

If our velocity function is more complicated, how do we find the area under the graph?

Idea: approximate the region with rectangles



underestimate
"lower Riemann sum"



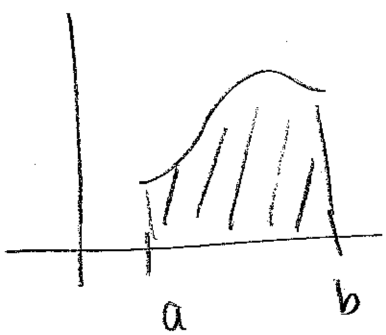
overestimate
"upper Riemann sum"

If we use thinner rectangles, we would expect the upper and lower estimates to both be close to the actual area.

Let's be precise about what are upper/lower Riemann sums.

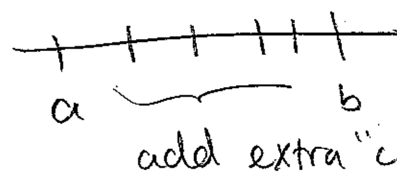
Start with: A continuous function $f(x)$ on $[a,b]$.

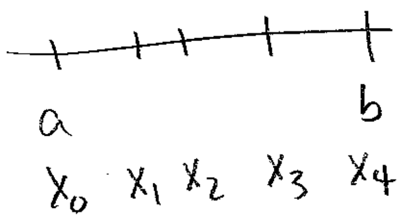
We want the area under f between a and b:



How do we form the rectangles?

First we partition the interval $[a,b]$:



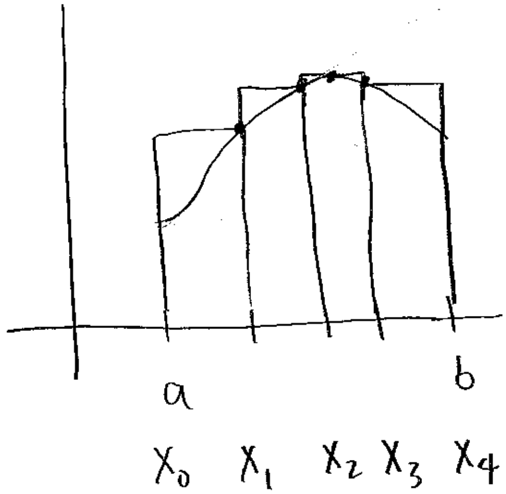


Let's give each of the points a name:

$$a = x_0 < x_1 < x_2 < x_3 < x_4 = b.$$

So we have split $[a,b]$ into 4 intervals:

$$[x_0, x_1],\ [x_1, x_2],\ [x_2, x_3],\ [x_3, x_4]$$



Let M_1 be max value of f on $[x_0, x_1]$

" " " M_2 " " " " "

" " " " " $[x_1, x_2]$

M_i " " " $[x_{i-1}, x_i]$.

Let $\Delta x_i = x_i - x_{i-1}$

So the i^{th} rectangle has $\left.\begin{array}{l}\text{base } \Delta x_i \\ \text{height } M_i\end{array}\right] \to$ area $M_i \Delta x_i$.

So the total area of the 4 rectangles is

$$M_1 \Delta x_1 + M_2 \Delta x_2 + M_3 \Delta x_3 + M_4 \Delta x_4.$$

This number is called the upper Riemann sum of f with respect to the partition.

More notation $P = \{x_0, x_1, x_2, x_3, x_4\}$ $(x_0 = a,\ x_4 = b)$

We write $U_f(P)$ = the number above.

In general, a partition of $[a,b]$ is just something of the form $\{x_0, x_1, \ldots, x_n\}$ with $a = x_0 < x_1 < \cdots < x_n = b$.

So if $P = \{x_0, x_1, \ldots, x_n\}$, we define $U_f(P)$ in the same way.

We also define the lower Riemann sum $L_f(P)$ similarly, except we consider

$$m_1 \Delta x_1 + \cdots + m_n \Delta x_n$$

where m_i = min value of f on $[x_{i-1}, x_i]$.

<u>Fact</u>: If f is continuous on $[a,b]$, then there is a unique number I which satisfies:

1. $I \leq U_f(P)$ for all partitions P of $[a,b]$.
2. $I \geq L_f(P)$ for all partitions P of $[a,b]$

This number I is called the <u>integral</u> of f from a to b and it is denoted by $\int_a^b f(x)\,dx$.

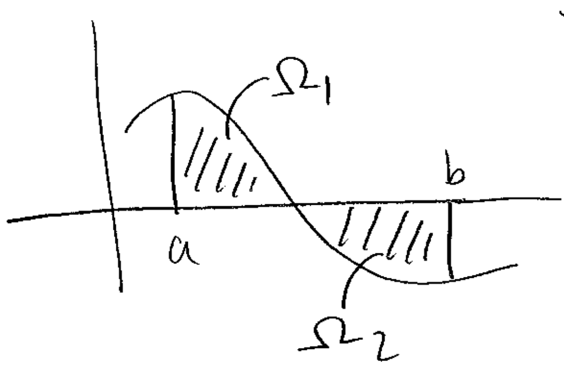
Why do we go through all this trouble?

- Need a mathematical way to define "area under a curve"
- "Area" isn't always as nice as it seems. (This is what "measure theory" studies.)

The upper/lower Riemann sums are useful for defining the integral $\int_a^b f(x)\,dx$, but doing any calculations with them is very hard. So now we'll try to find an easier way.

First note:

$$\int_a^b f(x)\,dx = (\text{area of } \Omega_1) - (\text{area of } \Omega_2).$$



Also, if $a > b$, we define $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

and $\int_c^c f(x)\,dx = 0$.

* Lecture 5

Suppose f is a continuous function on $[a,b]$.

We can define a new function on $[a,b]$

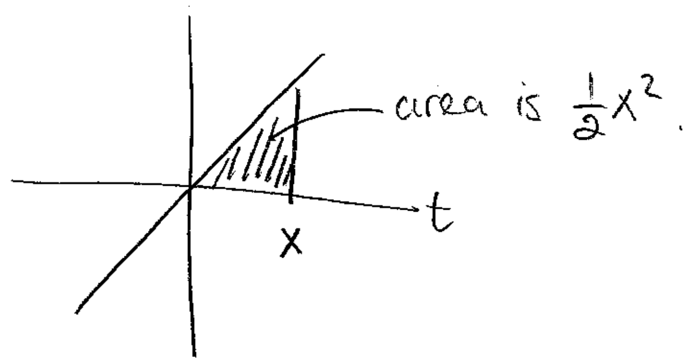
as follows:

① Fix a number $c \in [a,b]$.

② Define $F(x) = \int_c^x f(t)\,dt$

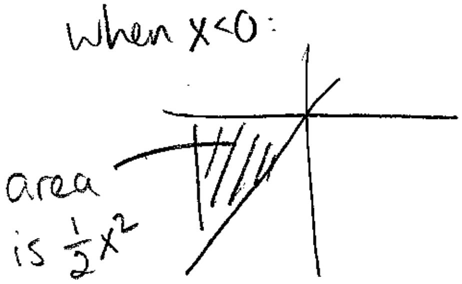
Example. $f(t) = t$. $c = 0$.

$$F(x) = \int_0^x t\,dt.$$



when $x > 0$: $F(x) = \frac{1}{2}x^2$.

when $x < 0$:



$$F(x) = \int_0^x t\,dt = -\int_x^0 t\,dt.$$

$$= -\left(-\frac{1}{2}x^2\right).$$

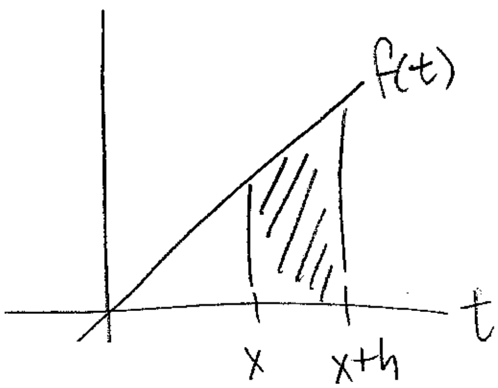
$$= \frac{1}{2}x^2.$$

but it's below the x-axis.

So $\quad F(x) = \int_0^x t\,dt = \frac{1}{2}x^2$ for all x.

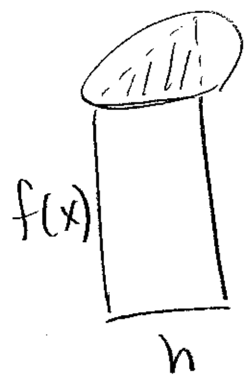
Note: $\quad F'(x) = f(x)$. Is this is a coincidence?

$$\frac{F(x+h) - F(x)}{h} = \frac{1}{h}\left[\int_0^{x+h} f(t)\,dt - \int_0^x f(t)\,dt\right].$$



$$= \frac{1}{h}\int_x^{x+h} f(t)\,dt.$$

Let's just approximate this area by a rectangle:



$$\int_x^{x+h} f(t)\,dt \approx f(x)h.$$

There's a small error, but since f is continuous, we don't have to worry about it.

So: $\quad \frac{F(x+h)-F(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)\,dt \approx \frac{1}{h}\left[f(x)h\right] = f(x).$

So $\quad \boxed{F'(x) = f(x).}$

Fundamental Theorem of Calculus, Part 1

Let f be continuous on $[a,b]$ and let c be any number in $[a,b]$. Define

$$F(x)=\int_c^x f(t)\,dt$$

Then F is continuous on $[a,b]$, differentiable on (a,b), and

$$F'(x)=f(x) \quad \text{for all } x\in(a,b).$$

Or phrased another way: $\frac{d}{dx}\int_c^x f(t)\,dt = f(x)$

(If you integrate then differentiate, you get the original function back.)

Why is this true?

Intuitively:

$$\text{velocity} \xrightarrow[\text{curve}]{\text{area under}} \text{position} \xrightarrow{\text{differentiate}} \text{velocity}$$

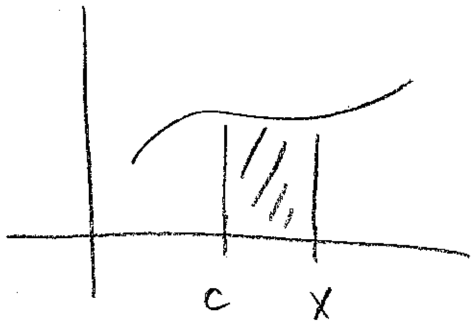
Another way to think about it:

Let $F(x)=\int_c^x f(t)\,dt$.

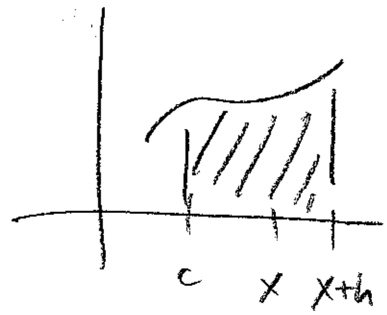
Then $$F'(x) = \lim_{h \to 0} \underbrace{\frac{F(x+h) - F(x)}{h}}$$

Question: what's a geometric interpretation of this quantity? Or of just $F(x+h) - F(x)$?

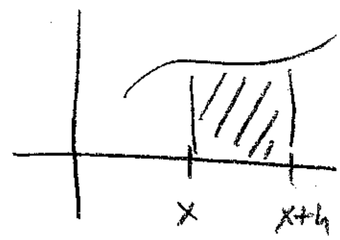
$\int_c^x f(t)\,dt = F(x)$



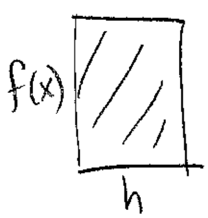
$\int_c^{x+h} f(t)\,dt = F(x+h)$



$\Longrightarrow \quad F(x+h) - F(x) = \int_x^{x+h} f(t)\,dt$:



What's a very simple estimate for this area?
Let's just approximate it with a rectangle:



$$\int_x^{x+h} f(t)\,dt \approx f(x)h$$

So $\quad F(x+h) - F(x) \approx f(x)h$

But this means $\quad F'(x) = f(x)$. That's all!
(Note: I think this is really important!)

The FTC, part 1 tells you how to differentiate an integral.

The FTC part 2 (below) tells you how to integrate a derivative.

FTC part 2: Let f be continuous on $[a,b]$.
If G is any antiderivative for f on $[a,b]$,
then $\int_a^b f(t)\,dt = G(b)-G(a)$

The following statements all mean the same thing:

1. G is an antideriv for f
2. $G'(x)=f(x)$
3. $\int f(x)\,dx = G(x)+C$

Or put another way:

$$\underbrace{\int_a^b G'(t)\,dt}_{\text{integral of a derivative.}} = G(b)-G(a)$$

Why is this true? We'll use FTC part 1.

Let $F(x)=\int_a^x G'(t)\,dt$. Then $F'(x)=G'(x)$ for all $x\in(a,b)$

But (by MVT) this means $F(x)=G(x)+C$.

Let $x=a$: $\underbrace{F(a)}_{\text{this is zero.}}=G(a)+C$

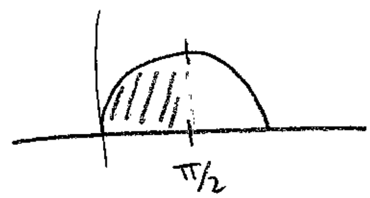
so $C = -G(a)$.

so $F(x) = G(x) - G(a)$.

let $x = b$: $\quad F(b) = G(b) - G(a)$

$\Rightarrow \int_a^b G'(t)\,dt = G(b) - G(a)$. Done!

Example: $\int_0^{\pi/2} \sin x\,dx = ?$



let $G(x) = -\cos x$. Then $G'(x) = \sin x$

so $\int_0^{\pi/2} \sin x\,dx = \left[-\cos x\right]_0^{\pi/2} = \left[-\cos\left(\frac{\pi}{2}\right)\right] - \left[-\cos 0\right]$

$= -0 + 1 = 1$.

(Recall $\left[G(x)\right]_a^b$ is just notation for $G(b) - G(a)$.)

Basic properties of the definite integral

① $\int_a^b \alpha f(x)\,dx = \alpha \int_a^b f(x)\,dx$

② $\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

These are true because the derivative satisfies the same kinds of properties

① $\frac{d}{dx}[\alpha f(x)] = \alpha \frac{d}{dx}[f(x)]$

② $\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$

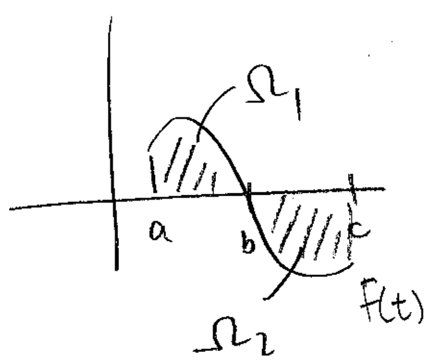
However: $$\int_a^b f(x)g(x)dx \neq \int_a^b f(x)dx \int_a^b g(x)dx.$$

This is because $(fg)' \neq f'g'$.

The rule that corresponds to the product rule for derivatives is called "integration by parts." We'll talk more about that later.

Back to finding area:

Example:



Q: what is (area of Ω_1) + (area of Ω_2)?

(or: what is the area of the shaded region?)

$$(\text{area of } \Omega_1) = \int_a^b f(t)\,dt.$$

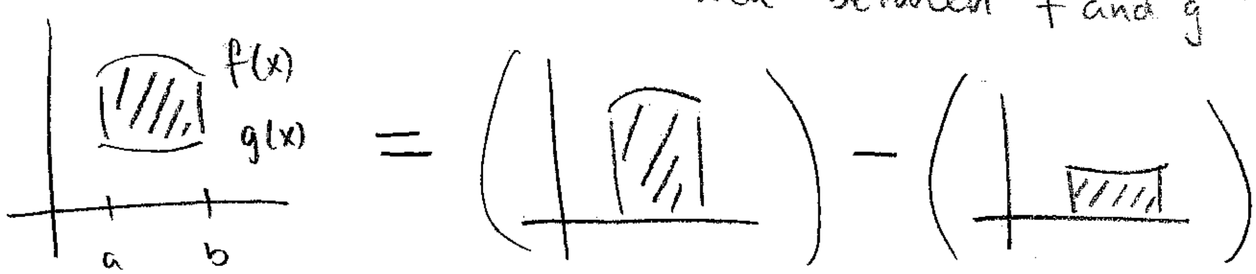
$$(\text{area of } \Omega_2) = -\int_b^c f(t)\,dt.$$

So $$(\text{total area}) = \int_a^b f(t)\,dt - \int_b^c f(t)\,dt = \int_a^c |f(t)|\,dt.$$

useful conceptually, but in practice, it's hard to compute antiderivatives of the form $|f(t)|$ directly.

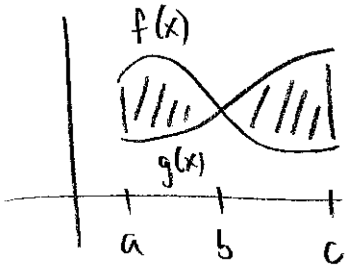
Example 2:

what is the area between f and g?



It's just $\int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b [f(x) - g(x)]\,dx.$

Example 3:



area of shaded region

$$= \int_a^b [f(x)-g(x)]\,dx + \int_b^c [g(x)-f(x)]\,dx$$

$$= \int_a^c |f(x)-g(x)|\,dx.$$

Key takeaway: When you have to find the area between two curves $y=f(x)$ and $y=g(x)$ between a and b, you have to split into intervals where $f>g$ and intervals where $f<g$.

(Lecture 6)

Next: indefinite integrals.

In general, finding antiderivatives can be very hard... ☹.

Let's start with some basic integrals that come from working backwards from derivative formulas.

$$\frac{d}{dx}[x^n] = nx^{n-1} \longrightarrow \int x^n dx = \frac{x^{n+1}}{n+1} + C$$

for all numbers n for all $\boxed{n \neq -1.}$

$$\frac{d}{dx}[\sin x] = \cos x \qquad \int \sin x\, dx = -\cos x + C$$

$$\frac{d}{dx}[\cos x] = -\sin x \qquad \int \cos x\, dx = \sin x + C$$

Example 1: An object moves on the x-axis with velocity $v(t) = 2 - 3t + t^2$. Its initial position is at $x = 2$. Find the position of the object at $t = 4$.

