Math 132  
Spring 2013  Final Exam

1. Let $F(x) = \int_{0}^{x} \frac{2 \ln(1 + t^2 + t^3)}{\ln(1 + \sqrt{t})} \, dt$. Calculate $F'(4)$, the derivative of $F(x)$ at $x = 4$.

   a) 0       b) 1       c) 2       d) 3       e) 4
   f) 5       g) 6       h) 7       i) 8       j) 9

Solution: i

> F := (x) -> Int(2*ln(1+t^2+t^3)/ln(1+sqrt(t)), t = 0 .. x);

> D(F)(x);

# This is calculated using the Fundamental Theorem of Calculus
# without any integration

> derivative := D(F)(4);

Answer = simplify( derivative );

2. Calculate $\int_{1}^{e} \frac{12 \sqrt{1 + 8 \ln(x)}}{x} \, dx$.

   a) 21       b) 22       c) 23       d) 24       e) 25
Solution: f

\[
J := \int_1^e \frac{12 \sqrt{1 + 8 \ln(x)}}{x} \, dx
\]

\[
K := \text{changevar}(u = 1 + 8 \ln(x), J, u);
\]

\[
K = \text{value}(K);
\]

Verification using Maple's built-in integrator

\[
J = \int_1^e \frac{12 \sqrt{1 + 8 \ln(x)}}{x} \, dx = 26
\]

3. Calculate

\[
\int_1^2 \frac{4x + 3}{x(x + 1)} \, dx
\]
Solution:  g

\[
J := \int_{1}^{2} \frac{4x + 3}{x(x + 1)} \, dx
\]

\[
\text{processedIntegrand} := \frac{3}{x} + \frac{1}{x + 1}
\]

antiderivative := int(processedIntegrand, x);

\[
\text{antiderivative} := 3 \ln(x) + \ln(x + 1)
\]

answer := simplify(subs(x = 2, antiderivative) - subs(x = 1, antiderivative));

\[
\text{answer} := 2 \ln(2) + \ln(3)
\]

Display\_answer = combine(answer, ln);

\[
\text{Display\_answer} = \ln(12)
\]

Verification using Maple's built-in integrator:

\[
\text{Int}((4x+3)/x/(x+1), x = 1 .. 2) = \text{int}((4x+3)/x/(x+1), x = 1 .. 2);
\]
\[
\int_1^2 \frac{4x + 3}{x(x + 1)} \, dx = 2 \ln(2) + \ln(3)
\]

\[\text{testeq(} 2*\ln(2)+\ln(3) = \ln(12) \); true

4. Calculate
\[
\int_0^1 \frac{4x^2 + x + 4}{(x^2 + 1)^2} \, dx.
\]

a) \(\pi + \frac{1}{4}\)  b) \(\pi + \frac{1}{2}\)  c) \(\pi + 1\)  d) \(2\pi + \frac{1}{4}\)  e) \(2\pi + \frac{1}{2}\)
f) \(2\pi + 1\)  g) \(\frac{\pi}{4} + 1\)  h) \(\frac{\pi}{4} + 2\)  i) \(\frac{\pi}{2} + 1\)  j) \(\frac{\pi}{2} + 2\)

Solution: a

\[\text{J := Int( (4*x^2+x+4)/(x^2+1)^2, x = 0 .. 1)};\]
\[
J := \int_0^1 \frac{4x^2 + x + 4}{(x^2 + 1)^2} \, dx
\]

\[\text{processedIntegrand := convert( student[integrand](J), parfrac, x);}\]
\[
#
# \text{Partial fraction form of given integrand}
processedIntegrand := \frac{x}{(x^2 + 1)^2} + \frac{4}{x^2 + 1}
\]

\[\text{antiderivative := int(processedIntegrand, x);}\]
\[
#
# \text{Antiderivative for integrand of given integral}
antiderivative := -\frac{1}{2(x^2 + 1)} + 4 \arctan(x)
\]

\[\text{Answer := simplify(subs(x = 1, antiderivative) - subs(x = 0,}\]
antiderivative));

\( \text{Answer} := \frac{1}{4} + \pi \)

