Math 11200/20 homework 4
Due date: Friday, October 21, 2016

Note: You shouldn't need to use a calculator for these problems.

Please present your solutions clearly and in an organized way. Think of it this way: if you show it to another student in this class, he/she should be able to understand it without needing to ask you questions.

Problem 4.1. In Friday 10/14's class, we proved that for any $a, b \in \mathbb{Z}$, we have $(a, b) = (a+b, b)$. Take the proof we gave and modify it slightly to show the following: "for any $a, b, c \in \mathbb{Z}$, we have $(a, b) = (a + cb, b)$."[1]

Problem 4.2. If we divide 1001 by 105, the quotient is 9 and the remainder is 56. So $1001 = 9 \cdot 105 + 56$. Now use Problem 4.1 to determine $(1001, 105)$. (What should a, b, c be?)

Problem 4.3. Exercise 3.12 in the textbook. (Hint: use the theorem we proved in class on Friday 10/14.)

Problem 4.4. Exercise 4.22(a),(c) in the textbook. You don't need anything from Chapter 4 to do this problem. This problem is intended to be fun(???) and to show an example where the prime factorization is not unique.

Problem 4.5. Let H be as in Exercise 4.22. Show that every element of H is divisible by some Hilbert prime. (Hint: if you understand the proof on Monday, you should be able to do this problem.)

Problem 4.6.

(a) Recall Problem 4.2. Since $1001 = 9 \cdot 105 + 56$, we know by Problem 4.1 that $(1001, 105) = (105, 56)$. We can repeat this procedure with 105 and 56: $105 = 1 \cdot 56 + 49$, so $(105, 56) = (56, 49)$. If we keep repeating this, what do we get in the end?
(b) Can you try the same thing for $(55, 34)$?

Problem 4.7. As we'll see later, this problem is related to finding multiplicative inverses in $\mathbb{Z}_n$. However, it might be a good idea to do some computations now to get a feel for what is going on.

(a) What is the smallest possible **positive** number of the form $50x + 70y$ that you can make, where x, y are integers? For example, if we let $x = -1, y = 1$, then $50x + 20y = 50(-1) + 70(1) = 20$. Is 20 the smallest? No, because if we let $x = -4, y = 3$, then we get 10, which is smaller than 20. Is 10 the smallest? If not, please find the smallest number. Show that the smallest number is indeed the smallest.
(b) Do the same with $4x + 36y$. (No need to show your computations, but please show that the smallest number you find is indeed the smallest.)
(c) Do the same with $4x + 36y$.
(d) Do the same with $8x + 36y$.
(e) Do the same with $3x + 10y$.
(f) Do the same with $7x + 16y$.
(g) Do you notice a pattern? What do you think the answer is for $ax + by$, where a, b are any two positive integers? (No need to prove anything.)

[1] This is Theorem 3.7, part 2 in the textbook. I guess you could copy the proof there, but the point of this exercise is to try to work it out yourself.