Let $x(t)$ = position of object at time t.

Solution 1: $x'(t) = v(t)$, so

$$x(t) = \int v(t)\,dt = \int (2 - 3t + t^2)\,dt = 2t - \frac{3}{2}t^2 + \frac{1}{3}t^3 + C$$

since $x(0) = 2$, we have $C = 2$

so $x(t) = 2t - \frac{3}{2}t^2 + \frac{1}{3}t^3 + 2$

$\Rightarrow$ $x(4) = 7\frac{1}{3}$.

Solution 2:

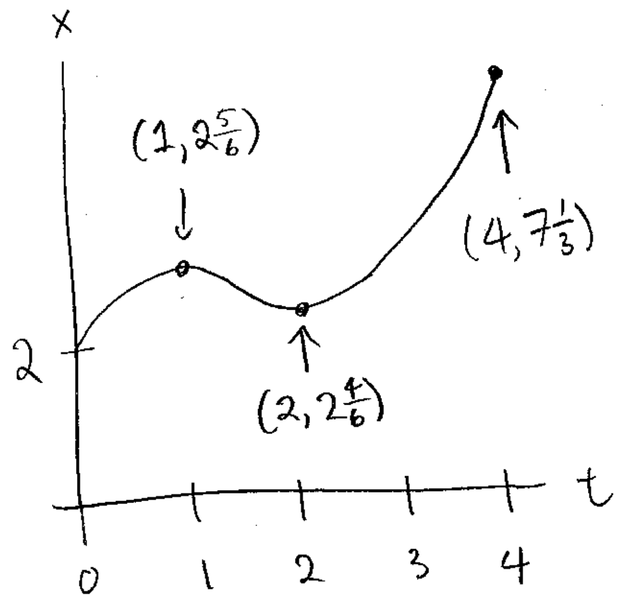
displacement $= x(4) - x(0)$

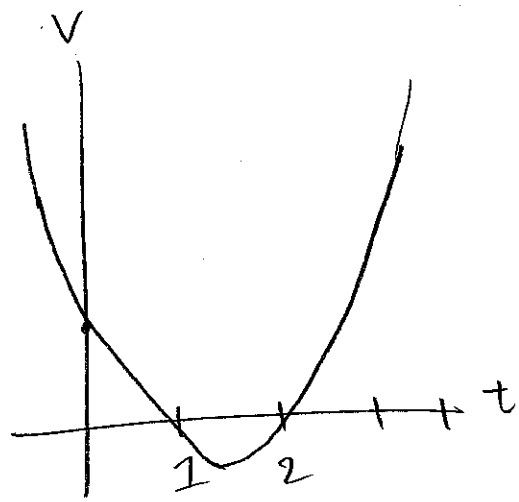
$$= \int_0^4 v(t)\,dt = \left[2t - \frac{3}{2}t^2 + \frac{1}{3}t^3\right]_0^4 = 7\frac{1}{3}.$$

Here's a graph of

$x(t) = 2t - \frac{3}{2}t^2 + \frac{1}{3}t^3 + 2$

$v(t) = 2 - 3t + t^2$





Note: distance traveled from time 0 to 4

$$= \left(2\tfrac{5}{6} - 2\right) + \left(2\tfrac{5}{6} - 2\tfrac{4}{6}\right) + \left(7\tfrac{1}{3} - 2\tfrac{4}{6}\right).$$

$$= \int_0^1 v(t)\,dt - \int_1^2 v(t)\,dt + \int_2^4 v(t)\,dt$$

$$\left(= \int_0^4 |v(t)|\,dt. \right)$$

This is one application of finding area between a curve and the x-axis.

Example 2: Find the equation of motion for an object that moves along a straight line with constant acceleration a from an initial position x_0 with initial velocity v_0.

So, we're given:

$$\begin{cases} x''(t) = a \\ x(0) = x_0 \\ x'(0) = v_0 . \end{cases}$$

$$x'(t) = \int x''(t)\,dt = \int a\,dt = at + C.$$

Let $t=0$: $\quad v_0 = x'(0) = a\cdot(0) + C = C$

$$\Rightarrow \boxed{x'(t) = at + v_0}$$

$$x(t) = \int x'(t)\,dt = \int (at + v_0)\,dt$$

$$= \tfrac{1}{2}at^2 + v_0 t + C$$

different C.

Let $t=0$: $\quad x_0 = x(0) = \tfrac{1}{2}a0^2 + v_0 0 + C = C$

$$\Rightarrow \boxed{x(t) = \tfrac{1}{2}at^2 + v_0 t + x_0}$$

Next topic: u-substitution.

Example 1: $\int (x^2-1)^4 x \, dx$?

We know $\frac{d}{dx}\left[(x^2-1)^5\right] = 5(x^2-1)^4(2x)$.

$= 10(x^2-1)^4 x$.

so $\int (x^2-1)^4 x \, dx = \frac{1}{10}(x^2-1)^5 + C$.

Observe: $\int (x^2-1)^4 x \, dx$ If this x were not here, this would not have worked.

Another way to see this calculation:

Let $f(t) = t^4$ $\quad F(t) = \frac{1}{5}t^5$.

$u(x) = x^2-1 \Rightarrow (u'(x) = 2x)$.

Then $(x^2-1)^4 x = \frac{1}{2} f(u(x)) \cdot u'(x)$.

$= \frac{1}{2} F'(u(x))\, u'(x)$.

$= \frac{1}{2} \frac{d}{dx}\left[F(u(x))\right]$

chain rule

so $$\int (x^2-1)^4 x\,dx = \frac{1}{2}\int \frac{d}{dx}\left[F(u(x))\right] dx$$

$$= \frac{1}{2} F(u(x)) + C$$

$$= \frac{1}{10}(x^2+1)^5$$

usually, when doing calculations like this, we write:

$$\int (x^2-1)^4 x\,dx.$$

let $u(x) = x^2 - 1$

$du = 2x\,dx$ $\longrightarrow \frac{du}{dx}$.

$$= \int u^4 \frac{du}{2}$$

$$= \frac{1}{2}\int u^4\,du = \frac{1}{10}u^5 + C = \frac{1}{10}(x^2-1)^5 + C.$$

<u>Example 2</u>: $\int x^2\sqrt{4+x^3}\,dx = ?$

let $u = 4+x^3$

$du = 3x^2\,dx.$

so $$\int x^2\sqrt{4+x^3}\,dx = \frac{1}{3}\int \sqrt{u}\,du = \frac{1}{3}\int u^{1/2}\,du.$$

$$= \frac{1}{3}\,\frac{u^{3/2}}{3/2} + C = \frac{2}{9}\cdot(4+x^3)^{3/2} + C$$

Example 3 : $\int \sec^3 x \tan x \, dx$.

Recall $\frac{d}{dx}(\sec x) = \sec x \tan x$.

so let $u(x) = \sec x$

$du = \sec x \tan x \, dx$.

$$\int \sec^3 x \tan x \, dx = \int u^2 \, du = \frac{1}{3} u^3 + C$$

$$= \frac{1}{3} \sec^3 x + C.$$

---

u-substitutions for definite integrals:

$$\int_a^b f(u(x)) u'(x) \, dx = \int_{u(a)}^{u(b)} f(u) \, du$$

you plug in a, b for x

you plug in u(a), u(b) for u.

Example: $\int_0^2 (x^2-1)(x^3-3x+2)^3 dx$

$$\begin{cases} u = x^3 - 3x + 2 \\ du = (3x^2 - 3) \, dx \\ u(2) = 4 \\ u(0) = 2 \end{cases}$$

$$= \int_2^4 u^3 \cdot \frac{1}{3} \, du$$

$$= \frac{1}{12} \left[u^4 \right]_2^4 = 20.$$

(Lecture 7)

Additional properties of the definite integral.

Let f and g be continuous functions.

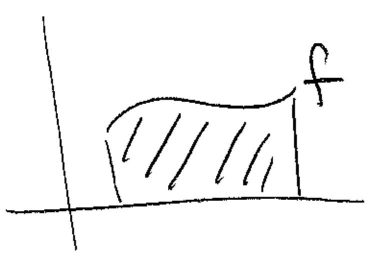
$\boxed{\#1}$ If $f(x) \geq 0$ for all $x \in [a,b]$ then $\int_a^b f(x)\,dx \geq 0$

$\boxed{\#2}$ If $f(x) > 0$ for all $x \in [a,b]$ then $\int_a^b f(x)\,dx > 0$

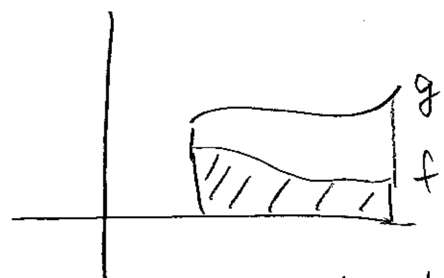
$\boxed{\#3}$ If $f(x) \leq g(x)$ for all $x \in [a,b]$ then $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$

$\boxed{\#4}$ If $f(x) < g(x)$ for all $x \in [a,b]$ then $\int_a^b f(x)\,dx < \int_a^b g(x)\,dx$.

Draw pictures!



#1 and #2



#3 and #4.

Question: what about the converses?

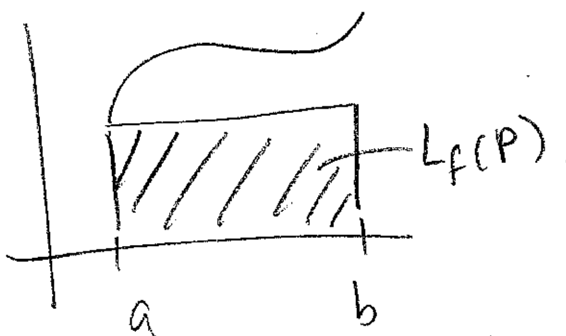
$\boxed{\#5}$ $$\left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx.$$

(ask: which inequalities are true? $\leq, \geq$?)

$\boxed{\#6}$ Let m = min value of f on $[a,b]$

M = max - - - - -

Then $$\underbrace{m(b-a)}_{L_f(P) \text{ where } P=\{a,b\}} \le \int_a^b f(x)\,dx \le M(b-a).$$



$\boxed{\#7}$ $$\frac{d}{dx}\int_a^{u(x)} f(t)\,dt = f(u(x))\,u'(x).$$

This is the chain rule.

Let $F(x) = \int_a^x f(t)\,dt.$ $\Longrightarrow$ $F'(x) = f(x).$

Then $$\frac{d}{dx}F(u(x)) = \frac{d}{dx}\int_a^{u(x)} f(t)\,dt$$

$$= F'(u(x))\,u'(x)$$

$$= f(u(x))\,u'(x)$$

Example: $\dfrac{d}{dx}\displaystyle\int_0^{x^3} \frac{1}{1+t}\,dt$

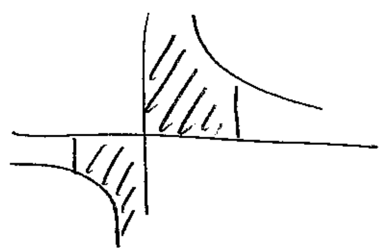
$$= \left(\frac{1}{1+x^3}\right)\frac{d}{dx}(x^3) = \frac{3x^2}{1+x^3}$$

$\boxed{\#8}$ If f is odd on $[-a,a]$ then $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$.

If f is even on $[-a,a]$ then $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$.

RECALL: we're always assuming (in #1 — #8) that the functions f, g, etc. are continuous.

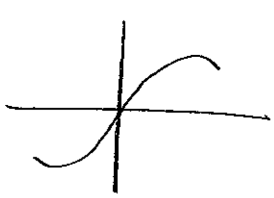
$$\int_{-1}^{1} \frac{1}{x}\,dx \neq 0.$$



even though $\frac{1}{x}$ is odd.

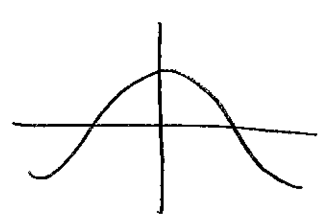
Recall:

odd means: $f(x) = -f(-x)$.



180° rotation Symmetry.

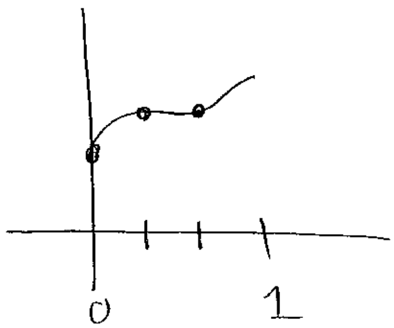
even means $f(x) = f(-x)$



reflection about y-axis symmetry.

Average of a function on an interval.

Q: What's the average value f on [0, 1].
(Assume f is continuous.)



Why don't we just "sample" f at n different, well-spaced points?

$$(\text{average of } f) \overset{?}{\approx} \frac{f(\frac{0}{n})+f(\frac{1}{n})+\dots f(\frac{n-1}{n})}{n}.$$

This is just a left Riemann sum!

$$\lim_{n\to\infty} \frac{1}{n}\left(f(\tfrac{0}{n})+f(\tfrac{1}{n})+\dots+f(\tfrac{n-1}{n})\right) = \int_0^1 f(x)\,dx.$$

so $$(\text{average of } f \text{ on } [0,1]) = \int_0^1 f(x)\,dx$$

This is a definition.

In general,

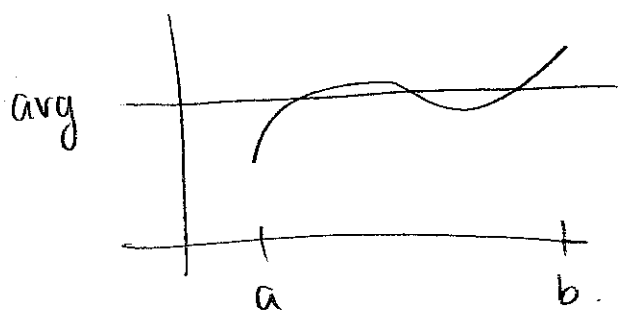
$$(\text{average of } f \text{ on } [a,b]) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Mean-value theorem of integrals

If f is continuous on $[a,b]$, then there is a $c \in (a,b)$ such that:

$$f(c) = (\text{avg of } f \text{ on } [a,b]) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

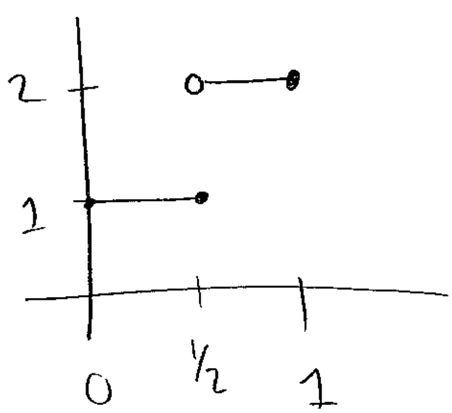
Why is this true?



min $\leq$ avg $\leq$ max

so since f is continuous, we can use IVT.

If f is not continuous, then it's no longer true.



Here the average value is $3/2$.

Remark: weighted averages

$\rightarrow$ It's also possible to talk about weighted averages of f on $[a,b]$.

Some applications of the integral:

f continuous on $[a, b]$.

$P = \{x_0, x_1, \ldots, x_n\}$ a partition of $[a,b]$

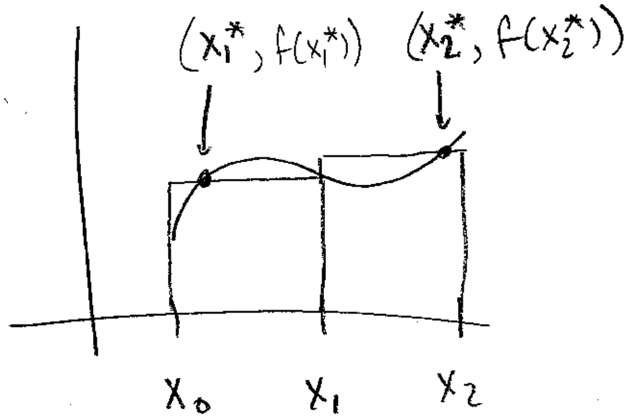
$(a = x_0 < x_1 < \cdots < x_n = b)$.

Recall:

$$\int_a^b f(x)\,dx = \lim_{\|P\| \to 0} \left[f(x_1^*)\Delta x_1 + \cdots + f(x_n^*)\Delta x_n \right]$$

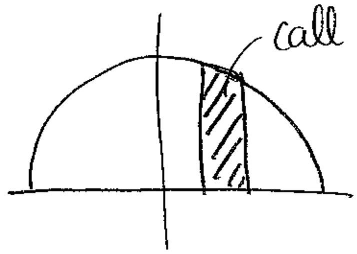
("$\|P\| \to 0$" means as the partition gets finer)

$\Delta x_i = x_i - x_{i-1}$, $x_i^* \in [x_{i-1}, x_i]$.

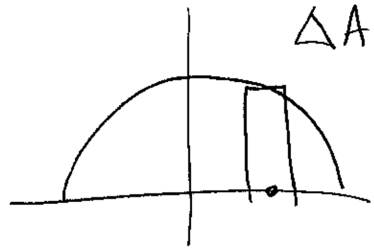


Idea: Riemann sums can be used to approximate area under curves. But they can be used to approximate other things, which means we can use definite integrals to evaluate other things.

Let's take a closer look at area.



① we cut our region into many thin slices.



