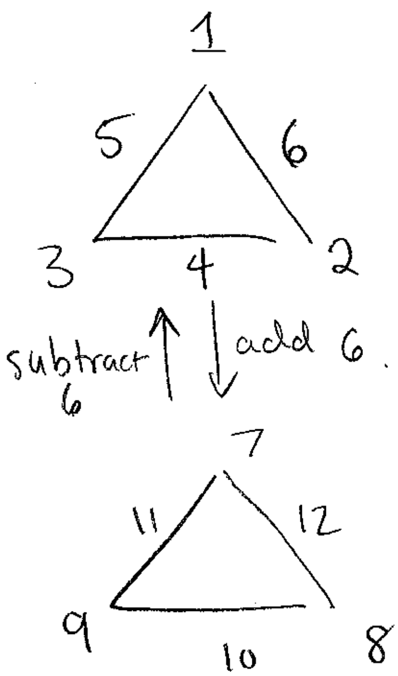
HW 1 sample solutions

[comments are written in brackets]

1.2(a)

(Solution 1) If we take any solution with $\{1, \ldots, 6\}$ and add 6 to every number, we get a solution with $\{7, \ldots, 12\}$. This is because the side sum changes from x to $x+6+6+6 = x+18$. We can also go backwards, by subtracting 6.



$\Longrightarrow$ The # of solutions with $\{1, \ldots, 6\}$ is the same as the # with $\{7, \ldots, 12\}$.

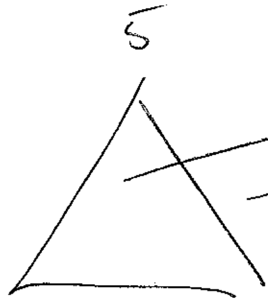
$\boxed{\text{Answer} = 24}$.

(solution 2) [read through chapter 0. You can apply the same arguments there to solve this problem]

!! Note: writing this is <u>not</u> a solution!

1.2 (b) There are 2 cases.

Case 1: 5 is on a corner



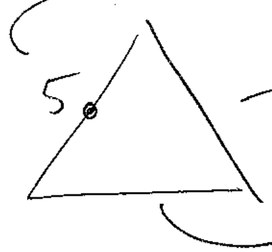
the products on these two sides are divisible by 5.

this product is not divisible by 5.

so the the products cannot be the same.

Case 2: 5 is on a side

this product is div by 5



these two are not

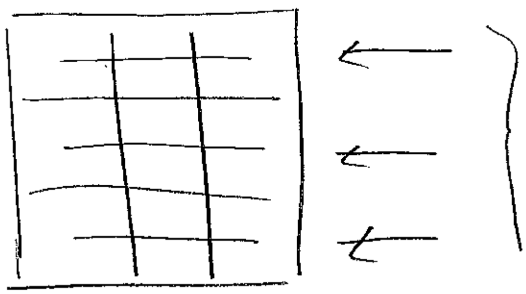
So the products cannot be the same

Conclusion: there are no solutions.

1.3 (solution 1)

Claim 1: The common sum must be 15.

Justification: Let S = common sum



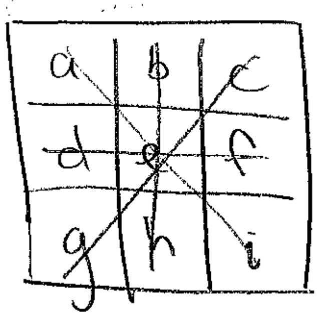
the three row sums are all S.

If we sum up the sum of all 3 rows, we get $3S = 1+2+\ldots+9 = 45$

so $S = 15$. $\square$

Claim 2: The middle number must be 5.

Justification: [more algebraic method]



look at these four sums:

$a+e+i = 15$
$b+e+h = 15$
$c+e+g = 15$
$d+e+f = 15$

Sum them up to get

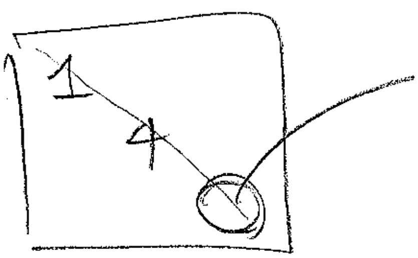
$\underbrace{a+b+c+d+e+f+g+h+i}_{\text{this is } 1+2+\dots+9=45.} + 3e = 4\cdot 15 = 60$

so $\quad 45+3e=60 \implies e=5$ $\quad\square$.

[less algebraic method]

suppose the center was 4:

we need to put 1 somewhere.



the number opposite from 1 has to be

$15-4-1=10$.

Not possible.

so it doesn't work with 4 in the middle.

Same argument works for any number smaller than 4.

Now suppose the center was 6:

We need to put 9 somewhere.