Verification using Maple's built-in integrator:

\[
\int_{0}^{\frac{\pi}{2}} \frac{4x^2 + x + 4}{(x^2 + 1)^2} \, dx = \frac{1}{4} + \pi
\]

5. Given that \( \int_{0}^{2} 5 e^{(2x)} \cos(x) \, dx = e^\pi - 2 \), what is \( \int_{0}^{\frac{\pi}{2}} 5 e^{(2x)} \sin(x) \, dx \)?

a) \( e^\pi - 3 \)  b) \( e^\pi - 1 \)  c) \( e^\pi + 1 \)  d) \( e^\pi + 2 \)  e) \( e^\pi + 3 \)

f) \( 2 e^\pi - 3 \)  g) \( 2 e^\pi - 1 \)  h) \( 2 e^\pi + 1 \)  i) \( 2 e^\pi + 2 \)  j) \( 2 e^\pi + 3 \)

Solution: h

\[
\text{J := Int}(5*\exp(2*x)*\sin(x), x=0..\Pi/2);
\]

\[
\\text{K := intparts(J, exp(2*x))};
\]

\[
\text{K has the same value as J}
\]
\[ K := 5 - \int_{0}^{\frac{\pi}{2}} -10e^{(2x)} \cos(x) \, dx \]

\[ \text{Answer := } 5 - (-2*(\exp(\pi)-2)); \]

# Substitute the given value for the definite integral in K
\[ \text{Answer := } 1 + 2e^{x} \]

Verification using Maple's built-in integrator:

\[ J = \text{value}(J); \]

\[ \int_{0}^{\frac{\pi}{2}} 5e^{(2x)} \sin(x) \, dx = 1 + 2e^{x} \]

6. What is the area of the region that lies under the graph of \( y = 4 - x^2 \) and above the graph of \( y = 2 + x \)?

a) 2  b) \frac{5}{2}  c) 3  d) \frac{7}{2}  e) 4  

f) \frac{9}{2}  g) 5  h) \frac{11}{2}  i) 6  j) \frac{13}{2} 

Solution: f
> Area = Int((4-x^2)-(2+x), x = -2 .. 1);
#
# Required area is obtained by integrating (4-x^2) - (2+x) from
x = -2 to x = 1

\[ \text{Area} = \int_{-2}^{1} \left(2 - x^2 - x\right) \, dx \]

> Answer := int((4-x^2)-(2+x),x=-2..1);

\[ \text{Answer} := \frac{9}{2} \]

7. The region in the first quadrant below the horizontal line \( y = 4 \) and above the curve \( y = \sqrt{x} \) is rotated about the vertical line \( x = 4 \). If \( V \) is the volume of the resulting solid of revolution, then what is \( 15V \)?

a) \( 152\pi \)  b) \( 160\pi \)  c) \( 168\pi \)  d) \( 176\pi \)  e) \( 184\pi \)

f) \( 192\pi \)  g) \( 200\pi \)  h) \( 208\pi \)  i) \( 216\pi \)  j) \( 224\pi \)

**Solution: j**

By washers

> Volume = Int(Pi*(4^2-(4-y^2)^2),y=0..2);
Volume = \int_{0}^{2} \pi (16 - (4 - y^2)^2) \, dy

> Pre_Answer := int(Pi*(4^2-(4-y^2)^2),y=0..2);

\[
Pre_Answer := \frac{224 \pi}{15}
\]

By cylindrical shells

> Volume = Int(2*Pi*(2-sqrt(x))*(4-x),x=0..4);

\[
Volume = \int_{0}^{4} 2 \pi (2 - \sqrt{x}) (4 - x) \, dx
\]

> Pre_answer := int(2*Pi*(2-sqrt(x))*(4-x),x=0..4);

\[
Pre_answer := \frac{224 \pi}{15}
\]

Either way you slice it, the answer is

> Answer := 15*Pre_answer;

\[
Answer := 224 \pi
\]

8. The p.d.f. of a certain random variable X with values in \([0, \infty)\) is \(f(x) = \frac{x^2 e^{-x}}{2}\).

What is the mean of \(X\)? The reduction formula

\[
\int x^n e^{-x} \, dx = -x^n e^{-x} + n \int x^{n-1} e^{-x} \, dx
\]

might be useful.