$\Delta A_i \approx f(x_i^*)\Delta x_i$

② For each slice, the area is approximately $f(x_i^*)\Delta x_i$.

$$\Delta A_i \approx f(x_i^*)\Delta x_i \longrightarrow A = \int_a^b f(x)\,dx$$

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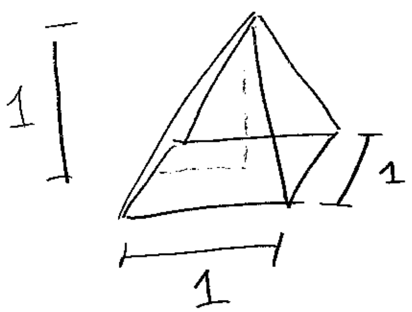
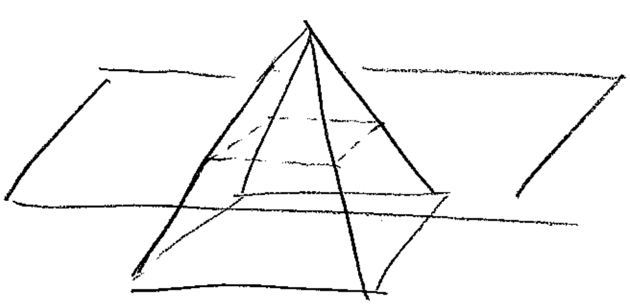
*Lecture 8: (10/29/19)

Another example of Riemann sum:

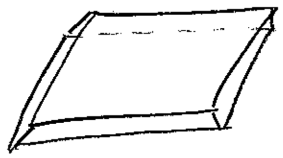


consider a pyramid with square base (side length of square $=1$) and with height 1

Q: What is the volume of the pyramid?



① cut our region into many thin horizontal slices.

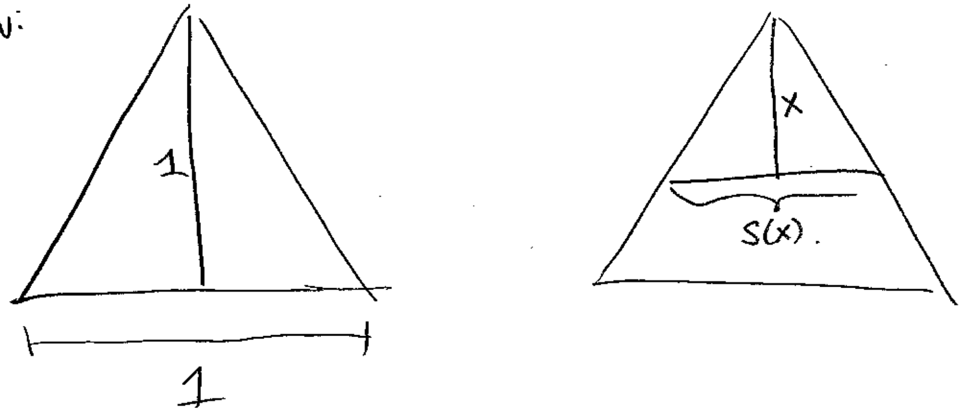


thin slab.

thickness: Δx_i

side length?

2D view:



By geometry, $s(x) = x$.

So the volume of the thin slab is

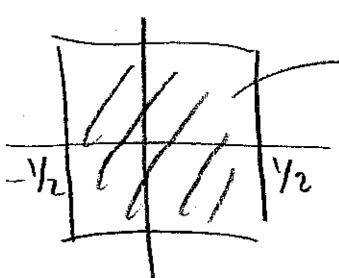
$$\approx \underbrace{s(x)^2}_{\text{base}} \underbrace{\Delta x_i}_{\text{height}}.$$

$$\Delta V_i \approx s(x)^2 \Delta x_i \longrightarrow V = \int_0^1 x^2 \, dx = \frac{1}{3}$$

<u>Example</u>: Area of unit square

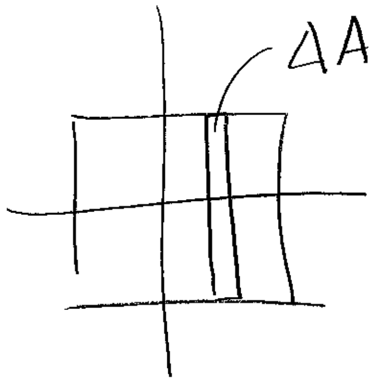
How can we slice a square?

① use vertical slices.



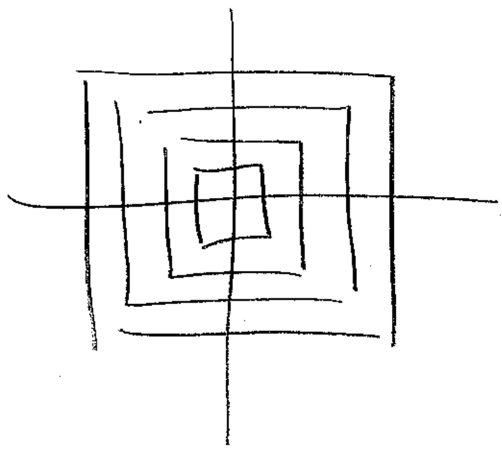
$|x| \le \frac{1}{2}$

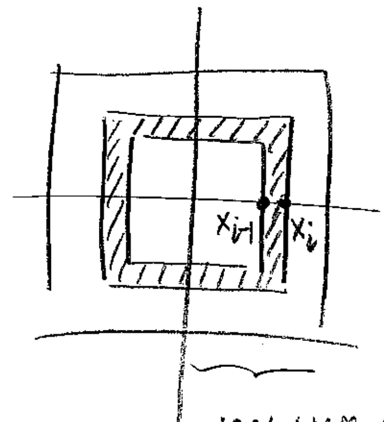
$|y| \le \frac{1}{2}$



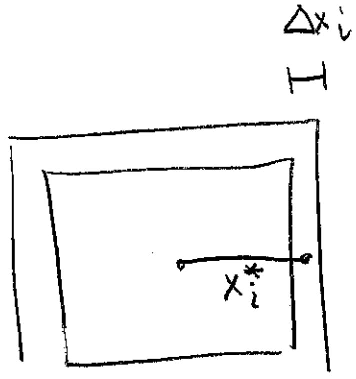
$$\Delta A_i = \underbrace{1}_{\text{height}} \cdot \overbrace{\Delta x_i}^{\text{base}} \longrightarrow A = \int_{-1/2}^{1/2} dx = 1.$$

(2). use thin hollow square "shells"





now we are partitioning $[0, \frac{1}{2}]$.



$$\Delta A_i \approx \underbrace{8 x_i^*}_{\text{perimeter}} \overbrace{\Delta x_i}^{\text{thickness}}$$

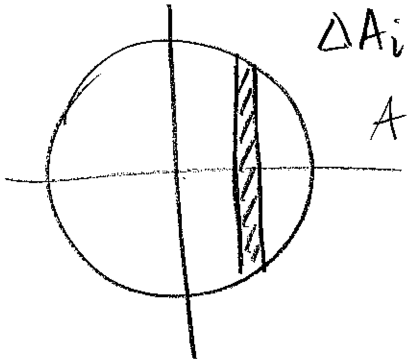
$$\rightarrow A = \int_0^{1/2} 8x \, dx = [4x^2]_0^{1/2} = 1.$$

Make sure you understand these examples!

1. Why is first method $\int_{-1/2}^{1/2}$ and second $\int_0^{1/2}$?
2. Where do the expressions for ΔA_i come from?

Now let's try the same thing for circles!
consider a unit circle ($x^2+y^2=1$)

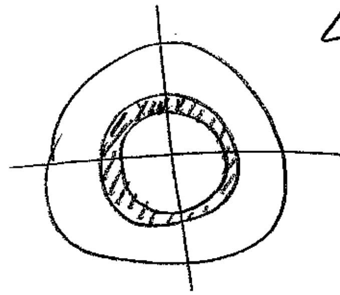
① vertical slices



$\Delta A_i = ?$

$A = \int ?$

② circular shells

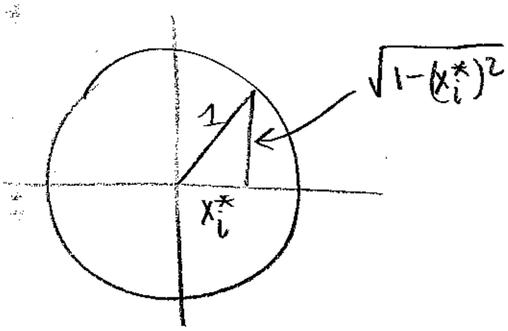


$\Delta A_i = ?$

$A = \int ?$

(Ask the class to work these out).

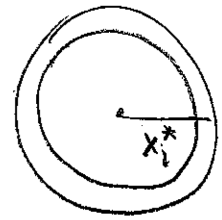
①



$$\Delta A_i \approx 2\sqrt{1-(x_i^*)^2}\,\Delta x_i$$

$$A = \int_{-1}^{1} 2\sqrt{1-x^2}\,dx$$

②

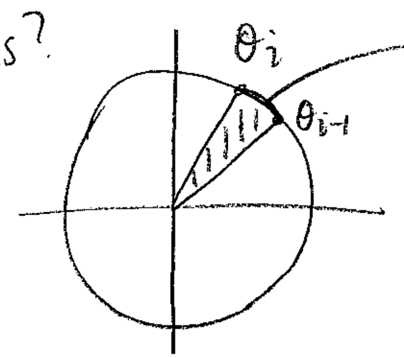


$$\Delta A_i \approx 2\pi x_i^* \,\Delta x_i$$

$$A = \int_0^1 2\pi x\,dx.$$

---

③ bonus?

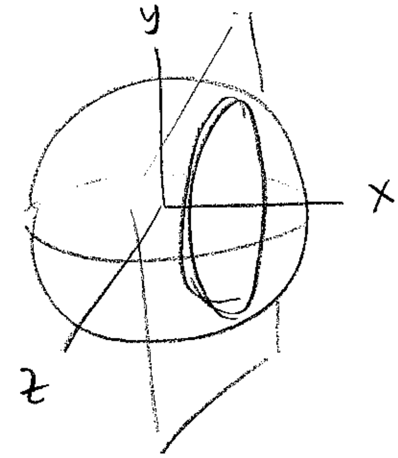


$$\Delta A_i \approx \underbrace{\frac{1}{2}\underbrace{1}_{\text{height}}\cdot\underbrace{\Delta\theta_i}_{\text{base}}}_{\text{area of triangle.}}$$

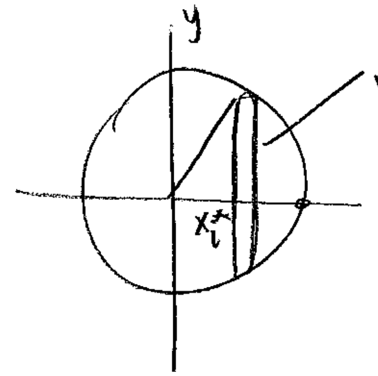
$$A = \int_0^{2\pi} \frac{1}{2}\,d\theta$$

Next example: sphere.

1. vertical slices.

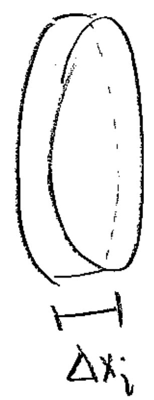


2D view:



$\sqrt{1-(x_i^*)^2}$ = radius of thin disc.

volume of thin disc?



$$\Delta V_i \approx \underbrace{\pi\left(\sqrt{1-(x_i^*)^2}\right)^2}_{\text{area of circular base}} \cdot \underbrace{\Delta x_i}_{\text{height}}$$

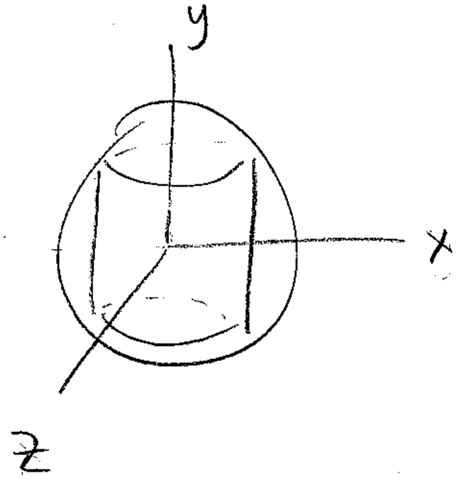
$$= \pi\left(1-(x_i^*)^2\right)\Delta x_i .$$

$$\longrightarrow \quad V = \int_{-1}^{1} \pi\left(1-x^2\right)dx$$

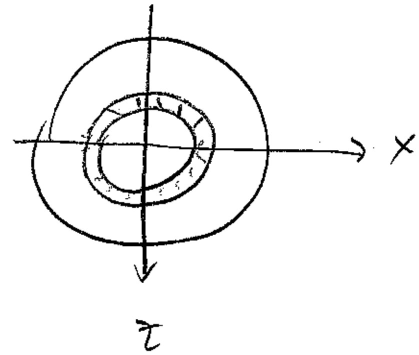
$$= \pi\left[x-\frac{x^3}{3}\right]_{-1}^{1}$$

$$= \frac{4}{3}\pi.$$

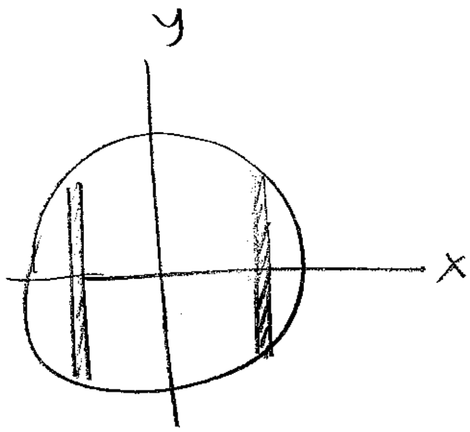
② cylindrical shells.



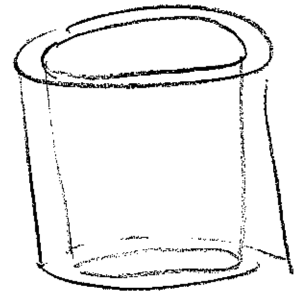
2D view from top:



The xy-plane: $(z=0)$.

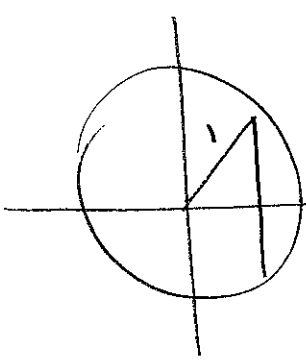


The shell:

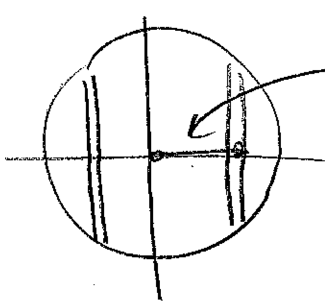


$$\Delta V_i \approx \underbrace{2\pi x_i^*}_{\text{perimeter of circle}} \cdot \underbrace{2\sqrt{1-(x_i^*)^2}}_{\text{height of cylinder}} \cdot \overbrace{\Delta x_i}^{\text{thickness}}$$

"unroll into a thin rectangular prism"



height of cylinder.
$= 2\sqrt{1-(x_i^*)^2}$



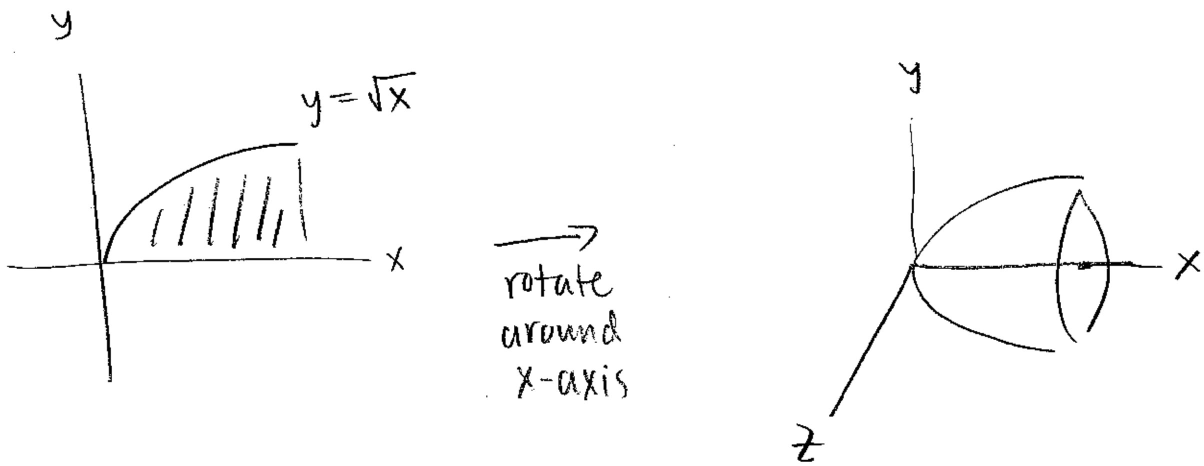
radius of cylinder
$= x_i^*$

$$\Delta V_i \approx 2\pi x_i^* \cdot 2\sqrt{1-(x_i^*)^2} \cdot \Delta x_i$$

$$V = \int_0^1 2\pi x \cdot 2\sqrt{1-x^2}\, dx = \frac{4}{3}\pi$$

u-substitution.

