WHAT IS PREDICATIVISM?

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Predicativism is a foundational philosophy that was developed by Poincaré, Russell, and Weyl. It was originally conceived in response to the discovery of the classical paradoxes of naive set theory.

1. The vicious circle principle

The official slogan of predicativism is what Russell called the "vicious circle principle": no class can contain elements which are definable only in terms of the class itself. Another way to say this is that we do not accept any entity which cannot be defined without referring to a class to which it belongs. It is hard to make this principle precise, but the underlying sentiment is that the elements of any collection should be understood as in some way logically preceding the collection itself. Operationally, this means that the comprehension axiom

$$(\exists x)(y \in x \leftrightarrow \phi(y)) \tag{$*$}$$

(informally: the set $\{y : \phi(y)\}$ exists) is not assumed to hold in general. We are typically allowed to infer the existence of the set x in $(*)$ only if all variables in the predicate $\phi(y)$ are restricted to range over sets whose existence has previously been established.

Ramsey gave "the tallest man in a group" as an example of a kind of circularity that should not be forbidden ([5], p. 41). Here an individual (the tallest man) is identified by means of a condition which quantifies over a collection (the group) to which that individual belongs. There is clearly nothing wrong with this definition because it merely selects an individual from a preexisting collection. The predicativist worry arises only when one is constructing new, previously unavailable individuals — hence the ban on elements which are definable *only* in terms of a class to which they belong, which is not the case in Ramsey's example. (There are surely other, non-circular descriptions of the man who happens to be the tallest.)

This shows that predicativism is a kind of constructivism. If the entire universe of sets were available from the start, then set definitions could always be interpreted as performing a selective function as in Ramsey's example, and there would be no need to prohibit any comprehension axioms. The vicious circle principle is only reasonable if we believe that sets have to be constructed in some sense.

2. The natural numbers

But unlike other constructivists, most predicativists have seen nothing wrong with infinite constructions. For instance, Russell said that running through the entire expansion of π is not logically impossible, but merely "medically" impossible ([7], p. 143). Weyl somewhat more obscurely asserted that "the intuition of iteration assures us that the concept 'natural number' is extensionally determinate" ([15],

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p. 110). Thus, the cogency of countable constructions is alleged to derive from our basic mathematical intuition.

It is tempting to suppose that the constructibility of the natural numbers follows directly from the basic idea of predicativism. After all, one might say, we can surely build up ω (taken to be the set of finite von Neumann ordinals) in a step-by-step fashion; each natural number is fashioned out of smaller natural numbers, so there is no circularity. But this presumes the cogency of countably long constructions, and that issue is completely independent of the vicious circle principle. Thus, the legitimacy of the natural numbers as a set is *not* entailed by the idea that sets are to be built up from below.

Having noted that there are weaker forms of predicativism which do not admit any infinite constructions, I will from here on use the word "predicative" in its primary historical sense according to which constructions of length ω are accepted.

3. The power set of ω

In contrast, predicativism does not sanction power sets of infinite sets, because if the power set of x is not already available then there is no sensible way to restrict the range of the variable y in the definition

$$\mathcal{P}(x) = \{y : y \subseteq x\}.$$

It has to range over the entire universe of sets, violating the vicious circle principle. If x is finite then we can explicitly list its subsets, but we have no analogous procedure when x is infinite. Weyl's "intuition of iteration" is clearly not sufficient here.

The circularity inherent in infinite power sets can be illustrated in a sharper way ([11], p. 11). Let (A_n) enumerate all the sentences in the language of second order arithmetic. This is a language with two kinds of variables, one type representing individual numbers and the other representing sets of numbers. Now define

$$S = \{n : A_n \text{ is true}\}.$$

Observe that for any given n, if A_n contains any set variables then in order to diagnose whether it is true we need to quantify over $\mathcal{P}(\omega)$. If the entire power set of ω is not already available then we may not be able to determine the truth value of each A_n and hence we are not in a position to construct S. In this sense, S is a set of natural numbers that only becomes available after we have all sets of natural numbers. Thus it is impredicative. Of course it is conceivable that the set S might also be definable in some totally different, predicatively acceptable way. But we have no reason to assume this is the case.