a) 1  
b) 2  
c) 3  
d) 4  
e) 6  
f) 8  
g) 9  
h) 12  
i) 16  
j) 18

Solution: c

Here is the answer using Maple's built-in integrator
\[ f := x \rightarrow \frac{1}{2} x^2 e^{-x} \]

\[ \text{testeq( int(f(x), x = 0 .. infinity) = 1);} \]

\[
\#
\#	ext{ You do not have to do this step}
\#	ext{ THIS is just a verification that the given function really is a p.d.f.}
\]

\[ \text{true} \]

\[ \text{mean = Int(x*f(x), x = 0 .. infinity);} \]

\[ \text{mean} = \int_{0}^{\infty} \frac{1}{2} x^3 e^{-x} \, dx \]

\[ \text{mean} = 3 \]

Alternatively:

The given reduction formula for \( 1 \leq n \), converted to a formula for definite integrals over the interval from 0 to infinity is

\[ \text{Int(x^n*exp(-x),x = 0 .. infinity) =} \]

\[ n*\text{Int(x^(n-1)*exp(-x),x=0..infinity);} \]

\[ \int_{0}^{\infty} x^n e^{-x} \, dx = n \int_{0}^{\infty} x^{(n-1)} e^{-x} \, dx \]

Thus

\[ \text{Int(x^3*exp(-x)/2,x = 0 .. infinity) =} \]

\[ 3*\text{Int(x^(3-1)*exp(-x)/2,x=0..infinity);} \]

\[ \int_{0}^{\infty} \frac{1}{2} x^3 e^{-x} \, dx = 3 \int_{0}^{\infty} \frac{1}{2} x^2 e^{-x} \, dx \]
and

\[ \int_{0}^{\infty} \frac{1}{2} x^3 e^{-x} \, dx = 6 \int_{0}^{\infty} \frac{1}{2} x e^{-x} \, dx \]

and

\[ \int_{0}^{\infty} \frac{1}{2} x^3 e^{-x} \, dx = 6 \int_{0}^{\infty} \frac{1}{2} e^{-x} \, dx \]

Because

\[ \int_{0}^{\infty} \frac{1}{2} e^{-x} \, dx = \frac{1}{2} \]

we see that the mean is 6/2, or 3.

\[ \int_{1}^{\infty} \frac{6 x^2}{(1 + x^3)^2} \, dx. \]
Solution: a

\[ J := \int_{1}^{\infty} \frac{6x^2}{(1+x^3)^2} \, dx \]

\[ K := \int_{2}^{\infty} \frac{2}{u^2} \, du \]

Verification using Maple's built-in integrator
10. If $y(x)$ is the solution of the initial value problem \[ \frac{d}{dx}y(x) - \frac{2y(x)}{x} = x^2, \quad y(2) = 4, \]
then what is $y(3)$?

a) 5   b) 6   c) 8   d) 9   e) 10
f) 12   g) 15   h) 16   i) 18   j) 20

Solution: i

```maple
> ode := diff(y(x),x) - (2/x)*y(x) = x^2;
# This is a linear differential equation
ode := \( \frac{d}{dx}y(x) - \frac{2y(x)}{x} = x^2 \)
> eqn := dsolve({ode, y(2)=4}, y(x));
# Solution to ode using Maple's built-in ode solver
eqn := y(x) = (x - 1)x^2
> subs(x=3, eqn);
y(3) = 18
```

Using the method shown in the text

```maple
> P := x -> -2/x;
Q := x -> x^2;
P := x \rightarrow -\frac{2}{x}
Q := x \rightarrow x^2
> intFactor := exp(int(P(x), x));
intFactor := \( \frac{1}{x^2} \)
> y(x) = (1/intFactor)*Int(intFactor*Q(x), x) + C/intFactor;
y(x) = x^2 \int 1 \, dx + Cx^2
```
Consider the following three statements about a series \( \sum_{n=1}^{\infty} a_n \) with positive terms:

I: The series converges because \( \lim_{n \to \infty} a_n = 0. \)

II: The series converges because \( \lim_{n \to \infty} \frac{a_n}{b_n} = 0 \) and \( \sum_{n=1}^{\infty} b_n \) converges.