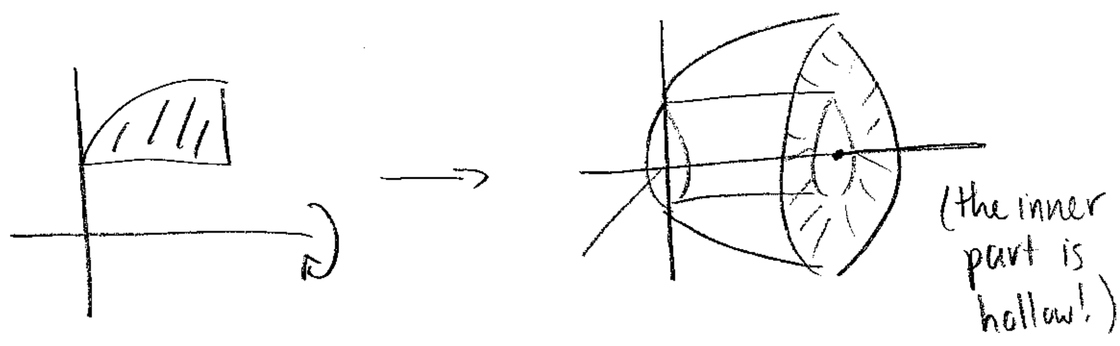
The 2 techniques we used to find the volume of the sphere work for a class of shapes (in 3D) called solids of revolution:



method 1: thin disks $\longrightarrow$ disk/washer method

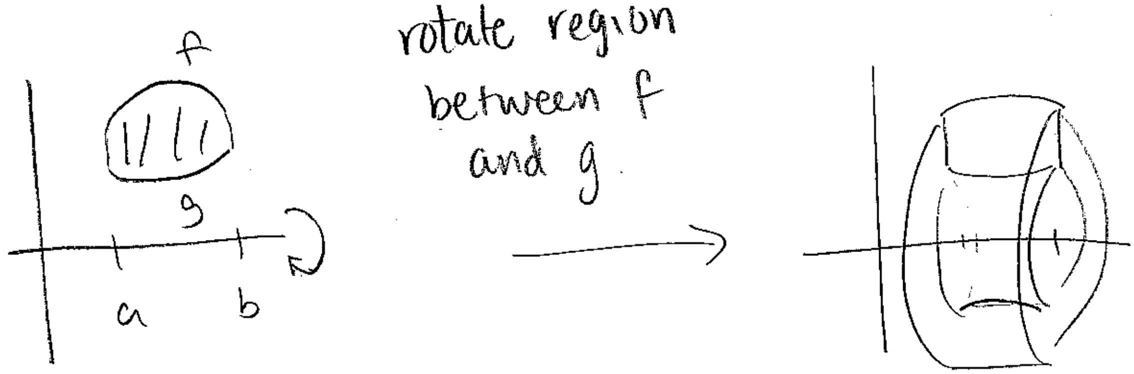
method 2: thin cylindrical shells $\longrightarrow$ shell method.

another picture

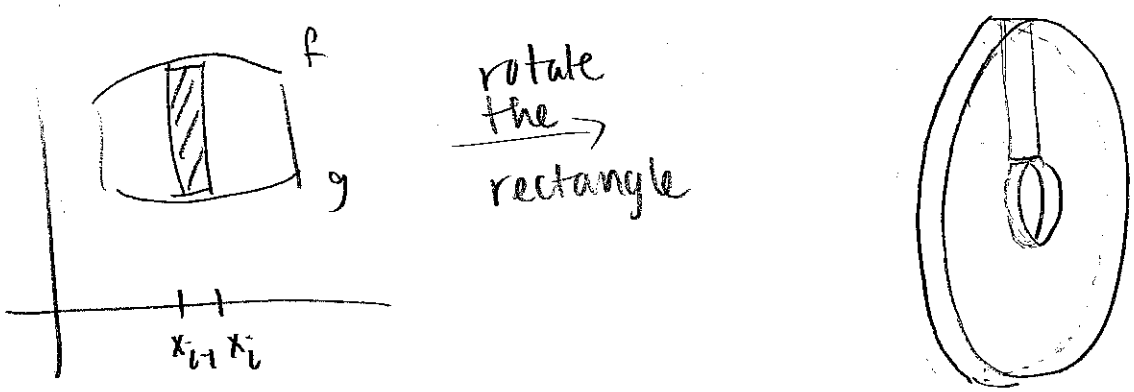


Let's always rotate around x-axis.

Washer / disk method:



Let's look at a small cross section.



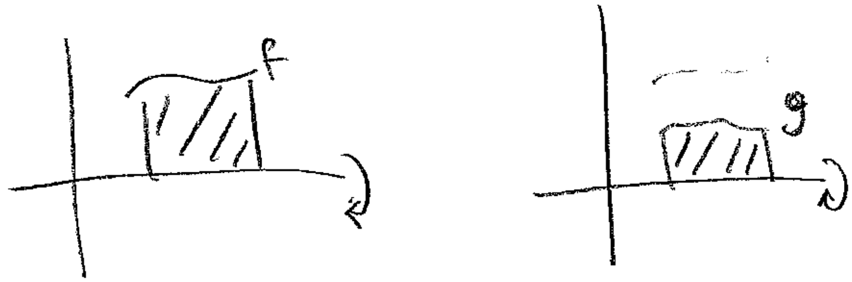
outer radius: $f(x_i^*)$
inner radius: $g(x_i^*)$ $\Bigg] \rightarrow$ area of base $= \pi f(x_i^*)^2 - \pi g(x_i^*)^2$

thickness: Δx_i.

$$\Rightarrow \Delta V_i \approx \pi \left[f(x_i^*)^2 - g(x_i^*)^2 \right] \Delta x_i$$

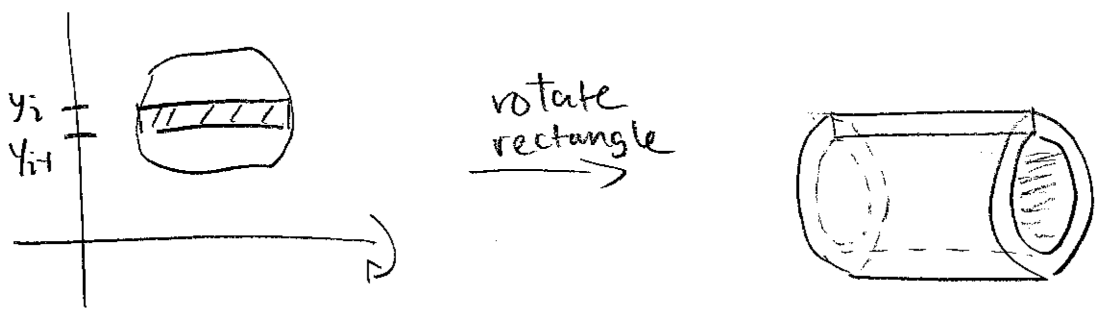
$$\Rightarrow V = \int_a^b \pi \left(f(x)^2 - g(x)^2 \right) dx.$$

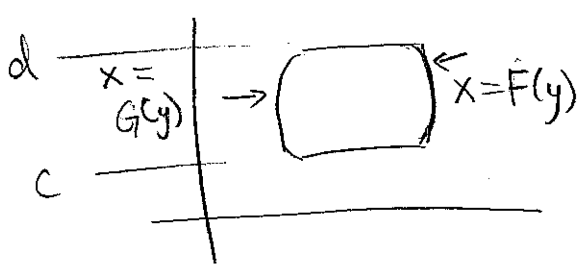
$$= \underbrace{\int_a^b \pi f(x)^2\,dx} - \underbrace{\int_a^b \pi g(x)^2\,dx}$$



Shell method.

Still, let's revolve around x-axis.





height of cylinder
$\approx F(y_i^*) - G(y_i^*)$.

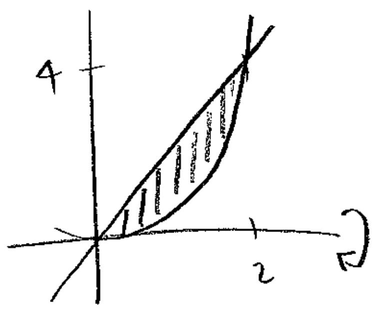
radius of base
$\approx y_i^*$

$$\Delta V_i \approx 2\pi y_i^* \left(F(y_i^*) - G(y_i^*)\right) \Delta y_i.$$

$$\Rightarrow \quad V = \int_c^d 2\pi y \left(F(y) - G(y)\right) dy.$$

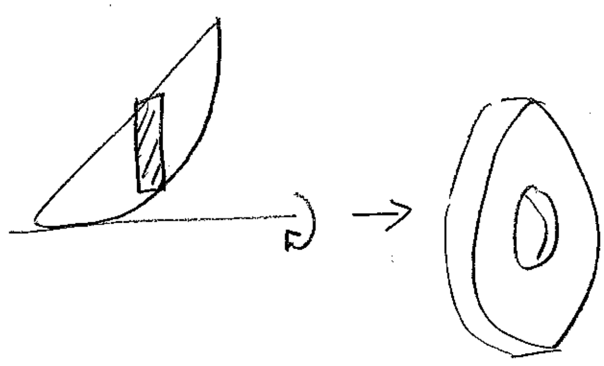
Example.

Find the volume of the solid generated by revolving the region between $y=x^2$ and $y=2x$ about the x-axis.



method 1: washer:

- we'll partition the interval $[0,2]$ on the x-axis
- draw a rectangle used to "generate" a washer:



upper:

$y=f(x)=2x$

lower:

$y=g(x)=x^2$

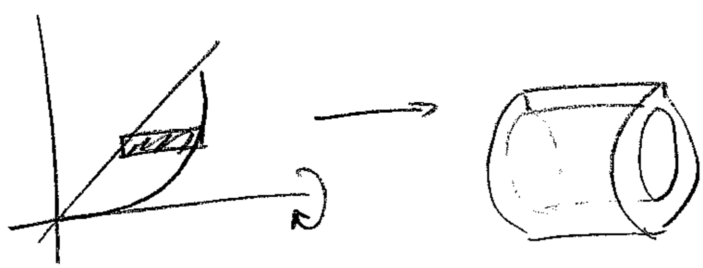
$$\Delta V_i \approx \pi\left(f(x_i^*)^2-g(x_i^*)^2\right)\Delta x_i .$$

$$\Rightarrow V=\int_0^2 \pi\left[f(x)^2-g(x)^2\right]dx=\int_0^2 \pi\left[(2x)^2-(x^2)^2\right]dx$$

$$=\frac{64}{15}\pi .$$

method 2: shell.

- partition the interval $[0,4]$ on y-axis.
- draw thin rectangle used to generate a cylindrical shell.



upper:
$x = F(y) = \sqrt{y}$.

lower:
$x = G(y) = \frac{1}{2}y$.

$$\Delta V_i = 2\pi y_i^*(F(y_i^*) - G(y_i^*))\,\Delta y_i$$

$$\rightarrow V = \int_0^4 2\pi y\left(\sqrt{y} - \tfrac{1}{2}y\right)dy = \frac{64}{15}\pi.$$

If revolve around x-axis:

washer	shell
• partition x-axis	• partition y-axis
• need y in terms of x	• need x in terms of y.

(If revolve around y-axis, then interchange x, y above)

Not helpful to memorize these! It's easier to draw "representative rectangles" and rotate them every time.

Next topic: Inverse functions:

Idea: a function is like:

(input) $\qquad\qquad\qquad\qquad\qquad\qquad$ (output)

$$x \longrightarrow \boxed{\; f \;} \longrightarrow y.$$

x is in domain of f $\qquad\qquad$ y is in range of f.

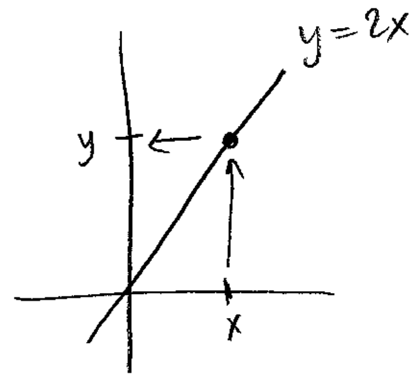
We write $y = f(x)$. for the relation between the input x and the output y.

Q: when can we go backwards?

input $\qquad$ output

$$x \xrightarrow{f} y$$

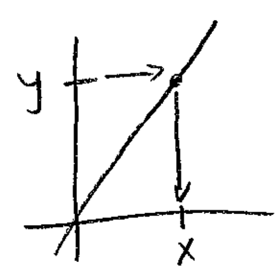
$$x \xleftarrow{??} y$$



Example:

$$f(x) = 2x.$$

$$y = 2x \implies \tfrac{1}{2}y = x.$$

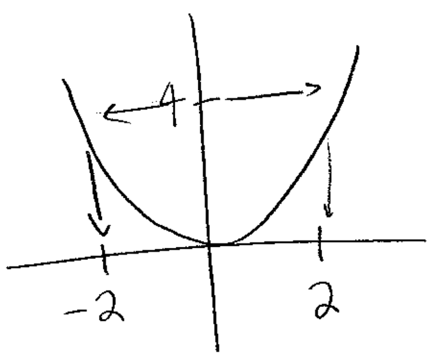


$$x \longrightarrow \boxed{f(x)=2x} \longrightarrow y$$

$$x \longleftarrow \boxed{f^{-1}(y)=\tfrac{1}{2}y} \longleftarrow y$$

But this is not always possible.

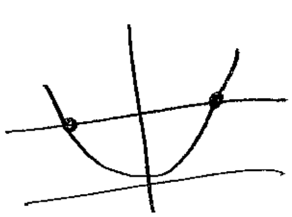
$f(x)=x^2$ $\qquad\qquad$ $f^{-1}(4) = ?$

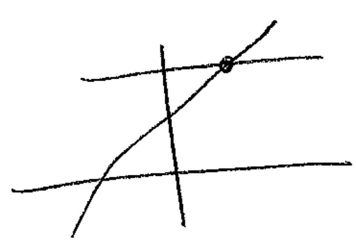


There are 2 possible values (± 2), so this is not a function.

<u>Def</u>: A function f is <u>one-to-one</u> if there are no two distinct numbers in the domain of f at which f takes on the same value.

You can use the "horizontal line test" to determine if a function is one-to-one.


fail


pass

The inverse function f^{-1} satisfies:

- $f(f^{-1}(x)) = x$ for all x in the range of f
- $f^{-1}(f(x)) = x$ for all x in the domain of f.

$$x \longrightarrow \boxed{f} \longrightarrow y \longrightarrow \boxed{f^{-1}} \longrightarrow x$$

$$x \longrightarrow \boxed{f^{-1}} \longrightarrow y \longrightarrow \boxed{f} \longrightarrow x.$$

Geometric picture: what's the relationship between the graphs $y=f(x)$ and $y=f^{-1}(x)$?

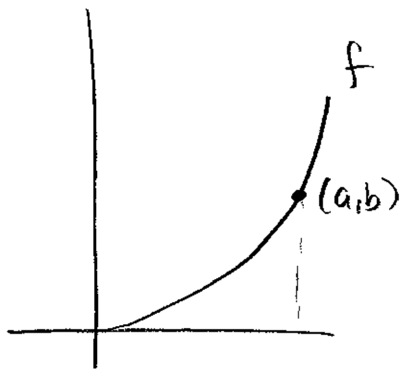
(a,b) is on the graph of f

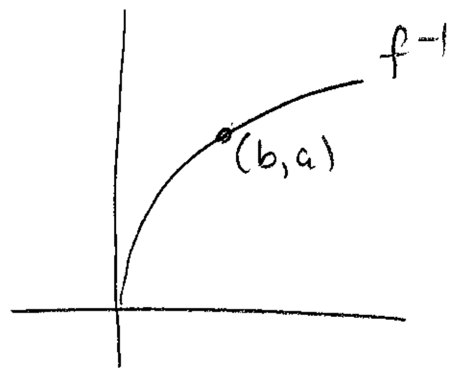
$\iff f(a)=b$

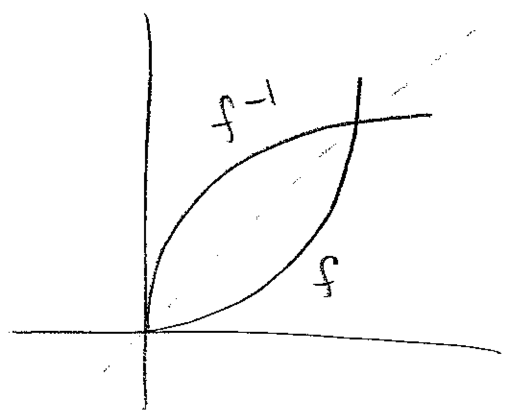
$\iff a=f^{-1}(b)$

$\iff (b,a)$ is on the graph of f^{-1}

(Lecture #11, 11/5/19).







The graph of f^{-1} is the graph of f reflected in the line $y = x$. ← THIS is VERY USEFUL

Q: What are the domain and range of f^{-1}?

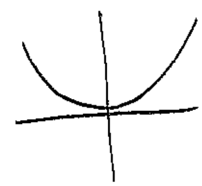
When we reflect in the line $y=x$, we interchange the x- and y- axes. So:

domain of f^{-1} = range of f
range of f^{-1} = domain of f.

Note / caution: Usually, when we state a function, we don't state its domain. But, sometimes we want to state it explicitly.

Example 1:

$f(x) = x^2$.



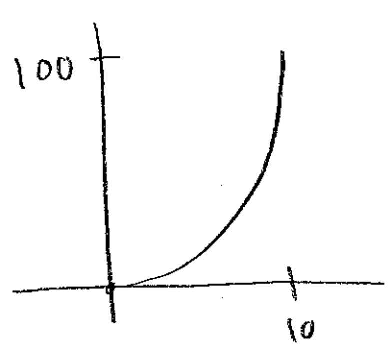
If we do not state the domain, it is implicitly assumed to be $(-\infty, \infty)$.

The range is then $[0, \infty)$.

Example 2:

$f(x) = x^2, \underbrace{x \in [0, 10]}_{\text{here, we explicitly state the domain.}}$

So: $f(3) = 9$. $f(-3)$ is not defined.

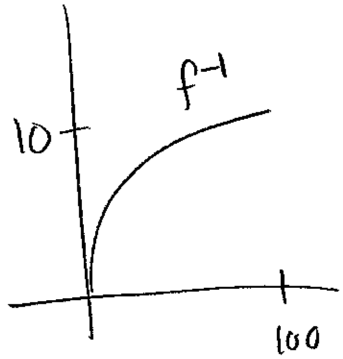


domain $= [0, 10]$

range $= [0, 100]$.

f is one-to-one.

$$f^{-1}(x) = \sqrt{x}, \quad x \in [0, 100]$$



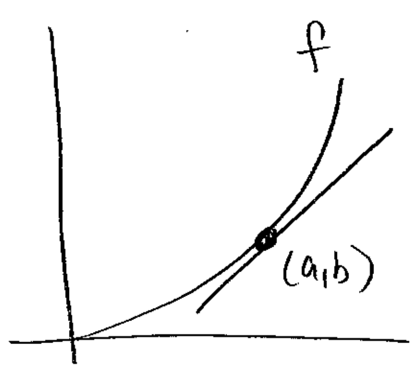
domain $= [0, 100]$

range $= [0, 10]$.