Thus, we reject any definition of a set of natural numbers which involves quantification over all sets of natural numbers. Should we also reject any definition of a single natural number which involves quantification over all natural numbers? No, because we have already agreed that ω is constructible in principle. Once ω is available, there is no problem with a definition of a number that involves quantification over ω; like Ramsey's "tallest man in a group", such a definition does not construct anything, it merely selects an individual out of the set ω.

The power set of ω is different, according to the predicativist, because we have no comparable intuition for building up $\mathcal{P}(\omega)$ in a step-by-step fashion. Therefore $\mathcal{P}(\omega)$ is not available in full and hence in general we have no choice but to interpret definitions of sets of numbers in constructive, not selective, terms.

It may be suggested that the axiom of choice entails that $\mathcal{P}(\omega)$ can be well-ordered, and that this gives us a picture of how it can be built up one element at a time. But the well-ordering theorem cannot be proven without using the power set axiom, which is precisely what is in question here. So this argument is not well-taken.

4. Predicative mathematics

The illegitimacy of infinite power sets has the striking consequence that in predicative mathematics apparently all sets are countable. All known methods for producing uncountable sets either explicitly rely on the existence of infinite power sets or else involve the same sort of circularity. On the other hand, since we do accept countable constructions, within the realm of countable mathematics predicativism is practically indistinguishable from platonism.

Structures such as the power set of ω or the real line are not absolutely barred from predicative mathematics, however. They may be regarded as proper classes. Although we cannot construct the real line in its entirety, we can produce successively richer countable approximations: the rationals, the algebraic numbers, the recursive reals (any real number whose binary expansion is the output of some finite computer program; this includes transcendental numbers like π and e), the arithmetical reals (any real number whose binary expansion is determined by the truth values of some formula $\phi(n)$ of first order arithmetic; this includes exotica like Chaitin's constant), and so on.

As a general rule of thumb, uncountable classical structures which are separable for some reasonable topology can usually be predicatively recast as proper classes. For instance, the sequence space l^∞ is separable for the weak* topology, and using a bijection between ω and $\omega \times \omega$ we can convert a single real number into a sequence of reals, and vice versa, with minor adjustments to handle nonuniqueness of expansions. Thus l^∞ is predicatively on a par with the real line. The same is true of any of the standard Banach spaces which are commonly used in modern analysis.

It is widely believed that very little of modern mathematics is predicatively valid. This is not true. In fact it is not so easy to think of a really important classical mathematical structure which cannot be predicatively modelled as, at worst, a proper class. Probably the best example is $\beta\omega$, the Stone-Čech compactification of the natural numbers. Even in this case, individual elements of $\beta\omega$ (i.e., ultrafilters on ω) are predicatively available as proper classes. (Of course, we need some choice principle to ensure the existence of nontrivial ultrafilters.)

A separate question is how much we can prove about these structures in a predicatively acceptable way. The answer is that the deductive strength of predicative mathematics is without question far weaker than that of platonist mathematics, in the technical sense measured by proof-theoretic ordinals. However, it is still the case that the overwhelming bulk of mainstream mathematics is, with inessential modifications, predicatively valid. This has been verified extensively; see, e.g., [9] or [14].

5. Predicative well-ordering

We think of the predicative universe as being available in stages, and it has generally been assumed that these stages should be well-ordered. Thus, in order to better understand the extent of predicative mathematics, we need to consider the well-ordering concept from a predicative point of view.

