Final Exam

DECEMBER 10, 2019
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Please present your solutions clearly and in an organized way. Simplify all your final answers. If an answer box is given, write your final answer in the box. If you run out of room, continue on the extra pages provided at the end. **The use of a calculator is not allowed.** Good luck!! ◡̈

Full Name: Sample Solutions

Student ID:

Question	Points	Score
1	3	
2	12	
3	80	
4	30	
5	10	
6	10	
7	14	
8	5	
9	20	
10	10	
11	4	
Total:	198	

This exam has 11 questions, for a total of 198 points. The maximum possible score for each problem is given on the right side of the problem.

Here are some trigonometric identities.

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$

$$\sin(x-y) = \sin x \cos y - \cos x \sin y$$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

$$\cos(x-y) = \cos x \cos y + \sin x \sin y$$

$$\sin(2x) = 2\sin x \cos x$$

$$\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

$$\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$$

$$\cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$$

$$\sin x \sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$$

$$\cos x \cos y = \frac{1}{2}[\cos(x-y) + \cos(x+y)]$$

$$\sin x \cos y = \frac{1}{2}[\sin(x+y) + \sin(x-y)]$$

$$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

$$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

1. Write down the three Pythagorean trigonometric identities (i.e., the ones involving, $\sin^2\theta$, $\cos^2\theta$, $\tan^2\theta$, etc.). 3

$$\sin^2\theta + \cos^2\theta = 1$$

divide by $\cos^2\theta$

$$\tan^2\theta + 1 = \sec^2\theta$$

divide by $\sin^2\theta$

$$1 + \cot^2\theta = \csc^2\theta$$

2. Fill in the six blanks with the correct capital letters. (Some options may be used more than once.)

$$\begin{array}{llll}
(A) & c < x < c+\delta & (E) & L < f(x) < L+\epsilon \\
(B) & c-\delta < x < c & (F) & L-\epsilon < f(x) < L \\
(C) & c-\delta < x < c+\delta & (G) & L-\epsilon < f(x) < L+\epsilon \\
(D) & 0 < |x-c| < \delta & (H) & 0 < |f(x)-L| < \epsilon
\end{array}$$

(a) Two sided limit: $\lim_{x\to c} f(x) = L$ means 4

"For all $\epsilon > 0$, there exists a $\delta > 0$ such that if $\boxed{D}$, then $\boxed{G}$."

(b) Limit from the left: $\lim_{x\to c^-} f(x) = L$ means 4

"For all $\epsilon > 0$, there exists a $\delta > 0$ such that if $\boxed{B}$, then $\boxed{G}$."

(c) Limit from the right: $\lim_{x\to c^+} f(x) = L$ means 4

"For all $\epsilon > 0$, there exists a $\delta > 0$ such that if $\boxed{A}$, then $\boxed{G}$."

3. Evaluate the following. You can use any method you want.

(a) $\displaystyle\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx = \boxed{\pi/2}$ 10

$$= \left[\arcsin x\right]_0^1 = \arcsin 1 - \arcsin 0$$

$$= \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

(b) $\displaystyle\int \frac{x}{\sqrt{1-x^2}}\,dx = \boxed{-\sqrt{1-x^2} + C}$ 10

$u = 1-x^2$

$du = -2x\,dx$

$$= -\frac{1}{2}\int \frac{du}{u^{1/2}} = -\frac{1}{2}\int u^{-1/2}\,du$$

$$= -\frac{1}{2}\,\frac{u^{1/2}}{1/2} + C$$

$$= -\sqrt{1-x^2} + C$$

(c) $\displaystyle\int_0^1 \frac{x^2}{\sqrt{1-x^2}}\,dx =$ $\boxed{\pi/4}$

10

$x = \sin u$ upper: $1 = \sin u \rightarrow u = \pi/2$

$dx = \cos u\,du$ lower: $0 = \sin u \rightarrow u = 0.$

$\sqrt{1-x^2} = \sqrt{1-\sin^2 u} = \cos u$

$$= \int_0^{\pi/2} \frac{\sin^2 u}{\cos u} \cos u\,du$$

$$= \int_0^{\pi/2} \sin^2 u\,du$$

$$= \int_0^{\pi/2} \frac{1-\cos 2u}{2}\,du$$

$$= \left[\frac{1}{2}u - \frac{1}{4}\sin 2u\right]_0^{\pi/2}$$

$$= \left(\frac{1}{2}\cdot\frac{\pi}{2}\ \ \frac{1}{4}\sin\pi\right) - (0-0)$$

$$= \frac{\pi}{4}$$

(d) $\int \frac{x^3}{\sqrt{1-x^2}}\,dx = \boxed{-(1-x^2)^{1/2} + \frac{1}{3}(1-x^2)^{3/2} + C}$ 10