From the geometric picture, we see:

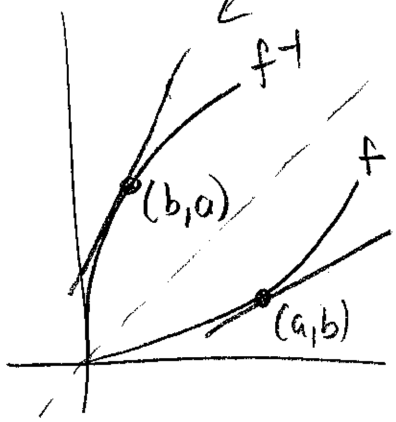
Let f be one-to-one. If f is continuous, then f^{-1} is continuous.

What about differentiability?



From this, what can we say about f^{-1}?

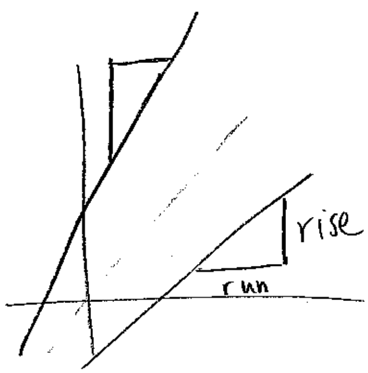
Reflect the picture!



slope $f'(a)$.

The reflected line has slope $\frac{1}{f'(a)}$.

(since we are interchanging rise and run).



So: $f(a)=b$.

$$\Rightarrow \boxed{(f^{-1})'(b) = \frac{1}{f'(a)}}$$

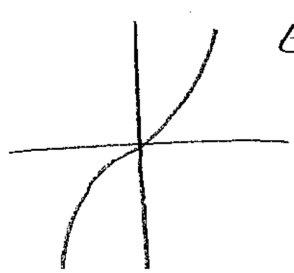
usually, we write this as $\boxed{(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}}$

<u>Example</u>:

$f(x) = x^3 + \frac{1}{2}x$.

(implicit: domain of f is $(-\infty, \infty)$.)

① Is f one-to-one?



$\leftarrow$ graph of f looks like this.

$f'(x) = 3x^2 + \frac{1}{2} > 0$

The derivative is positive

$\Rightarrow$ f is increasing

$\Rightarrow$ f passes the horizontal line test

$\Rightarrow$ f is one-to-one.

(2) $(f^{-1})'(9) = ?$

Note: $f^{-1}(x)$ is hard to write down
(need cubic formula...)

So: use the formula.

$$(f^{-1})'(b) = \frac{1}{f'(a)} \quad \text{where } f(a) = b.$$

$b = 9$. $f(a) = b$

$a^3 + \frac{1}{2}a = 9 \Rightarrow a = 2.$

$f'(x) = 3x^2 + \frac{1}{2}$. $f'(a) = f'(2) = 3 \cdot 2^2 + \frac{1}{2} = \frac{25}{2}$.

$$\Rightarrow (f^{-1})'(9) = \frac{1}{25/2} = \frac{2}{25}.$$

Another way to do part 2:

$$y = f^{-1}(x). \Rightarrow x = f(y) = y^3 + \tfrac{1}{2}y.$$

we want $\frac{dy}{dx}$, so do implicit differentiation:

$$1 = 3y^2 \frac{dy}{dx} + \frac{1}{2}\frac{dy}{dx}$$

$$1 = \left(3y^2 + \tfrac{1}{2}\right)\frac{dy}{dx}.$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{3y^2 + \frac{1}{2}}.$$

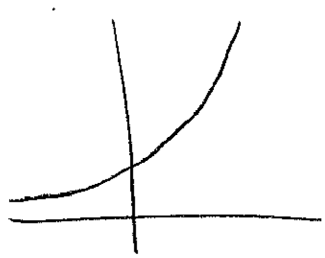
$x = 9$
$\Rightarrow y = 2.$

So $(f^{-1})'(9) = \dfrac{1}{3 \cdot 2^2 + \frac{1}{2}} = \dfrac{2}{25}.$

If you forget the formula for $(f^{-1})'(x)$, you can always use this method.

Next topic: logarithms

$f(x) = 10^x$ ← "exponential function"

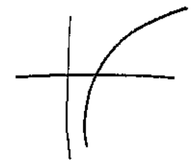


f is one-to-one, so it has an inverse.

The inverse is called $\log_{10} x$.

$f(x) = 10^x$
domain: $(-\infty, \infty)$
range: $(0, \infty)$

$f^{-1}(x) = \log_{10} x$
domain: $(0, \infty)$
range: $(-\infty, \infty)$



$$\left.\begin{array}{l} 10^0 = 1 \\ 10^1 = 10 \\ 10^2 = 100 \\ 10^{-1} = \frac{1}{10} \end{array}\right\} \Longrightarrow \begin{array}{l} \log_{10} 1 = 0 \\ \log_{10} 10 = 1 \\ \log_{10} 100 = 2 \\ \log_{10}\left(\frac{1}{10}\right) = -1. \end{array}$$

Fact: $\underbrace{10^a}_{x}\underbrace{10^b}_{y} = 10^{a+b}$

So $\log_{10}(xy) = \log_{10} 10^{a+b} = a+b = \log_{10} x + \log_{10} y$.

Basic properties of $\log_{10} x$.

① $\log_{10}(xy) = \log_{10} x + \log_{10} y \quad (x, y > 0)$.

② $\log_{10} 10 = 1$

For $\log_p x$ we just replace every "10" by "p" in ① and ②.

(Also: $(\underbrace{10^a}_{x})^b = 10^{ab} \Longrightarrow \log_{10}(x^b) = b \log_{10} x$.)

(11/7/19)

Can you think of other continuous functions that satisfy

$$f(xy) = f(x) + f(y) \text{ for all positive } x, y ?$$

No! If f satisfies the property above, (and is cts) then $f(x) = \log_p(x)$ for some base p.

(This is a fact. It requires proof...)

Warm-up: Solve for b:

$$\log_b 2 = \frac{1}{6}$$

Something seemingly unrelated: antiderivative of $\frac{1}{x}$.

Define $L(x) = \int_1^x \frac{1}{t}\,dt$. $(x > 0)$.

By FTC, $L'(x) = \frac{1}{x}$, so L is an antiderivative of $\frac{1}{x}$.

Remarkable property:

$$L(ab) = L(a) + L(b) \text{ for all positive } a, b.$$

<u>Proof:</u> (Ask the class.)

$$L(ab) = \int_1^{ab} \frac{1}{t}\,dt = \underbrace{\int_1^{a} \frac{1}{t}\,dt}_{L(a)} + \underbrace{\int_a^{ab} \frac{1}{t}\,dt}_{??}.$$

want:

$$\int_a^{ab} \frac{1}{t}\,dt = \int_1^{b} \frac{1}{t}\,dt.$$

Do a u-substitution!

$u = t/a$ upper $= u(ab) = b$

$du = \frac{1}{a}dt.$ lower $= u(a) = 1$

$$\Rightarrow \int_a^{ab} \frac{1}{t}\,dt = \int_1^{b} \frac{1}{au}\,a\,du = \int_1^{b} \frac{1}{u}\,du.$$

$\square$

and we're done!

L is continuous (since it is differentiable).

and $L(ab) = L(a) + L(b)$ for all $a, b > 0$.

This means that there is some base p such that $L(x) = \int_1^{x} \frac{dt}{t} = \log_p x.$ for all $x > 0$.

This special base is named e.

And $\log_e(x)$ is written $\ln(x)$.

("ln" stands for "natural logarithm").

It turns out $e \approx 2.71828\ldots$

How to calculate e using what we're given so far?

Method 1: Try to solve $\int_1^x \frac{dt}{t} = 1$ for x.

Method 2: $\log_e 2 = \int_1^2 \frac{dt}{t} \approx 0.693$

$\Rightarrow e^{0.693} \approx 2$

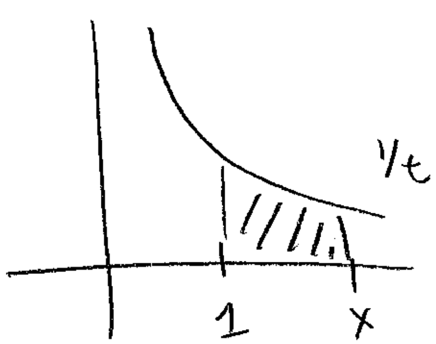
$\Rightarrow e \approx 2^{1/0.693}$

Basic properties of $\ln(x)$:

$$\left.\begin{aligned} &\ln(1) = 0 \\ &\ln(ab) = \ln(a) + \ln(b) \\ &\ln(a^b) = b\ln(a) \end{aligned}\right\} \text{ true for } \log_p(x)$$

$\ln(e) = 1 \longleftarrow$ only for $\log_e(x)$.

Recall: $\ln x = \int_1^x \frac{1}{t}\,dt$ (by definition)

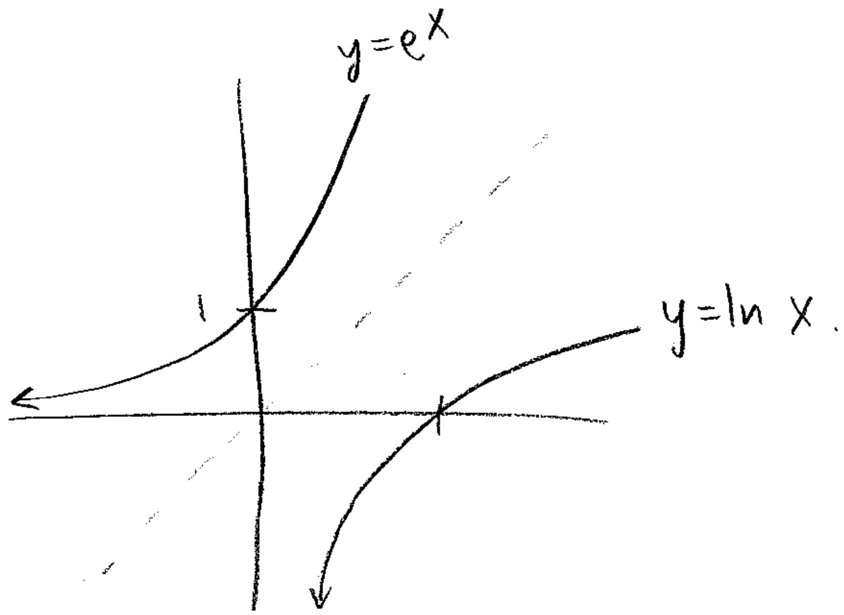


The domain of $\ln x$ is $(0, \infty)$.

FTC: $\boxed{\frac{d}{dx} \ln x = \frac{1}{x}}$ ← always positive.

$\Rightarrow$ $\ln(x)$ is an increasing function.

Graph of $\ln x$:



For $\ln x$:

domain: $(0, \infty)$

range: $(-\infty, \infty)$.

So: we've found/defined a function, defined on $(0,\infty)$, whose derivative equals $\frac{1}{x}$.

But the domain of $\frac{1}{x}$ is $(-\infty, 0) \cup (0, \infty)$.

So: can we find a function defined on $(-\infty,0)\cup(0,\infty)$ whose derivative equals $\frac{1}{x}$?

Yes! important: $\frac{d}{dx} \ln|x| = \frac{1}{x}$ for all $x \neq 0$.

Let's see why this is true:

Method 1: calculations.

For $x>0$, we have $|x|=x$, so

$$\frac{d}{dx}(\ln|x|)=\frac{d}{dx}(\ln x)=\frac{1}{x}.$$

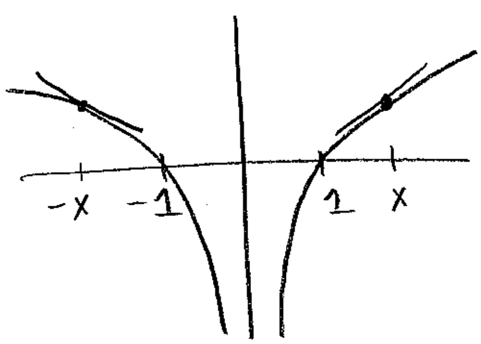
For $x<0$, we have $|x|=-x>0$, so.

$$\frac{d}{dx}(\ln|x|)=\frac{d}{dx}\ln(-x)\underset{\substack{\uparrow\\ \text{chain rule}}}{=}-\left(\frac{1}{-x}\right)=\frac{1}{x}$$

$\square$.

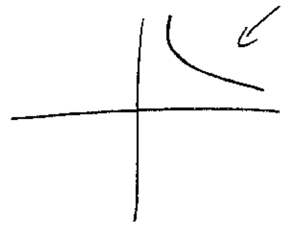
Method 2: graph.

$f(x)=\ln|x|$:



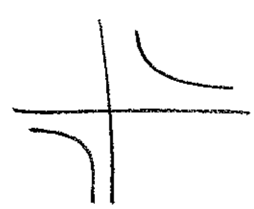
f is even

$\Longrightarrow$ f' is odd



we already positive part of f' looks like this.

$\longrightarrow$



since we know f' is odd, we can fill in the rest.

(11/14 Lecture)

So now we know: $\boxed{\int \frac{1}{x}\,dx = \ln|x| + C}$

Careful! you cannot integrate $\frac{1}{x}$ on an interval that contains 0.

WRONG: " $\int_{-1}^{1} \frac{1}{x}\,dx = [\ln|x|]_{-1}^{1} = \ln 1 - \ln 1 = 0$ "

Now we can do more integrals!

warmup:
$\ln n$ vs $1+2+\dots+(n-1)$
which is greater?
Hint: Riemann sums

Example: $\int \frac{x^2}{1-4x^3}\,dx$

$u = 1-4x^3$
$du = -12x^2\,dx$

$$= \int \left(-\frac{1}{12}\right)\frac{1}{u}\,du$$
$$= -\frac{1}{12}\ln|u| + C$$
$$= -\frac{1}{12}\ln|1-4x^3| + C.$$

Example: $\int \frac{\ln x}{x}\,dx.$

$u = \ln x$
$du = \frac{1}{x}.$

$$= \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}(\ln x)^2 + C.$$

Example: $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx.$

$u = \cos x$
$du = -\sin x\,dx$

$$= -\int \frac{du}{u} = -\ln|u| + C = -\ln|\cos x| + C$$

(similarly: $\int \cot x = \int \frac{\cos x}{\sin x}\,dx = \ln|\sin x| + C$).

Example: $\int \sec x\,dx$

??? I have no idea where the idea to do this comes from.

$$= \int \sec x \,\frac{\sec x + \tan x}{\sec x + \tan x}\,dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\,dx$$

$u = \sec x + \tan x$

$du = (\sec x \tan x + \sec^2 x)\,dx$

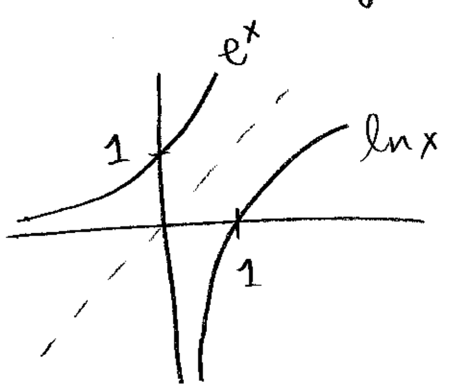
$$= \int \frac{du}{u} = \ln|u| + C = \ln|\sec x + \tan x| + C$$

(similarly, $\int \csc x\,dx = \ln|\csc x - \cot x| + C$.)

Next topic: The exponential function:

$f(x) = \ln x = \log_e x \implies f^{-1}(x) = e^x$

($e \approx 2.718$)

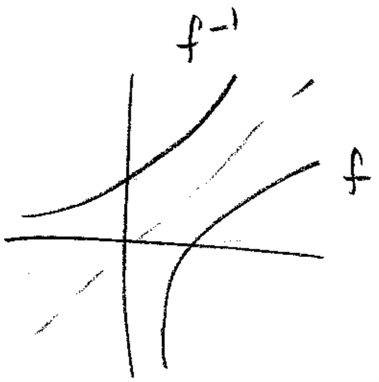


Since $\ln x$ and e^x are inverses of each other:

- $\ln(e^x) = x$ (for all $x \in (-\infty, \infty)$)
- $e^{\ln x} = x$ (for all $x > 0$).

Q: What is the derivative of e^x?

Let $f(x) = \ln x$. ($f^{-1}(x) = e^x$).



$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

$$= \frac{1}{f'(e^x)}$$

$$= \frac{1}{(1/e^x)} = e^x$$

$$\Rightarrow \boxed{\frac{d}{dx}[e^x] = e^x}$$

And: $\boxed{\int e^x \, dx = e^x + C}$

Example: What does $f(x) = e^{-x^2/2}$ look like?