This is slightly subtle. According to the usual definition, a totally ordered set x is well-ordered if every nonempty subset has a smallest element. Since this condition involves quantifying over the power set of x, which is effectively a proper class if x is infinite, is it even predicatively meaningful? Yes. For instance, take $x = \omega$ with its standard ordering. Even though we have no way to, as it were, separately inspect each nonempty subset of ω, we can still grasp the general principle that any nonempty subset must have a smallest element. This is clear from the intuition of iteration: if we were to search through any given infinite sequence of 0's and 1's, either we would find them all to be 0's or there would be an earliest point at which we would locate a 1. We conclude that, although we cannot generally construct sets of numbers using conditions which quantify over all sets of numbers (see Section 3), in at least some circumstances we can still regard such conditions as intelligible.

But the well-ordering property is even more subtle than this. The way it is typically used is in induction arguments: we have a well-ordered set x and a predicate ϕ, we know that

$$(\forall a \in x)((\forall b < a)\phi(b) \to \phi(a)),$$

and we want to conclude that $(\forall a \in x)\phi(a)$. Classically there is no problem in drawing this inference; just let $y = \{a \in x : \neg\phi(a)\}$, suppose that y is nonempty, use well-ordering to identify a smallest element of y, and deduce a contradiction. But this argument is generally not predicatively valid. Because we lack strong comprehension axioms, in general we cannot form the set y. Consequently, predicativists have trouble making induction arguments on well-ordered sets.

The obvious response to this difficulty is to change our definition of well-ordering. Say that x is *weakly well-ordered* if, as before, every nonempty subset has a smallest element, and *strongly well-ordered* if, whenever ϕ is any predicate such that $\phi(a)$ holds for some $a \in x$, there is a smallest a such that $\phi(a)$ holds. Under the hypothesis of strong well-ordering we would then recover our ability to carry out induction arguments for arbitrary predicates. Unfortunately, this hypothesis is not predicatively meaningful because the predicate "is strongly well-ordered" is defined by means of a condition which quantifies over all predicates. This violates a version of the vicious circle principle, and hence the definition of strong well-ordering is impredicative.

For any well-defined set $\mathcal{C}$ of predicates, we can legitimately define a predicate "is $\mathcal{C}$-well-ordered" to mean that induction holds for every $\phi \in \mathcal{C}$. But this new predicate will typically not itself belong to $\mathcal{C}$. So what we have is an open-ended hierarchy of well-ordering concepts.

6. The constructible universe

Gödel's "constructible hierarchy" (L_α), with α ranging over all ordinals, is defined by setting $L_0 = \emptyset$, $L_\alpha = \bigcup_{\beta<\alpha} L_\beta$ at limit stages, and

$$L_{\alpha+1} = \mathrm{def}(L_\alpha)$$

at successor stages. Here $\mathrm{def}(x)$ consists of all the subsets of x which can be defined by a set-theoretic formula with parameters from x and with variables restricted to range over x. Gödel viewed $L_{\alpha+1}$ as comprising the sets which are immediately predicatively accessible once L_α is available, and this interpretation has been generally accepted.

It is therefore reasonable to suppose that all predicative sets appear somewhere in the constructible hierarchy. This amounts to a kind of Church-Turing thesis, to the effect that there are no other predicative means of set construction besides those realized by iterating the definability operation. However, this conclusion has an important qualification: the extent of the predicative universe depends on which well-orderings of ω are predicatively available. In order to get up to L_α we first need to have some well-ordering of ω of order type α. (Cf. the discussion in Section 2 about needing to separately assume the cogency of constructions of length ω. We can restrict to countable indices because, as we noted in Section 4, predicatively all sets are countable.) In this connection it is important not to conflate different versions of the well-ordering concept (see Section 5), a point that has been badly underappreciated in the literature, as we will discuss below.