$x = \sin u$

$dx = \cos u \, du$

$$= \int \frac{\sin^3 u}{\cos u} \cos u \, du$$

$$= \int \sin^3 u \, du$$

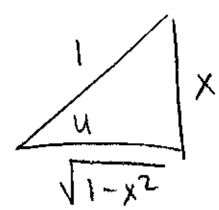
$$= \int (1-\cos^2 u) \sin u \, du$$

$v = \cos u$

$dv = -\sin u \, du$

$$= -\int (1-v^2)\, dv$$

$$= -v + \frac{v^3}{3} + C$$



$\cos u = \sqrt{1-x^2}$

$$= -\cos u + \frac{\cos^3 u}{3} + C$$

$$= -(1-x^2)^{1/2} + \frac{1}{3}(1-x^2)^{3/2} + C$$

(you can also factor this to get

$-\frac{1}{3}(1-x^2)^{1/2}(x^2+2) + C$.)

(e) $\dfrac{d}{dx}\displaystyle\int_0^{x^4} (1+t^2)^5\,dt = \boxed{(1+x^8)^5 \cdot 4x^3}$ 10

$$= (1+(x^4)^2)^5 \frac{d}{dx}(x^4)$$

$$= (1+x^8)^5 \cdot 4x^3$$

(f) $\dfrac{d}{dx}\left[\ln(e^{\sin x})\right] = \boxed{\cos x}$ 10

$$= \frac{d}{dx}[\sin x]$$

$$= \cos x$$

(g) $\int xe^x\,dx = \boxed{xe^x - e^x + C}$ 10

$$u = x \longrightarrow u' = 1$$
$$v' = e^x \qquad v = e^x$$

$$= xe^x - \int e^x \cdot 1\,dx$$

$$= xe^x - e^x + C$$

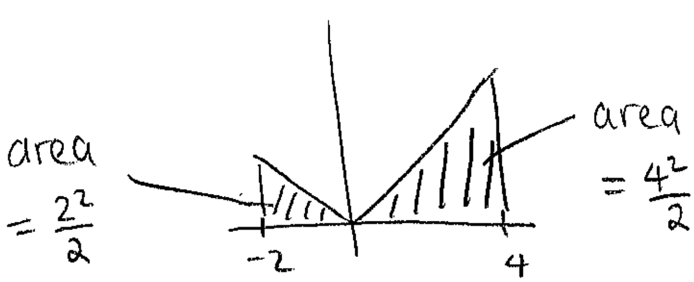
(h) $\int_{-2}^{4} |x|\,dx = \boxed{10}$ 10

Solution 1: $|x| = \begin{cases} x, & \text{if } x > 0 \\ -x, & \text{if } x < 0 \end{cases}$

$$\int_{-2}^{4} |x|\,dx = \int_{-2}^{0} |x|\,dx + \int_{0}^{4} |x|\,dx = \int_{-2}^{0} (-x)\,dx + \int_{0}^{4} x\,dx$$

$$= \left[-\frac{x^2}{2}\right]_{-2}^{0} + \left[\frac{x^2}{2}\right]_{0}^{4} = \frac{4}{2} + \frac{16}{2} = 10$$

Solution 2: Draw a picture.



total area

$$= \frac{2^2}{2} + \frac{4^2}{2} = 10$$

4. Make the required u-substitutions.

   Write down the integral you get after making the substitution. You do not have to simplify or evaluate the integral.

   (a) Make a substitution so that the integrand is a polynomial. There should be no trigonometric functions in your final answer. 10

   (Note that your final answer should be a definite integral.)