III: The series converges because \( \lim_{n \to \infty} \frac{a_n}{a_n + 1} = 0. \)

For each statement, determine whether the reasoning is correct (Y) or incorrect (N).

a) I: Y,  II: Y,  III: Y  

b) I: Y,  II: Y,  III: N  

c) I: Y,  II: N,  III: Y  

d) I: Y,  II: N,  III: N  

e) I: N,  II: Y,  III: Y  

f) I: N,  II: Y,  III: N  

g) I: N,  II: N,  III: Y  

h) I: N,  II: N,  III: N  

i) Wrong answer  

j) Bonus wrong answer
**Solution: f**

The reasoning of (I) is incorrect. If \( \lim_{n \to \infty} a_n \neq 0 \), then the series diverges. But no implication can be drawn from \( \lim_{n \to \infty} a_n = 0 \).

The reasoning of (II) is correct. The given limit implies that \( a_n < b_n \) for sufficiently large \( n \). Apply the Comparison Test to the tails.

The reasoning of (III) is incorrect. The given limit implies that \( \lim_{n \to \infty} \frac{a_n + 1}{a_n} = \infty \). The series must diverge by the Ratio Test.

12. Investigate the convergence/divergence of each of the four series

\[
\begin{align*}
\text{I)} & \quad \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \\
\text{II)} & \quad \sum_{n=1}^{\infty} \left( \frac{1}{n^3} \right) \\
\text{III)} & \quad \sum_{n=1}^{\infty} \frac{2^n}{2^n + n^2} \\
\text{IV)} & \quad \sum_{n=1}^{\infty} \frac{2^n}{2^n + 3^n}
\end{align*}
\]

using the Divergence Test, which yields a conclusion for at least one but no more than two of the series. State all the series for which the Divergence test is conclusive.

a) I only  
b) II only  
c) III only  
d) IV only  
e) I and II  
f) I and III  
g) I and IV  
h) II and III  
i) II and IV  
j) III and IV

**Solution: h**

Because

\[
\text{Limit ( } 1/\text{sqrt}(n) , \ n = \text{infinity} \text{) } = \text{limit ( } 1/\text{sqrt}(n) , \ n = \text{infinity) ;}
\]
\[
\lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0
\]

and

\[
\lim_{n \to \infty} \frac{2^n}{2^n + 3^n} = 0
\]

the Divergence Test says nothing about series (I) and (IV)

However, because

\[
\lim_{n \to \infty} \left( \frac{1}{n^3} \right)^{1/n} = 1
\]

and

\[
\lim_{n \to \infty} \frac{2^n}{2^n + n^2} = 1
\]

the Divergence Test is conclusive for series (II) and (III)

13. Consider the three series

\[
\text{I: } \sum_{n=1}^{\infty} \frac{1}{\sqrt{n!}}, \quad \text{II: } \sum_{n=0}^{\infty} \frac{\sqrt{1+n}}{\binom{5}{3}}, \quad \text{and III: } \sum_{n=3}^{\infty} \frac{1}{n \ln(n)}
\]

and the statements

(C) The series converges
(D) The series diverges

For each series, state which description, (C) or (D), is correct.

a) I: C, II: C, III: C
b) I: C, II: C, III: D
c) I: C, II: D, III: C
d) I: C, II: D, III: D
e) I: D, II: C, III: C
f) I: D, II: C, III: D
g) I: D, II: D, III: C
h) I: D, II: D, III: D
i) Wrong answer
j) Bonus wrong answer

Solution: b

Applying the Ratio Test to series I leads to a limit of 0, which is less than 1. Hence convergence.

```
> a := n -> 1/sqrt(n!);
a := n → \frac{1}{\sqrt{n!}}
> ratio := a(n+1)/a(n) ;
ratio := \frac{\sqrt{n!}}{\sqrt{(n + 1)!}}
> limit(ratio, n = infinity);
0
```