It now appears that the construction of the predicative universe involves a kind of feedback loop. We use well-orderings of ω in order to build up our total repertoire of sets, but as this repertoire expands new well-orderings become available. Wang was the first to think about this issue seriously. He suggested that since all recursive sets of natural numbers appear in $L_{\omega+1}$, we should consider all recursive well-orderings of ω to be predicatively available ([10], p. 47). Thus the predicative universe should be at least as large as

$$L_{\omega_1^{CK}} = \bigcup_{\alpha<\omega_1^{CK}} L_\alpha,$$

where ω_1^{CK} is the supremum of the recursive well-orderings of ω. Conversely, by a result of Spector, every well-ordering of ω appearing in $L_{\omega_1^{CK}}$ is recursive ([6], Corollary XXXVI). So as the predicativist works his way up to $L_{\omega_1^{CK}}$, he will never encounter a well-ordering that would enable him to get beyond that point. Thus, Wang tentatively proposed that the predicative universe of sets could be identified with $L_{\omega_1^{CK}}$.

Kreisel [3] criticized this conclusion on the grounds that in order for L_α to be considered predicatively available, it is not enough that the predicativist merely possess an ordering of ω isomorphic to α; he must also know that it is well-ordered. That is, there must be some predicatively valid formal system in which he is able to prove that the ordering is well-founded. Since, for example, first order Peano arithmetic only proves the well-foundedness of orderings with order types less than the (quite small) ordinal ε_0, this dramatically restricts the extent of the predicative universe. On the other hand, Spector's limitation does not apply at ε_0. Thus Kreisel's criticism leads to the notion of autonomous progressions of formal theories.

7. Autonomous progressions

Feferman and Schütte made Kreisel's ideas precise and came to the conclusion that an ordinal known as Γ_0 is the limiting ordinal of predicative reasoning [1, 8]. The simplest version of their analysis involves infinite formulas and proofs [4]. We work with a language in which formal expressions may include infinite conjunctions or disjunctions of other formulas. Thus formulas in general are represented by labelled well-founded trees. Also, given an infinite family of formulas we may infer their conjunction, so proofs generally also take the form of labelled well-founded trees.

The basic step by which a predicativist is supposed to bootstrap his way up to larger ordinals goes like this. Suppose that formulas and proofs of height α are available and accepted. We then give a proof of height α which establishes that some recursive total ordering of ω of a larger order type α' is well-founded. This gives us access to formulas and proofs of height α', and the process can now be repeated. Γ_0 is the supremum of the ordinals that can be reached in a finite number of steps, if we start with $\alpha = \omega + 1$.

The idea that a predicativist could apply the basic step any finite number of times but not infinitely many times is dubious on its face. But the really fatal objection to the preceding account is that in fact the predicativist could not even make the stated inference once. This is because Feferman and Schütte only prove that the order type of α' is weakly well-ordered, and from this premise we cannot infer the soundness of proof trees of height α'. We cannot carry out the necessary induction argument. Thus, the Feferman-Schütte analysis falls on a failure to distinguish weak and strong well-ordering.

Evidently aware of this error ([2], p. 85: "the well-ordering statement ... on the face of it only impredicatively justifies the transfinite iteration of accepted principles up to a"), Feferman subsequently proposed a variety of formal systems intended to model predicative reasoning, all of which have proof theoretic ordinal Γ_0. But, as was exhaustively documented in [13], each of these systems includes one obviously impredicative axiom whose function is to enable the inference from weak well-ordering to the validity of Σ^1_1 induction.

In [13] an attempt was made to rehabilitate the idea of autonomous progressions using a system which employs a transfinite hierarchy of Tarskian truth predicates. This system gets well beyond Γ_0, up to the small Veblen ordinal $\phi_{\Omega^\omega}(0)$, and could be modified to go further. (How much further is unclear.) Although I am not aware of any error in this analysis, it is perhaps too soon to definitively say whether the effort was successful. If not, then the maximal strength of known predicative theories would not go much beyond first order Peano arithmetic.

8. KPω and CZF

The essential impredicativity of Zermelo-Fraenkel set theory (ZFC) lies in the power set axiom. Some commentators have also located a problem in the use of unbounded formulas in the separation scheme, but this is only a concern if we are reasoning constructively. If we are using classical logic, then having granted the existence of power sets, we should regard the separation axioms as performing a selective rather than a constructive function, and hence as not violating any distinct predicativity restriction. Once we accept power sets, the separation axioms are only circular in the same unobjectionable way as Ramsey's "tallest man" example (see Section 1).