$$\int_0^{\pi/2} \sin^{21} x \cos^{60} x \, dx = \boxed{-\int_1^0 (1-u^2)^{10} u^{60} \, du}$$

$$= \int_0^{\pi/2} (\sin^2 x)^{10} \cos^{60} x \sin x \, dx$$

$$= \int_0^{\pi/2} (1-\cos^2 x)^{10} \cos^{60} x \sin x \, dx$$

$$= -\int_1^0 (1-u^2)^{10} u^{60} \, du$$

$u = \cos x$

$du = -\sin x$

upper $= \cos \pi/2 = 0$

lower $= \cos 0 = 1$

   (b) Make a substitution so that the integrand is a polynomial. There should be no trigonometric functions in your final answer. 10

$$\int \sec^{41} x \tan^{81} x \, dx = \boxed{\int u^{40} (u^2-1)^{40} \, du}$$

$$= \int \sec^{40} x (\tan^2 x)^{40} \sec x \tan x \, dx$$

$$= \int \sec^{40} x (\sec^2 x - 1)^{40} \sec x \tan x \, dx$$

$u = \sec x$

$du = \sec x \tan x \, dx$

$$= \int u^{40} (u^2-1)^{40} \, du$$

10

(c) Make a trigonometric substitution.
There should be no square roots in your final answer.

$$\int \frac{x}{\sqrt{x^2+2x-3}}\,dx = \boxed{\int \frac{2\sec u - 1}{2\tan u}\, 2\sec u \tan u \, du}$$

$$x^2+2x-3 = x^2+2x+1-4 = (x+1)^2-2^2$$

$$x+1 = 2\sec u$$

$$dx = 2\sec u \tan u \, du$$

$$\sqrt{(x+1)^2-2^2} = 2\tan u.$$

$$= \int \frac{2\sec u - 1}{2\tan u}\, 2\sec u \tan u \, du$$

10

5. Show that

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5} \quad \le \quad \ln 5 \quad \le \quad \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}$$

(Hint: Think about upper and lower Riemann sums.)

Consider the function $f(x) = \frac{1}{x}$.

and the partition $\{1,2,3,4,5\}$ of $[1,5]$.

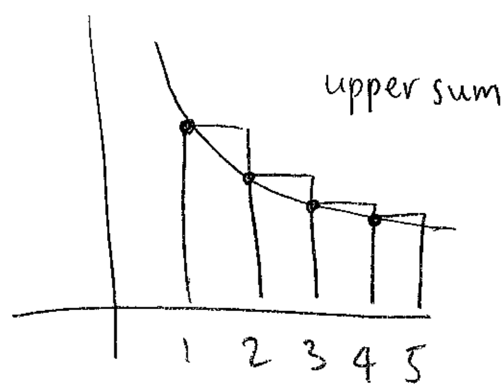
A basic property of the definite integral is

$$L_f(P) \le \underbrace{\int_1^5 f(x)\,dx}_{\text{this is } \ln 5} \le U_f(P) \qquad (*)$$

To find $L_f(P)$ and $U_f(P)$, note that f is a decreasing function so

$$U_f(P) = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$$

(see picture on right)



Similarly, $L_f(P) = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$.

So by $(*)$, we obtain the desired inequality.

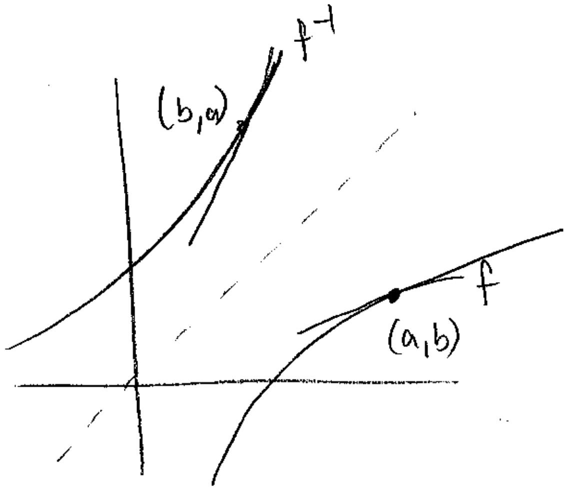
10

6. Suppose f is a one-to-one function, and that $f(a) = b$.

   What is an equation which relates $f'(a)$ and $(f^{-1})'(b)$? Give a geometric interpretation for this equation. Please include a picture.

equation: $\boxed{(f^{-1})'(b) = 1/f'(a)}$

The graph of $y = f^{-1}(x)$ is the reflection of $y = f(x)$ about the line $y = x$.



The tangent line gets reflected too.