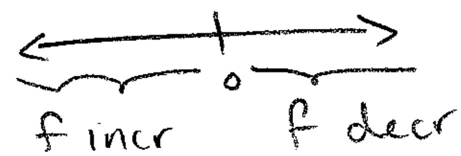
$$f'(x) = e^{-x^2/2}(-x) = -xe^{-x^2/2}$$

$$f''(x) = (-x)(-xe^{-x^2/2}) + (-1)e^{-x^2/2}$$

$$= (x^2-1)e^{-x^2/2}$$

Observations: • f is even

•



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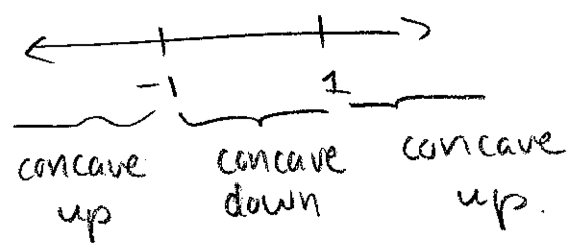
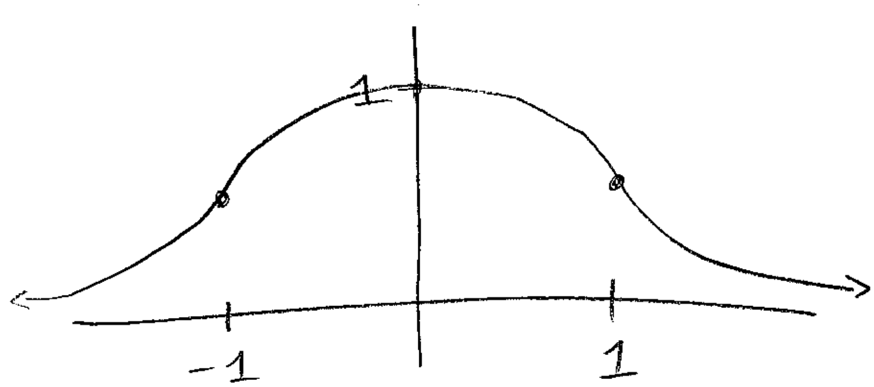


• $\lim_{x \to \infty} e^{-x^2/2} = 0$, $\lim_{x \to -\infty} e^{-x^2/2} = 0$

Graph:



Fun facts: • the total area between $f(x) = e^{-x^2/2}$ and the x-axis is $\sqrt{2\pi}$. (One way to calculate this is with multivariable calculus.)

• Let $g(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$. This function is the "probability density function for the standard normal distribution." (Very important for central limit theorem.)

Example: $\int \frac{e^{3x}}{e^{3x}+1}\,dx = ?$

$u = e^{3x}+1$
$du = 3e^{3x}\,dx$

$$= \frac{1}{3}\int \frac{du}{u} = \frac{1}{3}\ln|u| + C$$

$$= \frac{1}{3}\ln|e^{3x}+1| + C$$

(since $e^{3x}+1$ is never negative)

$$= \frac{1}{3}\ln(e^{3x}+1) + C$$

Logarithms and exponentials with other bases:

Q:

$f(x) = 2^x$. $f'(x) = ?$

Recall $2 = e^{\ln 2} \Rightarrow 2^x = (e^{\ln 2})^x = e^{(\ln 2)x}$.

so $f'(x) = \frac{d}{dx}\left(e^{(\ln 2)x}\right) = (\ln 2)e^{(\ln 2)x}$

$= (\ln 2) \cdot 2^x$.

In general, if p is some positive constant,

then $\boxed{\frac{d}{dx}(p^x) = (\ln p) \cdot p^x}$

Q: $f(x) = \log_2 x$ $f'(x) = ?$

Recall: $\log_2 x = \frac{\ln x}{\ln 2}$ "change of base rule"

(More generally: $\log_b a = \frac{\log_c a}{\log_c b}$)

so $f'(x) = \frac{d}{dx}\left[\frac{\ln(x)}{\ln 2}\right] = \frac{1}{\ln 2} \cdot \frac{1}{x}$.

So for any positive constant p,

$$\boxed{\frac{d}{dx}\left(\log_p x\right) = \frac{1}{\ln p} \cdot \frac{1}{x}.}$$

$$p^x = e^{(\ln p)x} \qquad \log_p x = \frac{\ln x}{\ln p}$$

Idea: convert exponentials / logarithms into base e to make them easier to differentiate.

<u>Another example</u>.

$$\frac{d}{dx}\left[x^x\right] = ?$$

convert into something easier

$$x^x = \left(e^{\ln x}\right)^x = e^{x \ln x}$$

(chain rule)

So $\frac{d}{dx}\left[x^x\right] = \frac{d}{dx}\left[e^{x\ln x}\right] = e^{x \ln x}\, \frac{d}{dx}\left[x \ln x\right]$

$$= \left(x^x\right) \cdot \left(\ln x + x \cdot \frac{1}{x}\right)$$

$$= x^x\left(\ln x + 1\right)$$

(11/19 lecture)

Exponential growth and decay.

Warm-up:
$\frac{d}{dx}[x^x] = ?$
Hint: $x^x = (e^{\ln x})^x$

$P(t) =$ population at time t.

Suppose the rate of change of P is proportional to the size of P.

i.e., $P'(t) = kP(t)$
where k is a constant.

What do we know about the function $P(t)$?

Example: suppose P satisfies $P'(t) = 2P(t)$.

Consider the function $e^{-2t}P(t)$.

Differentiate it: $\frac{d}{dt}[e^{-2t}P(t)]$

$$= e^{-2t}P'(t) - 2e^{-2t}P(t)$$
$$= e^{-2t}(2P(t)) - 2e^{-2t}P(t)$$
$$= 0.$$

So $e^{-2t}P(t) = C$ for some constant C.

$$\Rightarrow \boxed{P(t) = Ce^{2t}}$$

(Where does the expression $e^{-2t}P(t)$ come from?

This is an example of the method of "integrating factors" in ODE.)

In general:

If $f'(t) = kf(t)$,

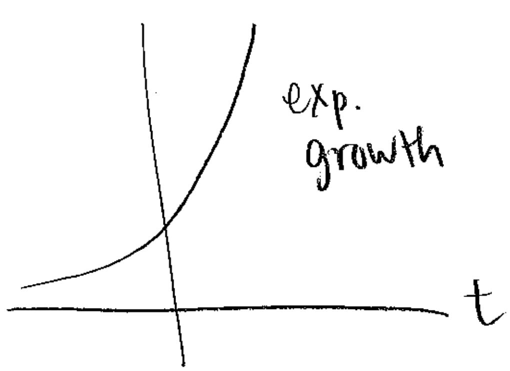
then there's a constant C such that
$$f(t) = Ce^{kt}$$

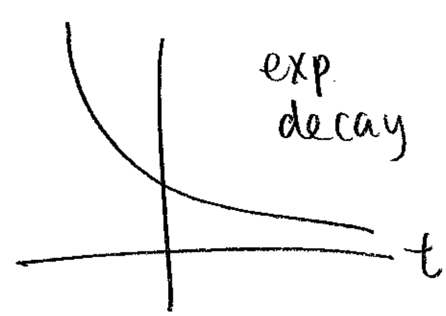
$f(t) = Ce^{kt}$: If $k > 0$: exponential growth.

(e.g. ideal population growth
continuous compounding)

If $k < 0$: exponential decay

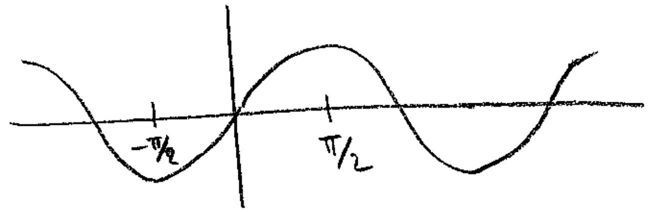
(e.g. radioactive decay).





Inverse trig functions:

$f(x) = \sin x$



This function is not invertible.

But: $\underbrace{f(x) = \sin x, \quad x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]}_{\text{this function is invertible.}}$

The inverse is called the "arc sine function"

$f(x)$:

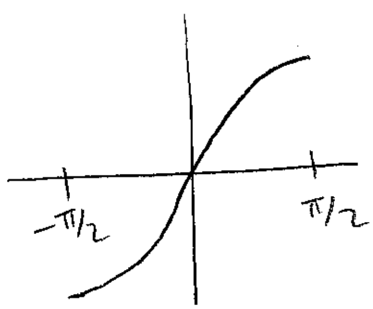
domain. $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

range: $[-1, 1]$

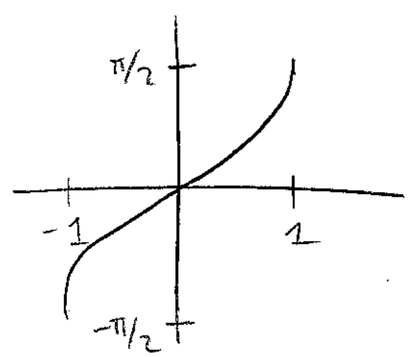
$f^{-1}(y) = \arcsin y$.

domain: $[-1, 1]$

range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

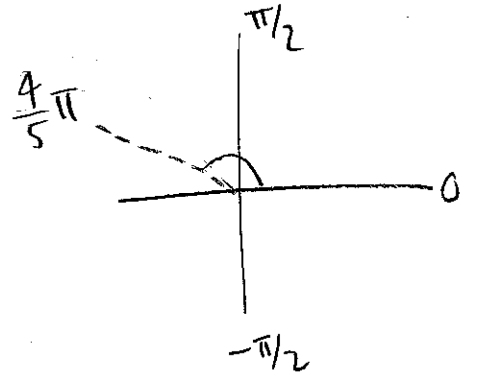


reflect $\longrightarrow$



- For all $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\arcsin(\sin x) = x$. $[f^{-1}(f(x)) = x]$
- For all $y \in [-1, 1]$, $\sin(\arcsin y) = y$ $[f(f^{-1}(y)) = y]$

Q: What if $x \notin [-\frac{\pi}{2}, \frac{\pi}{2}]$?



$$\arcsin(\sin \tfrac{4}{5}\pi) = \arcsin(\sin \tfrac{\pi}{5}) = \frac{\pi}{5}$$

Q: What if $y \notin [-1, 1]$?

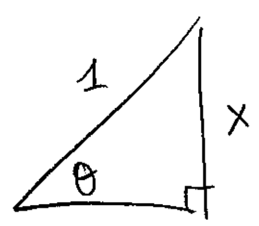
$\sin(\arcsin 2) = ?$

Note that arcsin 2 is undefined!

Derivative of arcsine?

$f(x) = \sin x, \ x \in [-\frac{\pi}{2}, \frac{\pi}{2}] \longrightarrow f^{-1}(x) = \arcsin x.$

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{\cos(\arcsin x)}$$

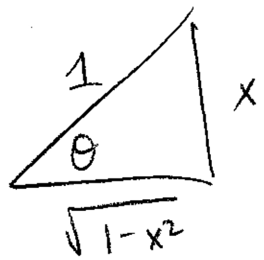


Let $\theta = \arcsin x$.

So: $\sin\theta = x$

$\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

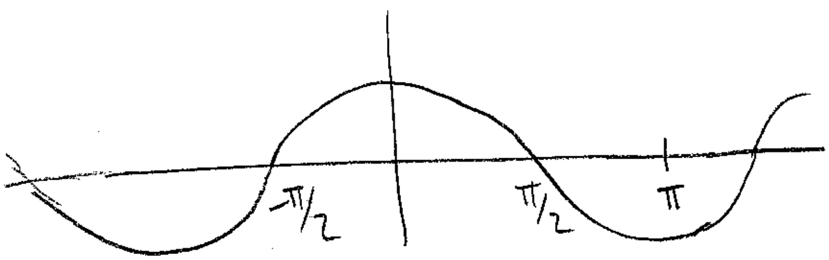
use pythagorean theorem.



$\Longrightarrow \quad \cos\theta = \dfrac{1}{\sqrt{1-x^2}}$

so: $\boxed{\dfrac{d}{dx}[\arcsin x] = \dfrac{1}{\sqrt{1-x^2}}}$

Now let's look at arccos:



$g(x) = \cos x, \ x \in [0, \pi]$. $\longleftarrow$ This function is one-to-one

g:

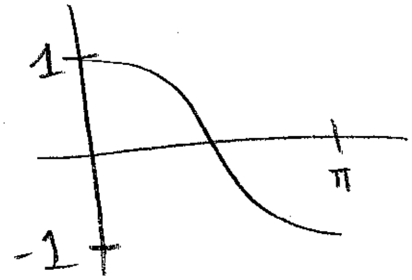
domain: $[0, \pi]$

range: $[-1, 1]$.

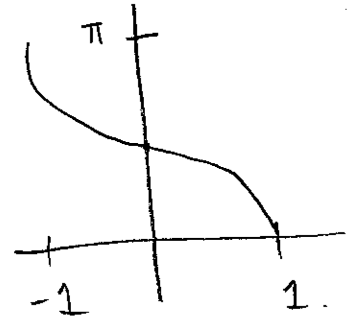
$g^{-1}(y) = \arccos y$.

domain: $[-1, 1]$

range: $[0, \pi]$.



flip $\longrightarrow$



ask class

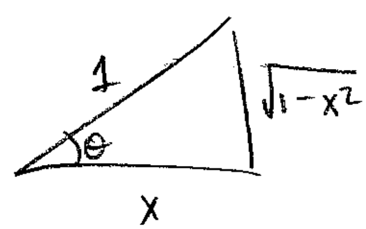
So: $\bullet$ $\arccos(\cos x) = x$ for all $x \in [0, \pi]$

$\bullet$ $\cos(\arccos y) = y$ for all $y \in [-1, 1]$

$\frac{d}{dx}[\arccos x] = ?$ (ask class).

Method 1: $g(x) = \cos x,\ x \in [0, \pi]$.

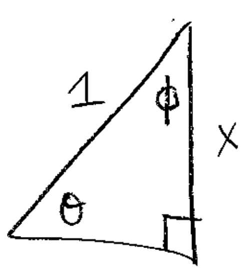
$$(g^{-1})'(x) = \frac{1}{g'(g^{-1}(x))} = \frac{-1}{\sin(\arccos x)} = -\frac{1}{\sqrt{1-x^2}}$$



$\sin(\arccos x) = \sin\theta = \sqrt{1-x^2}$

Method 2:

$\sin\theta = x \qquad \cos\phi = x$



$\theta = \arcsin x \qquad \phi = \arccos x$.

$\Longrightarrow \arcsin x + \arccos x = \frac{\pi}{2}$

$\Longrightarrow \frac{d}{dx}[\arcsin x] + \frac{d}{dx}[\arccos x] = 0$

so either method gives: $\boxed{\dfrac{d}{dx}[\arccos x] = \dfrac{-1}{\sqrt{1-x^2}}}$

Also: $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$ ← more commonly used.

$\left(\text{and } \displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx = -\arccos x + C\right)$

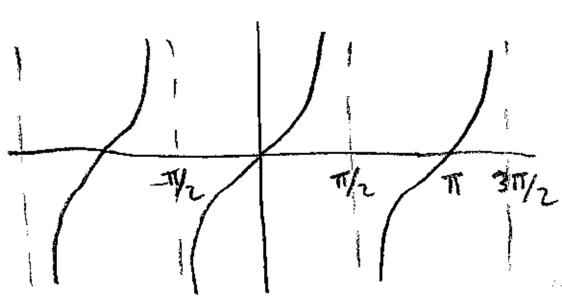
Q: $\displaystyle\int \frac{1}{\sqrt{9-x^2}}\,dx$?

$$\int \frac{1}{\sqrt{9-x^2}}\,dx = \int \frac{1}{\sqrt{9}\cdot\sqrt{1-\frac{x^2}{9}}}\,dx = \frac{1}{3}\int \frac{1}{\sqrt{1-\left(\frac{x}{3}\right)^2}}\,dx.$$

let $u = x/3$ $\quad du = dx/3$.

$$= \int \frac{1}{\sqrt{1-u^2}}\,du = \arcsin u + C = \arcsin \frac{x}{3} + C.$$

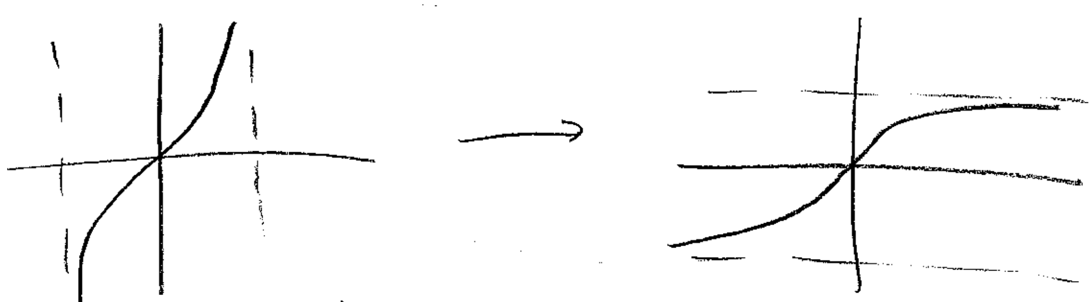
Next, let's take a look a the tangent function:



$f(x) = \tan x, \quad x \in (-\frac{\pi}{2}, \frac{\pi}{2})$ ← This is 1-to-1.

$f(x)$
domain: $(-\frac{\pi}{2}, \frac{\pi}{2})$
range: all reals

$f^{-1}(x) = \arctan x$
domain: all reals
range: $(-\frac{\pi}{2}, \frac{\pi}{2})$.

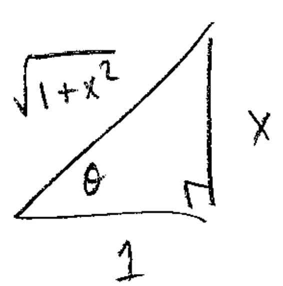


- For all $\overbrace{x \in (-\frac{\pi}{2}, \frac{\pi}{2})}^{\text{ask}}$ $\quad \arctan(\tan x) = x$.
- For all $y \in (-\infty, \infty)$ $\quad \tan(\arctan y) = y$.

$\frac{d}{dx} \arctan x = ?$ (ask class).