However, since predicativists do not accept power sets, it is true that ZFC can only be made predicative by either weakening or eliminating both the power set axiom and the separation scheme. The obvious way to predicatively weaken the separation scheme is by restricting it to formulas with bounded quantifiers.

Kripke-Platek set theory with infinity (KPω) and Aczel's Constructive Zermelo-Fraenkel set theory (CZF) both weaken separation in this way, and both have been widely considered to be predicative systems. However, they are not, because both systems also strengthen the foundation axiom to a "set induction" scheme consisting of the axioms

$$(\forall x)[(\forall a \in x)\phi(a) \to \phi(x)] \to (\forall x)\phi(x)$$

for all formulas ϕ. What could the predicative justification for these axioms be? Granting that every set is well-founded, it might seem that we should be able to inductively infer $(\forall x)\phi(x)$ whenever the premise of a set induction axiom holds. But a moment's thought shows that in order to do this we need to use a comprehension axiom which, for most formulas ϕ, is predicatively unavailable. This is just the weak versus strong well-ordering issue we discussed in Section 5. Thus, the predicative acceptability of the set induction scheme fails for the same reason the Feferman-Schütte analysis fails, because predicatively there is a difference between weak and strong well-ordering.

The problem is fatal. If we take the universe of sets to be the union of the L_α as α ranges over the order types of weak well-orderings of ω, then we lack the inductive power to justify set induction for arbitrary formulas. We cannot even prove that L_α exists for all such α. We could try to restrict α to range over the order types of $\mathcal{C}$-well-orderings of ω, where $\mathcal{C}$ is some class of predicates which is broad enough to support the construction of the L_α. But this could never succeed in justifying set induction because the meanings of the formulas ϕ in the set induction scheme depend on what we take to be the universe of sets. That is, in order to make sense of these formulas we need to first specify what range of values of α are permitted for constructing sets. So we cannot use these formulas to restrict the range of α. That would be circular, i.e., impredicative.

The proposal that theories of generalized inductive definitions should be considered predicative is refuted in a similar way ([13], p. 22).

9. Conclusions

What, then, should we consider to be the standard predicatively valid formal systems? This depends on the setting:

- first order arithmetic — Peano Arithmetic (PA)
- second order arithmetic — Friedman's ACA
- set theory — KPω with Δ_0-foundation

The system CM presented in [14] should also be mentioned here. It is similar to ACA but it formalizes core mathematics more smoothly because it includes third order variables which play the role of proper classes (so its relationship to ACA is something like the relationship of NBG to ZFC). Current understanding of the limits of predicativity is represented in [13].

Because of the constructive nature of predicativism, a case can be made that intuitionistic logic should be used when reasoning predicatively. However, since countable constructions are accepted we should always at least have the law of excluded middle for arithmetical formulas, even when reasoning constructively.

The literature on predicativism has suffered from two major errors. The first is a gross exaggeration of its limitations relative to ordinary mathematical practice. This was initially a reasonable concern, but as early as the 1950s it was clear that essentially all bread-and-butter mathematics is predicative (see [12] for references). Nonetheless, the foundations community has been adamantly unreceptive to this point, while simultaneously being highly credulous toward fanciful claims to the contrary (e.g., that impredicative mathematics is needed in quantum field theory).

The second pervasive error, as highlighted in this article, has centered on the concept of well-ordering. Although it is well-known among proof theorists that in the absence of strong comprehension axioms the notion of well-ordering stratifies into a hierarchy of levels, this knowledge has not been critically applied. Both the Feferman-Schütte analysis and the claim that CZF is predicative remain widely accepted among a population that is quite aware of the preceding phenomenon. Nor has the fact that these positions contradict each other been a source of real concern, as workers have been happy to agree that in some vague way they reflect "different notions of predicativism".

At this point, the word "predicative" has been so badly and persistently misused that the best course of action for actual predicativists may be to simply abandon it in favor of a new term [11].

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