Math 11200/20 homework 6 hints

Problem 6.5. Let $a, b \in \mathbb{Z}$. Show that if $a^2 \mid b^2$, then $a \mid b$.
Hint: use the prime factorizations! Let $a = p_1^{a_1} \cdots p_k^{a_k}$ and $b = p_1^{b_1} \cdots p_k^{b_k}$. (Note that it's okay for exponents to be zero.) If $a^2 \mid b^2$, then what can we say about the exponents?

Hint for Problem 6.5.

- Let $a = p_1^{a_1} \cdots p_k^{a_k}$ and $b = p_1^{b_1} \cdots p_k^{b_k}$. (Here, $p_1, \ldots, p_k$ are distinct primes.)
- Note that $a^2 = (p_1^{a_1} \cdots p_k^{a_k})^2 = p_1^{2a_1} \cdots p_k^{2a_k}$ and similarly, $b^2 = p_1^{2b_1} \cdots p_k^{2b_k}$.
- Suppose that $a^2 \mid b^2$. Then "each prime factor of a^2 must appear in b^2." That is,

$$(*) \qquad 2a_1 \leq 2b_1, \quad 2a_2 \leq 2b_2, \quad \ldots, \quad 2a_k \leq 2b_k.$$

- To complete the proof, we need to deduce from $(*)$ that $a \mid b$. That is, "each prime factor of a must appear in b." Why is this true?

Problem 6.6. Let $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. (These are the Fibonacci numbers.) Show that

$$F_2 + F_4 + F_6 + \cdots + F_{2n} = F_{2n+1} - 1$$

(Hint: Recall in class that $1 + 2 + 4 + \cdots + 2^n = 2^{n+1} - 1$.)

Hint for Problem 6.6. Suppose we want to find $1 + 2 + 4 + 8 + 16$. Let x be the answer. Then

$$\begin{aligned} 1 + x &= 1 + 1 + 2 + 4 + 8 + 16 \\ &= 2 + 2 + 4 + 8 + 16 \\ &= 4 + 4 + 8 + 16 \\ &= 8 + 8 + 16 \\ &= 16 + 16 \\ &= 32 \end{aligned}$$

so $x = 32 - 1$. Try something similar for the Fibonacci sum above. Don't start with the general case. Try a specific case first. For example, for $n = 3$, we want to show $F_2 + F_4 + F_6 = F_7 - 1$. Can you show this?