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{\sec^2(\arctan x)} = \frac{1}{(\sqrt{1+x^2})^2}$$

$$= \frac{1}{1+x^2}.$$



So $\frac{d}{dx}[\arctan x] = \frac{1}{1+x^2}$

(and $\frac{d}{dx}[\text{arc}\cot x] = -\frac{1}{1+x^2}$).

$$\int \frac{1}{1+x^2}\,dx = \arctan x + C$$

Ex:

$$\int \frac{1}{9+x^2}\,dx = \int \frac{1}{9(1+\frac{x^2}{9})}\,dx = \frac{1}{9}\int \frac{1}{1+(\frac{x}{3})^2}\,dx$$

$u = \frac{x}{3}$ $\quad du = \frac{1}{3}dx.$

$$= \frac{1}{3}\int \frac{1}{1+u^2}\,du = \frac{1}{3}\arctan u + C = \frac{1}{3}\arctan \frac{x}{3} + C$$

Key integral formulas:

- $\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C \quad \left(\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin \frac{x}{a} + C\right)$
- $\int \frac{1}{1+x^2}\,dx = \arctan x + C \quad \left(\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan \frac{x}{a} + C\right)$

Not sure if these are worth memorizing...

using the same method, we can show

$$\frac{d}{dx}\operatorname{arcsec} x = \frac{1}{|x|\sqrt{x^2-1}}$$

but this seems to be less useful, so we won't be using it ...

(11/21 lecture)

Q: (Poll everywhere)

Which of the following can we integrate using the integration formulas we just learned?

(a) $\displaystyle\int \frac{1}{1+x^4}\,dx$

(b) $\displaystyle\int \frac{x}{1+x^4}\,dx$

(c) $\displaystyle\int \frac{x^2}{1+x^4}\,dx$

(d) $\displaystyle\int \frac{x^3}{1+x^4}\,dx.$

Answer: (b).

Let $u=x^2$.
$du = 2x\,dx$.
Get arctan formula.

(For (d), $u=1+x^4$ works.)

(For (a) and (c), um...)

(b) $\displaystyle\int \frac{x}{1+x^4}dx = \frac{1}{2}\int \frac{1}{1+u^2}du = \frac{1}{2}\arctan u + C = \frac{1}{2}\arctan x^2 + C.$

For the remaining 3 lectures, we'll cover:

- integration by parts
- powers and products of trig functions
- trig substitutions

Integration by parts

Recall: chain rule $\longleftrightarrow$ u-substitution

Now: product rule $\longleftrightarrow$ integration by parts.

$$(uv)'(x) = u(x)v'(x) + v(x)u'(x).$$

$$\Rightarrow \quad u(x)v'(x) = (uv)'(x) - v(x)u'(x).$$

$$\Rightarrow \quad \int u(x)v'(x)\,dx = \int (uv)'(x)\,dx - \int v(x)u'(x)\,dx.$$

$$\Rightarrow \quad \boxed{\int u(x)v'(x)\,dx = u(x)v(x) - \int v(x)u'(x)\,dx}$$

or: $\int u\,dv = uv - \int v\,du$

This is useful when we want to calculate $\int u\,dv$, but $\int v\,du$ is easier to calculate.

Example 1 : $\int x e^x dx = ?$

To use I.B.P., need to set something to $u(x)$ and something to $v'(x)$.

$\underbrace{u(x)}$ — This will be differentiated. | $\underbrace{v'(x)}$ — This will be integrated.

$$u(x) = x \longrightarrow u'(x) = 1$$

$$v'(x) = e^x \longrightarrow v(x) = e^x$$

So $\int \underset{u}{x}\,\underset{v'}{e^x}\,dx = \underset{u}{x}\,\underset{v}{e^x} - \int \underset{v}{e^x} \cdot \underset{u'}{1}\,dx$

$$\int x e^x dx = x e^x - \int e^x dx$$

$$= x e^x - e^x + C.$$

In this example, we chose $u = x$ because u' becomes "simpler". What if we had done something else?

$$u = e^x \longrightarrow u' = e^x$$

$$v' = x \longrightarrow v = \frac{x^2}{2}.$$

We get:

$$\int xe^x = e^x \cdot \frac{x^2}{2} - \underbrace{\int \frac{x^2}{2} \cdot e^x \, dx}$$

This is not any better...

<u>Example 2</u>:

$$\int x \sin 2x \, dx$$

We want to differentiate this

$$u(x) = x \longrightarrow u'(x) = 1$$

$$v'(x) = \sin 2x \longrightarrow v(x) = -\frac{1}{2} \cos 2x.$$

$$\int x \sin 2x \, dx = \underbrace{-\frac{1}{2} x \cos 2x}_{uv} + \int \frac{1}{2} \cos 2x \, dx = -\frac{1}{2} x \cos 2x + \frac{1}{4} \sin 2x + C.$$

Example 3: $\int x^2 e^x \, dx$.

$$u = x^2 \longrightarrow u' = 2x$$
$$v' = e^x \qquad v = e^x.$$

$$\int x^2 e^x \, dx = x^2 e^x - \int e^x 2x \, dx$$
$$= x^2 e^x - 2 \underbrace{\int x e^x \, dx}$$

now we integrate by parts again

$$= x^2 e^x - 2x e^x + 2e^x + C$$
$$= (x^2 - 2x + 2) e^x + C.$$

General pattern:

$\int x^n e^{ax} \, dx$. If n is a positive integer

$$u = x^n \longrightarrow u' = n x^{n-1} \quad (u' \text{ is "simpler"})$$
$$v' = e^{ax} \qquad v = \frac{1}{a} e^{ax} \quad (v \text{ is not more "complicated" than } v')$$

Example 4: $\int x \ln x \, dx = ?$

(Ask class to try)

Maybe $\begin{cases} u = x \\ v' = \ln x \end{cases}$? But what is v?

let's try $\begin{array}{l} u = \ln x \\ v' = x \end{array} \longrightarrow \begin{array}{l} u' = \frac{1}{x} \\ v = \frac{x^2}{2}. \end{array}$

$$\int x \ln x \, dx = (\ln x)\left(\frac{x^2}{2}\right) - \int \frac{x^2}{2} \cdot \frac{1}{x} \, dx$$

$$= \frac{x^2}{2} \ln x - \frac{1}{2} \int x \, dx$$

$$= \frac{x^2}{2} \ln x - \frac{1}{4} x^2 + C.$$

Method 2: (looks crazy, but it works!)

$\begin{array}{l} u = x \ln x \\ v' = 1 \end{array} \longrightarrow \begin{array}{l} u' = x \cdot \frac{1}{x} + \ln x = 1 + \ln x \\ v = x. \end{array}$

$$\int x \ln x \, dx = (x \ln x) x - \int x (1 + \ln x) \, dx.$$

$$= x^2 \ln x - \int x \, dx - \int x \ln x \, dx.$$

$$\Rightarrow \quad 2\int x\ln x\,dx = x^2\ln x - \int x\,dx$$
$$= x^2\ln x - \tfrac{1}{2}x^2 + C$$

$$\Rightarrow \quad \int x\ln x\,dx = \tfrac{1}{2}x^2\ln x - \tfrac{1}{4}x^2 + C.$$

Example 5: $\int e^x \cos x\,dx$.

Neither e^x nor $\cos x$ become "simpler" or "more complex" when you integrate/differentiate it.

$u = e^x \longrightarrow u' = e^x$

$v' = \cos x \qquad v = \sin x$

$$\int e^x\cos x\,dx = e^x\sin x - \int(\sin x)e^x\,dx$$

again!

$u = e^x \longrightarrow u' = e^x$

$v' = \sin x \qquad v = -\cos x.$

$$\int(\sin x)e^x\,dx = e^x(-\cos x) - \int(-\cos x)\,e^x\,dx.$$
$$= -e^x\cos x + \int e^x\cos x\,dx.$$

So we get:

$$\int e^x \cos x\,dx = e^x \sin x - \left[-e^x \cos x + \int e^x \cos x\,dx\right]$$

$$= e^x(\sin x + \cos x) - \int e^x \cos x\,dx.$$

$$2\int e^x \cos x\,dx = e^x(\sin x + \cos x) + C$$

$$\int e^x \cos x\,dx = \frac{1}{2} e^x(\sin x + \cos x) + C.$$

Example 6: $\int \arcsin x\,dx = ?$

$u = \arcsin x \longrightarrow u' = \frac{1}{\sqrt{1-x^2}}$

$v' = 1 \longrightarrow v = x.$

u-sub: $(u = 1-x^2)$

$$\int \arcsin x\,dx = x \arcsin x - \int x \cdot \frac{1}{\sqrt{1-x^2}}\,dx.$$

$$= x \arcsin x + \sqrt{1-x^2} + C$$

(You can do something similar with

$\int \ln x\,dx$ and $\int \arctan x$)

In general, what do you set as u and v'?

There's no general rule... you have to try and see!

Definite integrals and I.B.P.:

$$\int_0^1 xe^x \, dx = ?$$

For these, calculate the indefinite integral first. Then use FTC.

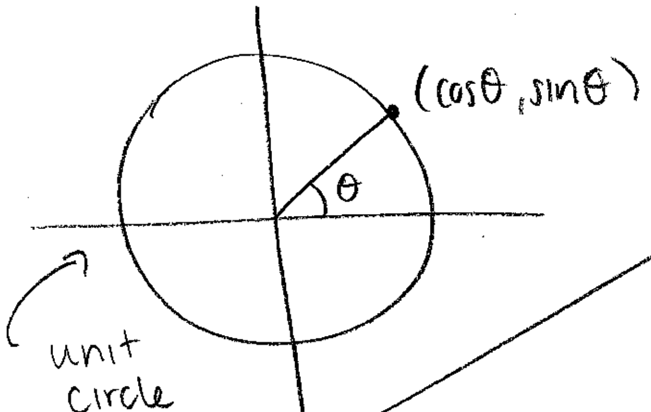
$$\int xe^x dx = xe^x - e^x + C$$

$$\Rightarrow \int_0^1 xe^x dx = \left[xe^x - e^x\right]_0^1 = \left[e^1 - e^1\right] - \left[0 - 1\right]$$
$$= 1.$$

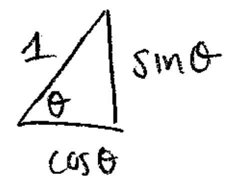
(There is a I.B.P. formula for definite integrals but it's not any easier than the approach here.)

Next topic: integrating products of trig functions.

Recall basic trig:



Pythagorean theorem:



$$\boxed{\sin^2\theta + \cos^2\theta = 1}$$

divide by $\sin^2\theta$

$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

$$\boxed{1 + \cot^2\theta = \csc^2\theta}$$

divide by $\cos^2\theta$

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\boxed{\tan^2\theta + 1 = \sec^2\theta}$$

It's also useful to know the power-reducing (half-angle) identities.

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

Example 1: $\int \sin x \, dx = -\cos x + C$ ...

Ex 2: $\int \sin^2 x \, dx = ?$

$$\int \sin^2 x \, dx = \int \frac{1-\cos 2x}{2} \, dx$$

$$= \frac{1}{2} \int (1-\cos 2x) \, dx$$

$$= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right] + C$$

$$= \frac{1}{2}x - \frac{1}{4}\sin 2x + C.$$

Ex 3: $\int \sin^3 x \, dx = ?$

$$\int \sin^3 x \, dx = \int \sin^2 x \, \sin x \, dx = \int (1-\cos^2 x) \sin x \, dx$$

$u = \cos x \quad du = -\sin x \, du$

$$= \int (1-u^2)(-du) = -\left[u - \frac{u^3}{3}\right] + C$$

$$= -u + \frac{u^3}{3} + C = -\cos x + \frac{\cos^3 x}{3} + C.$$

Ex 4:

$$\int \sin^4 x \, dx = ?$$

$$\sin^4 x = (\sin^2 x)^2 = \left(\frac{1-\cos 2x}{2}\right)^2$$

$$= \frac{1}{4}\left[1 - 2\cos 2x + \cos^2 2x\right]$$

$$= \frac{1}{4}\left[1 - 2\cos 2x + \frac{1+\cos 4x}{2}\right]$$

$$= \frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{8} + \frac{1}{8}\cos 4x$$

$$= \frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x.$$

$$\Rightarrow \int \sin^4 x \, dx = \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$$

Ex 5: $\int \sin^5 x \, dx = ?$ (Ask class)

$$\sin^5 x = (\sin^2 x)^2 \sin x = (1-\cos^2 x)^2 \sin x.$$

so let $u = \cos x$

$$\int \sin^5 x \, dx = -\int (1-u^2)^2 \, du = \int (-1+2u^2-u^4)\, du$$

$$= -u + \frac{2}{3}u^3 - \frac{1}{5}u^5 + C$$

$$= -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + C$$

<u>Example</u>: $\int \sin^2 x \cos^5 x \, dx$

$\cos^5 x$
$= \cos^4 x \cos x$
$= (\cos^2 x)^2 \cos x$
$= (1-\sin^2 x)^2 \cos x$

$$= \int \sin^2 (1-\sin^2 x)^2 \cos x \, dx$$

Now let $u = \sin x$
$du = \cos x \, dx$

$$= \int \underbrace{u^2(1-u^2)^2}_{\text{expand out this polynomial}} du$$

$$= \int (u^2 - 2u^4 + u^6) du$$

$$= \frac{1}{3}u^3 - \frac{2}{5}u^5 + \frac{1}{7}u^7 + C$$

$$= \frac{1}{3}\sin^3 x - \frac{2}{5}\sin^5 x + \frac{1}{7}\sin^7 x + C.$$

From this example, we see that this strategy works for $\int \sin^m x \cos^n x \, dx$ when at least one of m, n is odd.

If both are even, try the power reducing formulas...

For tangents and secants

Recall

- $1+\tan^2 x = \sec^2 x$
- $\frac{d}{dx}(\tan x) = \sec^2 x$
- $\frac{d}{dx}(\sec x) = \sec x \tan x.$

So for $\int \sec^m x \tan^n x \, dx$, we can try to use these identities to help us. See if we can either

(i) keep a "$\sec x \tan x$", and convert remaining "$\tan x$" to "$\sec x$." $(u = \sec x)$

or (ii) keep a "$\sec^2 x$" and convert remaining "$\sec x$" to "$\tan x$" $(u = \tan x)$

<u>Example</u>: $\int \tan^5 x \sec^3 x \, dx.$

Can we try $u = \tan x$? No, because

$$\sec^3 x = \underbrace{\sec^2 x}_{\text{keep for } du} \cdot \underbrace{\sec x}$$

we can't convert this.

What about $u = \sec x$?

$$\tan^5 x \sec^3 x = \underbrace{(\tan x \sec x)}_{\text{keep as } du} \underbrace{(\tan^4 x \sec^2 x)}_{\text{convert all to } \sec x.}$$

$$\int \tan^5 x \sec^3 x \, dx = \int \tan^4 x \sec^2 x \sec x \tan x \, dx.$$

$$= \int (\sec^2 x - 1)^2 \sec^2 x \underbrace{\sec x \tan x \, dx}_{du}$$

$$= \int (u^2-1)^2 u^2 \, du$$

$$= \cdots$$

Example: $\int \sec^6 x \, dx$

$$\sec^6 x = \sec^4 x \sec^2 x$$
$$= (1+\tan^2 x)^2 \sec^2 x$$

so let $u = \tan x$
$du = \sec^2 x \, dx$

$$\int \sec^6 x \, dx = \int (1+u^2)^2 \, du = \cdots$$

<u>Example</u>: $\int \sec^3 x \, dx$.

$u = \tan x$ doesn't work...

$u = \sec x$ doesn't work...

This one is crazy. integrate by parts with

$u = \sec x \qquad\longrightarrow\qquad u' = \sec x \tan x$

$v' = \sec^2 x \qquad\qquad\qquad v = \tan x$.

Final answer is $\frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C$ ...

(details are in the textbook).

Why do we care about these kinds of integrals?
One reason is that they come up in <u>trigonometric</u> substitution.

<u>Example</u>: $\int \sqrt{a^2 - x^2} \, dx$?

Let $x = a \sin u$

$dx = a \cos u \, du$.

$$\int \sqrt{a^2 - x^2} \, dx = \int \sqrt{a^2 - (a^2 \sin^2 u)} \; a \cos u \, du$$

$$= \int \sqrt{a^2(1-\sin^2 u)}\; a\cos u\, du$$

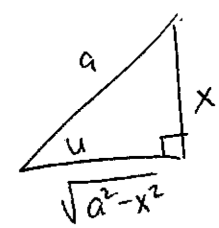
$$= a^2 \int \sqrt{\cos^2 u}\, \cos u\, du$$

$$= a^2 \int \cos^2 u\, du$$

$$= a^2 \int \left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2u\right) du$$

$$= a^2 \left[\tfrac{1}{2}u + \tfrac{1}{4}\sin 2u\right] + C$$

$x = a \sin u$



$\sin 2u$

$= 2\sin u \cos u$

$= 2 \cdot \frac{x}{a} \cdot \frac{\sqrt{a^2-x^2}}{a}$

$$= \frac{a^2}{2} \arcsin\frac{x}{a} + \frac{a^2}{4} \cdot 2 \cdot \frac{x}{a} \cdot \frac{\sqrt{a^2-x^2}}{a} + C$$

$$= \frac{a^2}{2} \arcsin\frac{x}{a} + \frac{1}{2} x\sqrt{a^2-x^2} + C$$

Maybe start with something simpler.

Example:

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx$$

We already know this is $\arcsin\frac{x}{a} + C$.

Here's another way to do this integral.

Let $x = a\sin u$. $\left(\begin{array}{c} -a \le x \le a \\ -\frac{\pi}{2} \le u \le \frac{\pi}{2} \end{array}\right)$

$dx = a\cos u\,du$.

$$\sqrt{a^2-x^2} = \sqrt{a^2-a^2\sin^2 u} = a\sqrt{1-\sin^2 u}$$

$$= a\sqrt{\cos^2 u} = a\cos u.$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \int \frac{1}{a\cos u}\,a\cos u\,du = \int du = u + C$$

$$= \arcsin\frac{x}{a} + C.$$

So: If you see a^2-x^2 you can try substituting

$x = a\sin u$ to get

$$a^2-x^2 = a^2-a^2\sin^2 u = a^2\cos^2 u$$

3 identities:

$$\cos^2 x = 1 - \sin^2 x$$
$$\sec^2 x = 1 + \tan^2 x$$
$$\tan^2 x = \sec^2 x - 1$$

each one is useful in a different situation. use the form of these identities to help you figure out which subs. to make.