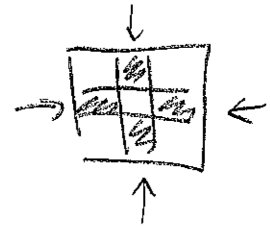
The number opposite has to be $15-6-9=0$.

Not possible

So it doesn't work with 6. same arg. shows it doesn't work for ≥ 6.

So 5 must go in the middle. $\square$

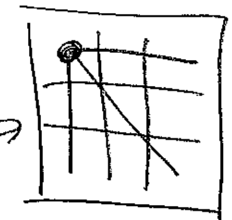
Claim 3: 1 must go on the side



Justification: There are only two ways to get 15 using 1:

$$(*)\begin{cases} 1+5+9 \\ 1+6+8 \end{cases}$$

So it cannot go in the corner



3 sums

$\square$

Because of symmetry, it doesn't matter which side we put 1.

6,8 must go here, by (*). by symmetry it doesn't matter which

1	5	9

$\rightsquigarrow$

6		
1	5	9
8		

$\downarrow$ now just fill in the rest

6	7	2
1	5	9
8	3	4

It works!

So we showed this works,

6	7	2
1	5	9
8	3	4

and every solution must be a rotation or flip of this one.
(this is b/c of the symmetry we used).

- If we rotate, we get 4 solutions
- If we flip and then rotatate, we get 4 solutions

$\longrightarrow$ total = 8 solutions

(solution 2)

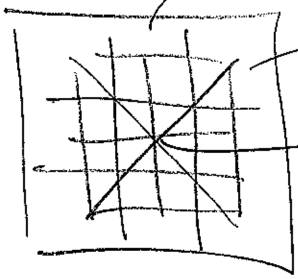
write out <u>all</u> the ways to get 15.

$1+5+9$
$1+6+8$
$2+4+9$
$2+5+8$
$2+6+7$
$3+4+8$
$3+5+7$
$4+5+6$

<u>Note</u>: We made this list with the smallest numbers first, so we did not miss a way of getting 15.

- appears 4 times: 5
- " " 3 times: 2, 4, 6, 8
- " " 2 times: 1, 3, 7, 9

Next, observe



side: belong to 2 sums
corner: - - - 3
center: - - - 4.

so 5 must go in the center
2, 4, 6, 8 - - - - - - corners
1, 3, 7, 9 - - - - - - sides.

Let's put 5 first. Then let's put 2
By symmetry it doesn't matter which corner

Let's just put it here

2		
	5	
		8

4, 6 must go in other two corners it doesn't matter which (by symmetry)

2		4
	5	
6		8

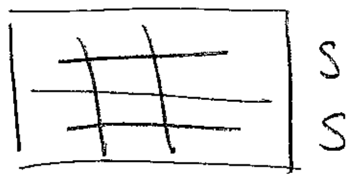
only one way to fill in the rest

2	9	4
7	5	3
6	1	8

[now, using the same argument as in solution 1, we can conclude there are 8 solutions].

$\boxed{1.4}$ (solution 1)

2×3:



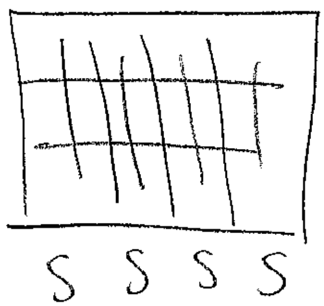
Let S = common sum

by same argument as 1.3,

$$2S = 1 + 2 + \dots + 6 = 21 \implies S = 10.5$$

impossible.

3×4



sum across columns

$$4S = 1 + 2 + \dots + 12 = 78$$

$$\implies S = \frac{39}{2} \quad \text{impossible}$$

4×5

sum across rows

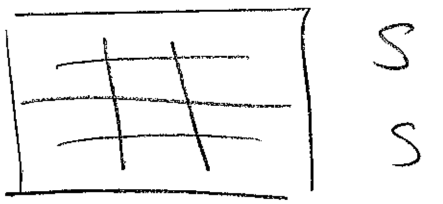
$$4S = 1 + 2 + \dots + 20 = 210$$

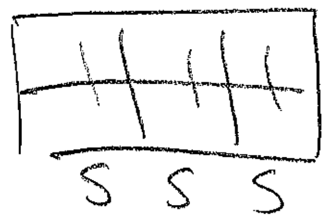
$$\implies S = \frac{105}{2} \quad \text{impossible.}$$

So there are no solutions.

(solution 2).

For 2x3: Let S = common sum





sum across rows

sum across columns

$$2S = 1 + 2 + \dots 6 = 3S.$$

so $2S = 3S. \Rightarrow S = 0.$

impossible.

same argument works for 3x4, 4x5.