Applying the Limit Comparison Test with \( \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^{7/6}} \) leads to the conclusion of convergence.

```
> a := n -> sqrt(1+n)/(1+n^(5/3));
a := n → \sqrt{\frac{n + 1}{1 + n^{5/3}}}
> b := n -> 1/n^(7/6);
b := n → \frac{1}{n^{7/6}}
```

```
\[ b := n \rightarrow \frac{1}{n^{(7/6)}} \]

\[ \text{Limit}(a(n)/b(n), n = \text{infinity}) = \text{limit}(a(n)/b(n), n = \text{infinity}); \]

\[ \lim_{n \to \infty} \frac{\sqrt[n+1]{n^{(7/6)}}}{1 + n^{(5/3)}} = 1 \]

The value 7/6 was obtained by comparing the numerator of \(a_n\) to \(n^{(1/2)}\) and the denominator to \(n^{(5/3)}\).

Note \(\frac{5}{3} - \frac{1}{2} = \frac{7}{6}\).

Applying the Integral Test to series III leads to the conclusion of divergence.

\[ \text{Int}(1/x/\ln(x), x=3..\text{infinity}) = \text{int}(1/x/\ln(x), x=3..\text{infinity}); \]

\[ \int_{3}^{\infty} \frac{1}{x \ln(x)} \, dx = \infty \]

14. Consider the two series

\[ \text{I: } \sum_{n=0}^{\infty} (-1)^n \left( \frac{n+2}{2n+1} \right)^n \quad \text{and} \quad \text{II: } \sum_{n=0}^{\infty} \frac{(-1)^n \sqrt[n]{n}}{1 + n \sqrt[n]{n}} \]

and the statements

i: The series converges absolutely

ii: The series converges conditionally

iii: The series diverges

For each series, decide which of statements i, ii, or iii is correct.

a) I: i, II: i
b) I: i, II: ii
c) I: i, II: iii
d) I: ii, II: i
e) I: ii, II: ii
Solution:  b

Series I converges absolutely by the Root Test

\[ a_n := (-1)^n \left( \frac{n+2}{2n+1} \right)^n \]

\[ \lim_{n \to \infty} \left( \left( \frac{n+2}{2n+1} \right)^n \right)^{1/n} = \frac{1}{2} \]

Series II converges conditionally

It converges by the Alternating Series Test. The series of absolute values diverges by comparison with the harmonic series.

15. Apply the Ratio Test to series I:  \[ \sum_{n=2}^{\infty} \frac{3^n (5^n + 1)}{n^2} \]

Apply the Root Test to series II:  \[ \sum_{n=1}^{\infty} \frac{10^n}{n^n} \]

For each, decide which of statements,

(C) The test establishes convergence,
(D) The test establishes divergence,
(X) The test is not conclusive,
is correct.

a) I: C, II: C
b) I: C, II: D
c) I: C, II: X
d) I: D, II: C
e) I: D, II: D
f) I: D, II: X
g) I: X, II: C
h) I: X, II: D
i) I: X, II: X
j) Wrong answer

Solution: g

The Ratio Test is inconclusive (X) when applied to Series I:

```maple
> a := n -> n^3*(5^n+1)/(n^5+1)/5^n;
a := n -> \frac{n^3 (5^n + 1)}{(n^5 + 1) 5^n}
> Limit(a(n+1)/a(n), n = infinity) = limit(a(n+1)/a(n), n = infinity);
n = \infty
\lim_{n \to \infty} \frac{(n+1)^3 (5^{(n+1)} + 1) (n^5 + 1) 5^n}{(n+1)^5 + 1} \frac{5^{(n+1)} n^3 (5^n + 1)}{n^5 + 1} = 1
```

The Root Test is conclusive, and yields the conclusion of "convergent" (C), when applied to Series I:

```maple
> a := n -> 10^n/n^n;
a := n \to \frac{10^n}{n^n}
> Limit(a(n)^(1/n), n = infinity) = limit(a(n)^(1/n), n = infinity);
n = \infty
\lim_{n \to \infty} \left( \frac{10^n}{n^n} \right)^\frac{1}{n} = 0
```
16. Calculate the interval of convergence of
\[
\sum_{n=1}^{\infty} \frac{(-1)^n (x + 3)^n}{n 5^n}.
\]

(a) (-8, 2)  
(b) [-8, 2]  
(c) (-8, 2)  
(d) [-8, 2)  
(e) [-2, 8]  
(f) [-2, 8)  
(g) (-2, 8)  
(h) [-2, 8)  
(i) (2, 8)  
(j) (-\infty, \infty)

**Solution: c**

Because \( x + 3 = x - (-3) \), the center is \( c = -3 \).

The next calculation determines the radius of convergence.

\[
> a := n \rightarrow 1/(n*5^n);
\]
\[
> R := \lim(a(n)/a(n+1), n = \text{infinity});
\]
\[
R := 5
\]

The left endpoint is \( x = c - R, \) or \( x = -3 - 5, \) or \( x = -8 \). When this value is substituted in \( \sum_{n=1}^{\infty} \frac{(-1)^n (x + 3)^n}{n 5^n} \), the result is \( \sum_{n=1}^{\infty} \frac{1}{n} \).

This series is Divergent - it is the harmonic series.
Thus, -8 is out.

The right endpoint is \( x = c + R, \) or \( x = -3 + 5, \) or \( x = 2 \). When this value is substituted in \( \sum_{n=1}^{\infty} \frac{(-1)^n (x + 3)^n}{n 5^n} \), the result is \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \).

This series is convergent by the Alternating Series Test.
Thus, 2 is in.
17. Let \( T(x) \) denote the order (degree) 3 Taylor series of 
\[ f(x) = \sqrt{2 + x} \] with base point \(-1\). What is \( T(3) \)?
(Caution: This value is not intended to be, and is not, a good approximation of \( f(3) \).)

a) \( 2 \)  
b) \( 3 \)  
c) \( 4 \)  
d) \( 5 \)  
e) \( 6 \)  
f) \( \frac{5}{2} \)  
g) \( \frac{15}{4} \)  
h) \( \frac{9}{2} \)  
i) \( \frac{16}{3} \)  
j) \( \frac{27}{4} \)

Solution: d

\[
\begin{align*}
&> f := x \rightarrow \sqrt{2 + x}; \\
&\quad f := x \rightarrow \sqrt{2 + x} \\
&> T := x \rightarrow f(-1) + D(f)(-1)(x+1) + (D@@2)(f)(-1)(x+1)^2/2! + (D@@3)(f)(-1)(x+1)^3/3!; \\
&\quad T := x \rightarrow (\frac{D(2)(f)(-1)(x+1)^2}{2!} + \frac{(D(3)(f)(-1)(x+1)^3}{3!}) \\
&> T(x); \\
&\quad \frac{3}{2} + \frac{x}{2} - \frac{(x+1)^2}{8} + \frac{(x+1)^3}{16} \\
&> T(3); \\
&\quad 5 \\
&> S := \text{series}(\sqrt{2+x}, x=-1, 4); \\
&\quad \# \quad \# \text{Verification using Maple's built-in Taylor series calculator} \\
&\quad S := 1 + \frac{1}{2}(x+1) - \frac{1}{8}(x+1)^2 + \frac{1}{16}(x+1)^3 + \text{O}((x+1)^4) \\
&> \text{subs}(x=3, \text{convert}(S, \text{polynom})); \\
&\quad 5
\end{align*}
\]

18. The Maclaurin series of \( x^3 \cos(x^2) \) is used to approximate \( \int_0^1 x^3 \cos(x^2) \, dx \) with an error less than 0.01. Use only as many terms as are necessary for the required accuracy (according to the estimate from the Alternating Series Test). What is the approximation?
Solution: d

\[ \text{eqn1 := } \cos(u) = 1 - \frac{u^2}{2!} + \frac{u^4}{4!} + \frac{u^6}{6!} + E; \]
\#
\# Here E is a small error term that results from truncating the Maclaurin series
\[ \text{eqn1 := } \cos(u) = 1 - \frac{u^2}{2} + \frac{u^4}{24} + \frac{u^6}{720} + E \]
\[ \text{eqn2 := subs(u = x^2, eqn1)}; \]
\[ \text{eqn2 := } \cos(x^2) = 1 - \frac{x^4}{2} + \frac{x^8}{24} + \frac{x^{12}}{720} + E \]
\[ \text{eqn3 := map(z->x^3*z, eqn2)}; \]
\#
\# This has the effect of multiplying every term of eqn2 by x^3
\[ \text{eqn3 := } x^3 \cos(x^2) = x^3 \left( 1 - \frac{x^4}{2} + \frac{x^8}{24} + \frac{x^{12}}{720} + E \right) \]
\[ \text{eqn4 := } \text{lhs(eqn3)} = \text{expand(subs(E=0, rhs(eqn3))));} \]
\#
\# We will discard E by setting it equal to 0
\[ \text{eqn4 := } x^3 \cos(x^2) = x^3 - \frac{x^4}{2} + \frac{x^{11}}{24} + \frac{x^{15}}{720} \]
\[ \text{eqn5 := } \int x^3 \cos(x^2) \, dx = C + \frac{x^4}{4} + \frac{x^8}{16} + \frac{x^{12}}{288} + \frac{x^{16}}{11520} \]
\[ \text{eqn6 := } \int_0^1 x^3 \cos(x^2) \, dx = \frac{3}{16} \]

# Because 1/288 < 0.01 and this is the first such term, we truncate the series at this term
19. What is the coefficient of \( x^3 \) in the Maclaurin series of \( \frac{12x}{e^{(2x)}} \)?

a) −6    b) −8    c) −12    d) −16    e) −24
f) 6    g) 8    h) 12    i) 16    j) 24

Solution: j

Brute force:

```maple
> f := x -> 12*x/exp(2*x);
f := x → \frac{12 x}{e^{(2x)}}
> Answer := (D@@3)(f)(0)/3!;
Answer := 24
```

A more elegant alternative:

```maple
> eqn1 := exp(u) = 1 + u + u^2/2! + u^3/3! + u^4/4!;
  # The Maclaurin series of exp(u) up to order 4
eqn1 := e^u = 1 + u + \frac{u^2}{2} + \frac{u^3}{6} + \frac{u^4}{24}
> eqn2 := subs(u = -2*x, eqn1);
eqn2 := e^{-2x} = 1 - 2x + 2x^2 - \frac{4x^3}{3} + \frac{2x^4}{3}
> eqn3 := map(z -> z*12*x, eqn2);
  # Multiplies each side of eqn2 by 12*x
eqn3 := 12 e^{-2x} x = 12 \left( 1 - 2x + 2x^2 - \frac{4x^3}{3} + \frac{2x^4}{3} \right) x
> eqn4 := lhs(eqn3) = expand(rhs(eqn3));
eqn4 := 12 e^{-2x} x = 12 x - 24 x^2 + 24 x^3 - 16 x^4 + 8 x^5
```
Verification using Maple's built-in Maclaurin series calculator

\[ \text{series}(12x/\exp(2x), x=0, 8); \]
\[
12x - 24x^2 + 24x^3 - 16x^4 + 8x^5 - \frac{16}{5}x^6 + \frac{16}{15}x^7 + O(x^8)
\]

20. What is the coefficient of \(x^3\) in the Maclaurin series of \( \frac{x}{\sqrt{1-x}} \)?

a) \(\frac{1}{8}\)  b) \(\frac{3}{16}\)  c) \(\frac{1}{4}\)  d) \(\frac{5}{16}\)  e) \(\frac{3}{8}\)

f) \(-\frac{1}{8}\)  g) \(-\frac{3}{16}\)  h) \(-\frac{1}{4}\)  i) \(-\frac{5}{16}\)  j) \(-\frac{3}{8}\)

**Solution:** e

The \(x^3\) term of \( \frac{x}{\sqrt{1-x}} \) is the \(x^2\) term of \( (1-x)^{-\frac{1}{2}} \).

The \(u^2\) term of \( (1+u)^{-\frac{1}{2}} \) is the binomial coefficient

\[ \text{series}(x/sqrt(1-x), x = 0, 5); \]
\#
\# Verification using Maple's built-in Taylor series function
\[ x + \frac{1}{2} x^2 + \frac{3}{8} x^3 + \frac{5}{16} x^4 + O(x^5) \]