① $a^2 - x^2$: let $x = a \sin u$.

$$a^2 - x^2 = a^2(1 - \sin^2 u) = a^2 \cos^2 u$$

② $a^2 + x^2$: let $x = a \tan u$

$$a^2 + x^2 = a^2(1 + \tan^2 u) = a^2 \sec^2 u$$

③ $x^2 - a^2$: let $x = a \sec u$.

$$x^2 - a^2 = a^2(\sec^2 u - 1) = a^2 \tan^2 u.$$

<u>Example</u>:
$$\int \frac{dx}{\sqrt{x^2 - 1}}.$$

let $x = \sec u$ $\longrightarrow$ For $x > 1$ take $0 \leq u < \frac{1}{2}\pi$

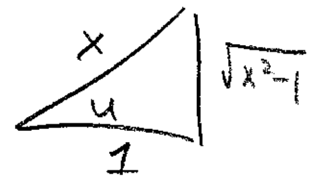
$dx = \sec u \tan u \, du$

For $x < -1$ take $\frac{3}{2}\pi < u \leq \pi$

$$\sqrt{x^2-1} = \sqrt{\sec^2 u - 1} = \tan u$$

$$\int \frac{dx}{\sqrt{x^2-1}} = \int \frac{\sec u \tan u}{\tan u}\, du = \int \sec u \, du$$

$$= \ln|\sec u + \tan u| + C$$



$$= \ln|x + \sqrt{x^2-1}| + C$$

Example: $\int \sqrt{a^2+x^2}\, dx.$

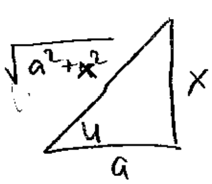
$$x = a \tan u$$
$$dx = a \sec^2 u \, du$$
$$\sqrt{a^2+x^2} = a\sqrt{1+\tan^2 u} = a \sec u$$

$$= \int a \sec u \cdot a \sec^2 u \, du = a^2 \int \sec^3 u \, du$$

$$= \frac{a^2}{2}\left(\sec u \tan u + \ln|\sec u + \tan u|\right) + C$$

$\vdots$

$$= \frac{1}{2} x\sqrt{a^2+x^2} + \frac{1}{2}a^2 \ln|x + \sqrt{a^2+x^2}| - \frac{1}{2}a^2 \ln a + C$$

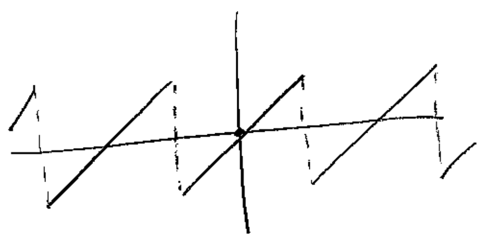


$$\left(\tan u = \frac{x}{a} \quad \sec u = \frac{\sqrt{a^2+x^2}}{a}.\right)$$

Final lecture!

"Fun" applications of integration:

① $(-1/2)! = ?$

② 

what is the equation for this function? ("sawtooth wave")

③ $e^{i\pi} = ?$

---

① Generalization of the factorial function.

Let $f(n) = \int_0^\infty x^n e^{-x}\,dx$.

We see $f(0) = \int_0^\infty e^{-x}\,dx = \left[-e^{-x}\right]_0^\infty = -0-(-e^0) = 1$.

For $n = 1, 2, \ldots$ we can integrate by parts.

Indefinite integral.

$$\int x^n e^{-x}\,dx \qquad \begin{array}{l} u = x^n \\ v' = e^{-x} \end{array} \longrightarrow \begin{array}{l} u' = nx^{n-1} \\ v = -e^{-x} \end{array}$$

$$= x^n(-e^{-x}) - \int nx^{n-1}(-e^{-x})\,dx$$

$$= -x^n e^{-x} + n\int x^{n-1} e^{-x}\,dx.$$

Definite integral:

$$\underbrace{\int_0^\infty x^n e^{-x}\,dx}_{f(n)} = \underbrace{\left[-x^n e^{-x}\right]_0^\infty}_{\text{this is } 0.} + n\underbrace{\int_0^\infty x^{n-1} e^{-x}\,dx}_{f(n-1).}$$

since $\lim_{x\to\infty} \frac{x^n}{e^x} = 0.$

we see that $\begin{cases} f(n) = n f(n-1) \\ f(0) = 1 \end{cases}$

so $f(1) = 1 \cdot f(0) = 1$

$f(2) = 2 \cdot f(1) = 2$

$f(3) = 3 \cdot f(2) = 6$

$f(4) = 4 \cdot f(3) = 24$

$\vdots$

$$f(n) = n \cdot (n-1) \cdots (2)(1) = n!$$

usually the factorial function is only defined for $n = 0, 1, 2, \ldots$ But we can use the integral definition when n is not a nonnegative integer!

For example:

$$(-1/2)! = \int_0^\infty x^{-1/2} e^{-x}\, dx$$

(Guessing game again! what do you think this is?)

let $u = \sqrt{x}$ $\quad du = \frac{1}{2} x^{-1/2}$.

$$= \int_0^\infty 2e^{-x} \cdot \frac{1}{2} x^{-1/2}\, dx = 2 \underbrace{\int_0^\infty e^{-u^2}\, du}$$

We can evaluate this integral with multivariable calculus. (It's related to the normal distribution)

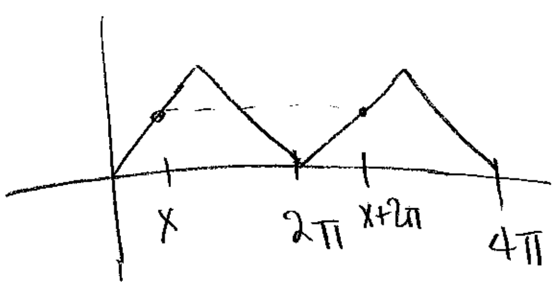
$$= 2 \cdot \frac{\sqrt{\pi}}{2} = \sqrt{\pi}.$$

So: $(-1/2)! = \sqrt{\pi}$ ← this is not a factorial, it is amazement.

Fact: Volume of n-dim unit ball is $(\pi^{n/2})/((n/2)!)$

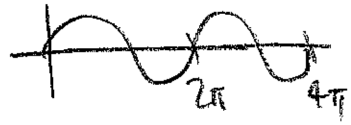
② Formulas for periodic functions:

<u>Def</u>: A function f has period 2π if it satisfies $f(x+2\pi) = f(x)$.



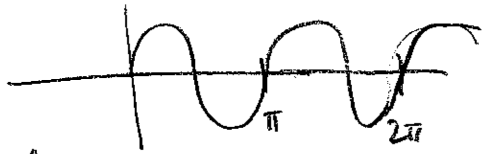
Examples:

- $f(x)=\sin x$ ←



- $f(x)=\cos x$

- $f(x)=\sin 2x$ ←



↖ the "fundamental period" is π, but $\sin 2x$ does have period 2π.

- $f(x)=1$ (constant function).

- If n is any integer,

$\left.\begin{matrix}\sin nx \\ \cos nx\end{matrix}\right\} \to$ both have period 2π.

Fact: If f, g have period 2π, then so does

$c_1 f + c_2 g$ where c_1, c_2 are constants.

e.g. $(3f+5g)(x+2\pi) = 3f(x+2\pi)+5g(x+2\pi)$

$= 3f(x)+5g(x)$

$= (3f+5g)(x)$.

Much more amazing fact:

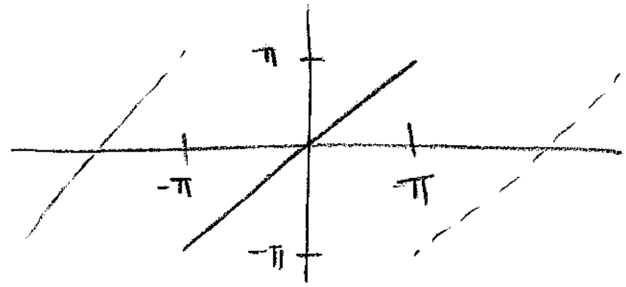
If f is any "nice" function with period 2π,

then f can be written in the form

$$f(x) = a_0 + (b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \dots)$$
$$+ (c_1 \cos x + c_2 \cos 2x + c_3 \cos 3x + \dots).$$

"Fourier series"

Consider the sawtooth wave:



$f(x) = x$ if $x \in (-\pi, \pi)$.
Repeat this periodically.

How do we write the $\underbrace{\text{sawtooth wave}}_{\text{(or any other periodic func)}}$ as a Fourier series?

We make the following remarkable observation.

Consider the following set of functions:

$$\{1, \quad \sin x, \sin 2x, \sin 3x, \dots, \cos x, \cos 2x, \cos 3x, \dots\}.$$

↳ const. function.

① If f and g are any two different functions chosen from the set, then

$$\int_{-\pi}^{\pi} f(x)g(x)\,dx = 0.$$

This is called "orthogonality"

Example: • $f(x)=1 \qquad g(x)=\sin nx$

$$\int_{-\pi}^{\pi} \sin nx \, dx = 0 \quad \text{since } \sin nx \text{ is odd.}$$

• $f(x)=1 \qquad g(x)=\cos nx$

$$\int_{-\pi}^{\pi} \cos nx \, dx = \left[\frac{\sin nx}{n}\right]_{-\pi}^{\pi} = \frac{0}{n} - \frac{0}{n} = 0.$$

• $f(x)=\sin 2x \qquad g(x)=\cos 3x$

$$\int_{-\pi}^{\pi} \sin 2x \cos 3x = 0$$

↳ you did this integral on your homework. You need to use some trig identities.

(2). $\displaystyle\int_{-\pi}^{\pi} \sin^2 nx = \pi \qquad \int_{-\pi}^{\pi} \cos^2 nx = \pi.$ for any $n=1,2,3,\ldots$

(Need to use power reducing identities.)

Now back to

$$f(x) = a_0 + (b_1 \sin x + b_2 \sin 2x + \ldots) + (c_1 \cos x + c_2 \cos 2x + \ldots).$$

Suppose we want to find c_1:

$$\int_{-\pi}^{\pi} f(x)\cos x\,dx = a_0\int_{-\pi}^{\pi}\cos x\,dx$$

$$+\ b_1\int_{-\pi}^{\pi}\sin x\cos x\,dx + b_2\int_{-\pi}^{\pi}\sin 2x\cos x\,dx + \cdots$$

$$+\ c_1\int_{-\pi}^{\pi}\cos x\cos x\,dx + c_2\int_{-\pi}^{\pi}\cos 2x\cos x\,dx + \cdots$$

all the integrals are 0, except $\int_{-\pi}^{\pi}\cos x\cos x\,dx$.

$$= c_1\int_{-\pi}^{\pi}\cos^2 x\,dx$$

$$= \pi c_1$$

$$\longrightarrow \quad c_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos x\,dx$$

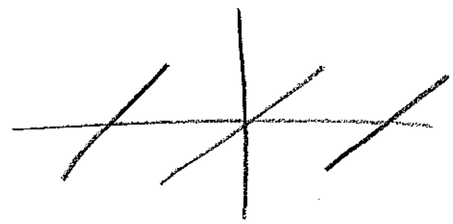
similarly:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$c_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx.$$

For $f(x) =$ sawtooth wave:



$f(x)$ is odd $\Rightarrow \begin{cases} a_0 = 0 \\ C_n = 0 \text{ for all } n. \end{cases}$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx \, dx$$

integrate by parts!

$u = x \longrightarrow u' = 1$

$v' = \sin nx \qquad v = -\frac{1}{n}\cos nx.$

$$\int x \sin nx \, dx = -\frac{x}{n}\cos nx + \int \frac{1}{n}\cos nx \, dx$$

$$= -\frac{x}{n}\cos nx + \frac{1}{n^2}\sin nx + C$$

$$b_n = \frac{1}{\pi}\left[-\frac{1}{n}\left[x\cos nx\right]_{-\pi}^{\pi} + \underbrace{\frac{1}{n^2}\left[\sin nx\right]_{-\pi}^{\pi}}_{\text{zero}}\right]$$

$$= -\frac{1}{n\pi}\left[x\cos nx\right]_{-\pi}^{\pi}$$

$$= -\frac{1}{n\pi}\left[\pi\cos n\pi + \pi\cos n\pi\right]$$

$$= -\frac{2}{n}\cos n\pi$$

$$\cos n\pi = \begin{cases} 1 & \text{if } n \text{ is even} \\ -1 & \text{if } n \text{ is odd} \end{cases} = (-1)^n.$$

so $\quad b_n = -(-1)^n \dfrac{2}{n}$.

$$b_1 = \frac{2}{1} \qquad b_2 = -\frac{2}{2} \qquad b_3 = \frac{2}{3} \qquad b_4 = -\frac{2}{4}, \ldots$$

$$f(x) = \frac{2}{1}\sin x - \frac{2}{2}\sin 2x + \frac{2}{3}\sin 3x - \frac{2}{4}\sin 4x + \cdots$$

$$= \sum_{n=1}^{\infty} \left[-(-1)^n \frac{2}{n}\right] \sin nx$$

we can approximate $f(x)$ by doing a partial sum.

see (and hear) an example of this at

youtu.be/K3D1fPjWAnc.

Fourier series are useful in:

- music (especially electronic music)
- differential equations
- signal processing (e.g. wireless communication).

(3) The exponential function with complex numbers.

<u>Q</u>: What is e^{ix}?

Idea: consider a certain type of ODEs:

"Constant-coefficient linear homogenenous ODE"

<u>Example</u>: $y'' + 3y' + 2y = 0$

Let's guess a solution! $y = e^{\alpha x}$?

$(y = e^{\alpha x} \qquad y' = \alpha e^{\alpha x} \qquad y'' = \alpha^2 e^{\alpha x})$.

$$\alpha^2 e^{\alpha x} - 3\alpha e^{\alpha x} + 2e^{\alpha x} = 0.$$

$$\underbrace{e^{\alpha x}}_{\text{This is never zero}}(\alpha^2 - 3\alpha + 2) = 0.$$

$\Rightarrow \quad \alpha^2 - 3\alpha + 2 = 0$

$(\alpha - 2)(\alpha - 1) = 0$

$\alpha = 2$ or $\alpha = 1$.

So: e^x and e^{2x} are both solutions.

In fact, so is $5e^x - 7e^{2x}$.

So is $c_1 e^x + c_2 e^{2x}$ for any c_1, c_2.

Fact: Any solution to $y''-3y'+2y=0$ is of the form $y=c_1e^x+c_2e^{2x}$.

(This makes sense: 2nd order ODE $\Rightarrow$ 2 degrees of freedom)

Next example: $y''+y=0$.

"simple harmonic oscillator".

$\Rightarrow$ $y=\sin x$ and $y=\cos x$ are solutions.

In fact, any solution is of the form

$y=c_1\cos x+c_2\sin x$.

Now, if $y=e^{\alpha x}$?

$$\alpha^2e^{\alpha x}+e^{\alpha x}=0$$

$$e^{\alpha x}(\alpha^2+1)=0$$

$$\alpha^2+1=0 \Rightarrow \alpha=\pm i$$

So e^{ix} and e^{-ix} are also solutions!

But what are these functions?

We know that any solution is of the form $c_1 \cos x + c_2 \sin x$.

So $e^{ix} = c_1 \cos x + c_2 \sin x$ for some constants.

Let's try to determine c_1, c_2.

Let $x=0$: $\underbrace{e^0}_{1} = c_1 \underbrace{\cos 0}_{1} + c_2 \underbrace{\sin 0}_{0}$

$\Rightarrow \boxed{1 = c_1}$

To get c_2, first differentiate both sides:

$$ie^{ix} = -c_1 \sin x + c_2 \cos x$$

Let $x=0$: $ie^{i0} = -c_1 \sin 0 + c_2 \cos 0$.

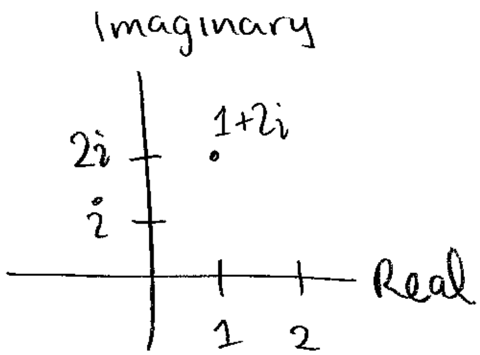
$$\boxed{i = c_2}$$

So we conclude $\boxed{e^{ix} = \cos x + i \sin x}$

Euler's formula.

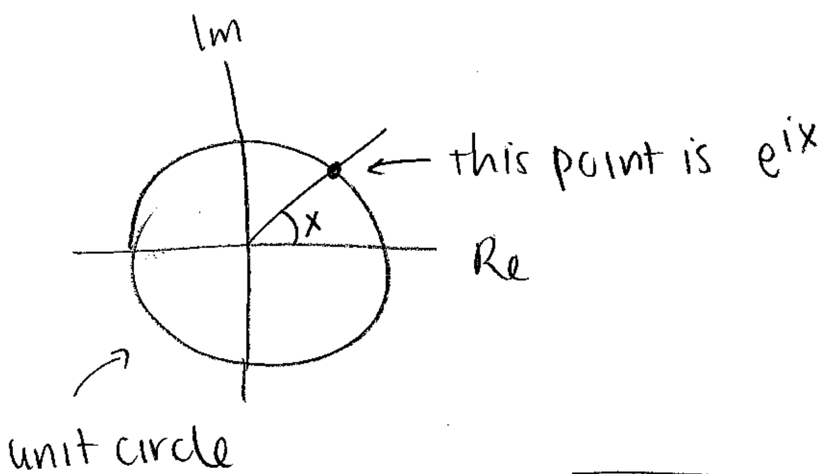
Geometric interpretation.

Recall the complex plane.



$(a,b) \longleftrightarrow a+bi$.

So $\quad e^{i\theta} = \cos\theta + i\sin\theta \longleftrightarrow (\cos\theta, \sin\theta)$.



$\boxed{e^{i\pi} = -1}$ Euler's formula.