The original tangent line has slope $f'(a)$, so the reflected one has slope $\frac{1}{f'(a)}$

7. (a) What are the domain and range of the function $\arctan x$? 4

domain = $(-\infty, \infty)$

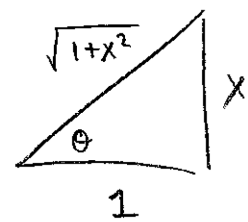
range = $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

(b) Use the formula for the derivative of an inverse to evaluate $\frac{d}{dx}[\arctan x]$. 10

$f(x) = \tan x, \ x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

$$\frac{d}{dx}(\arctan x) = (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

$$= \frac{1}{\sec^2(\arctan x)}$$



$$= \cos^2(\arctan x)$$

$$= \left(\frac{1}{\sqrt{1+x^2}}\right)^2 = \frac{1}{1+x^2}$$

8. A generic nondescript object moves on the x-axis. The velocity at time t is 5

$$v(t) = t\sin(1+t^4).$$

What is the displacement of the object between times $t = -1$ and $t = 1$? (Recall that "displacement" refers to the difference between the initial and final positions.)

displacement = 0

$$= \int_{-1}^{1} t\sin(1+t^4)\,dt = 0$$

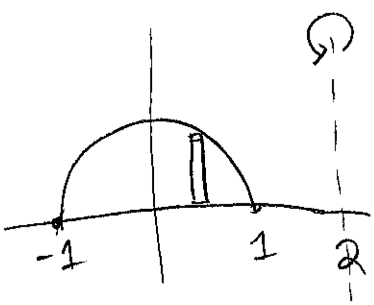
since the integrand is an odd function

9. Consider the region between $y = 0$ and $y = 1 - x^2$.

Write down an integral for the volume of this solids described below. You can use either the washer or shell method. You do not have to simplify or evaluate the integral.

(a) Write down an integral for the solid generated by revolving the region around the vertical line $x = 2$. $\boxed{10}$

integral expression = $\boxed{\int_{-1}^{1} 2\pi(2-x)(1-x^2)\,dx}$



use shell method:

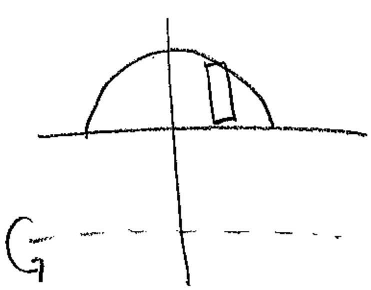
radius $= 2 - x$

height $= 1 - x^2$

$\Delta V \approx 2\pi r \cdot h \cdot \Delta x$

(b) Write down an integral for the solid generated by revolving the region around the horizontal line $y = -2$. $\boxed{10}$

integral expression = $\boxed{\int_{-1}^{1} \pi\left((3-x^2)^2 - 2^2\right)\,dx.}$



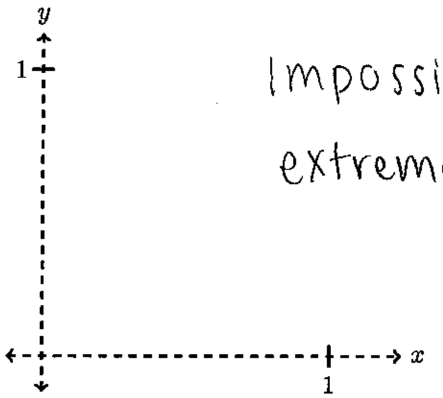
use washer method

outer radius: $1 - x^2 + 2 = 3 - x^2$

inner radius: 2

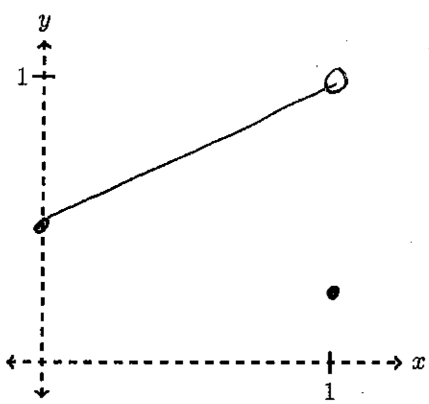
10. Sketch the graph of a function f that is **defined on** $[\mathbf{0}, \mathbf{1}]$ and meets the given conditions. If this is not possible, give a reason why.

(a) f is continuous on $[0, 1]$ $\boxed{5}$
f does **NOT** take on a minimum value



Impossible, by the extreme value theorem

(b) f is one-to-one $\boxed{5}$
f does **NOT** take on a maximum value



11. (There are no incorrect answers to this question.)

(a) Of all the material that we covered in this class, what was your favorite? $\boxed{2}$

integration by parts

(b) Least favorite? $\boxed{2}$

integration by parts

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Name a Pokémon (or something else if you prefer): Is Digimon a Pokémon?