1. The region in the first quadrant that is bounded above by \( y = 4x - x^2 \) and bounded below by \( y = x \) is rotated about the vertical line \( x = -1 \). What is the volume of the resulting solid of revolution?

a) \( \frac{15\pi}{2} \)  

b) \( 16\pi \)  
c) \( 18\pi \)  
d) \( 21\pi \)  
e) \( \frac{45\pi}{2} \)  
f) \( 20\pi \)  
g) \( \frac{27\pi}{2} \)  
h) \( \frac{25\pi}{2} \)  
i) \( 14\pi \) 
j) \( 15\pi \)

Solution: e

First we plot the planar region that is to be rotated.

\[
> f := x \rightarrow 4x-x^2; \\
g := x \rightarrow x;
\]

\[
f := x \rightarrow 4x-x^2 \\
g := x \rightarrow x
\]

The planar region bounded by \( y = x \) and \( y = 4x - x^2 \) is shown below:
Problem 1, Figure 1

When this region is rotated about the specified axis, the following solid results.
Calculation of the Volume by the Method of Cylindrical Shells

The next figure shows a typical cylindrical shell.
Here is the calculation of the volume by means of the Method of Cylindrical Shells:

\[
\text{radius}_{\text{of shell}} := x \to x + 1; \\
\text{height}_{\text{of shell}} := x \to f(x) - g(x); \\
\text{Volume} = \int_{0}^{3} 2\pi x (x + 1) (3 x - x^2) \, dx \\
\text{Volume} = 2\pi \int_{0}^{3} 2 x^2 - x^3 + 3 x \, dx \\
\text{antiderivative} := \int (2x^2-x^3+3x, x); \\
\]
antiderivative := \frac{2 x^3}{3} - \frac{x^4}{4} + \frac{3 x^2}{2}

Volume = \frac{45}{4}

Calculation of the Volume by the Method of Washers

It is possible (but more difficult) to obtain the volume using washers. A glance at Figure 2 above indicates that we have to divide the integration into two pieces: one for $0 \leq y \leq 3$ and one for $3 \leq y \leq 4$. Figures 4 and 5 below show the two types of washer. For $y < 3$, the outer radius of the washer extends to the curve $y = x$. For $3 < y$, the outer radius of the washer extends to the curve $y = 4 - x^2$.

The Solid of Revolution with a Washer at Height $y < 3$

Problem 1, Figure 4
The Solid of Revolution with a Washer at Height \( y > 3 \)

Problem 1, Figure 5

\[ \text{Volume} = \int_{0}^{3} \pi ((1+y)^2 - (1+2-\sqrt{4-y})^2) \, dy + \int_{3}^{4} \pi ((3+\sqrt{4-y})^2 - (3-\sqrt{4-y})^2) \, dy \]

\[ = \frac{45\pi}{2} \]

2. The region in the first quadrant bounded above by \( y = x \) and below by \( y = x^2 \) for \( 0 \leq x \leq 1 \) is rotated about the line \( y = 2 \). What is the volume of the solid of revolution that results?

a) \( \frac{8\pi}{15} \)
b) \( \frac{3\pi}{5} \)
c) \( \frac{2\pi}{3} \)
d) \( \frac{4\pi}{5} \)
e) \( \frac{14\pi}{15} \)
f) \( \frac{\pi}{2} \)
g) \( \frac{3\pi}{8} \)
h) \( \frac{5\pi}{8} \)
i) \( \frac{3\pi}{4} \)
j) \( \frac{7\pi}{8} \)
Solution: a

The region (shown with blue boundary) is rotated about $y = 2$.

In the next figure, we show the solid that results from the rotation. The inner boundary, in the shape of a conical frustum, is rendered with solid tan. The outer boundary, a paraboloid, is rendered using a brown wireframe. A washer is also shown.
Calculation of the Volume by the Method of Washers

The outer radius is the distance from $y = x^2$ to $y = 2$, namely $2 - x^2$. The inner radius is the distance from $y = x$ to $y = 2$, namely $2 - x$.

\[
> \text{volume} := \pi \int ((2-x^2)^2 - (2-x)^2, x = 0 .. 1);
\]

\[
\text{volume} := \frac{8\pi}{15}
\]

Calculation of the Volume by the Method of Cylindrical Shells
For a shell that has an edge at level $y$, the radius is $2 - y$ and the height of the shell is $\sqrt{y - y}$.

\[
\int 2\pi (2 - y) (\sqrt{y} - y), \quad y = 0..1;
\]

\[
\frac{8\pi}{15}
\]

3. Calculate the arc length of the graph of 
\[y = \frac{x^3}{3} + \frac{1}{4x} \quad \text{for} \quad \frac{1}{2} \leq x \leq 1.\]

\begin{align*}
\text{a) } & \frac{5}{12} \\
\text{b) } & \frac{11}{24} \\
\text{c) } & \frac{1}{2} \\
\text{d) } & \frac{13}{24} \\
\text{e) } & \frac{7}{12} \\
\text{f) } & \frac{5}{8} \\
\text{g) } & \frac{2}{3} \\
\text{h) } & \frac{17}{24} \\
\text{i) } & \frac{3}{4} \\
\text{j) } & \frac{5}{6}
\end{align*}

Solution: d

\[
f := x \rightarrow x^{3/3} + 1/(4x);
\]

\[
f := x \rightarrow \frac{1}{3}x^3 + \frac{1}{4x}
\]

\[
diff(f(x), x)^2;
\]

\[
\left(x^2 - \frac{1}{4x^2}\right)^2
\]

\[
expand(1+diff(f(x), x)^2);
\]

\[
\frac{1}{2} + x^4 + \frac{1}{16x^4}
\]

\[
eqn := 1+diff(f(x), x)^2 = (x^2+1/(4*x^2))^2;
\]

\[
eqn := 1 + \left(x^2 - \frac{1}{4x^2}\right)^2 = \left(x^2 + \frac{1}{4x^2}\right)^2
\]

\[
testeq( eqn );
\]

true

\[
`\text{arc length} = \int((x^2+1/(4*x^2)), \; x=1/2..1) = \int((x^2+1/(4*x^2)), \; x=1/2..1);
\]
The line segment with end points (2,3) and (6,0) is rotated about the y-axis. What is the surface area of the resulting conical frustum?

a) $12\pi$  b) $16\pi$  c) $20\pi$  d) $24\pi$  e) $25\pi$

f) $27\pi$  g) $28\pi$  h) $32\pi$  i) $36\pi$  j) $40\pi$

Solution: j

The average circumference of the frustum is $2\pi \left(\frac{2 + 6}{2}\right)$, or $8\pi$. The slant length is $\sqrt{3^2 + 4^2}$, or 5. The surface area is the product of the average circumference and the slant length: $40\pi$.

The graph of $y = 3\sqrt{x}$ for $4 \leq x \leq 10$ is rotated about the x-axis. What is the surface area of the resulting figure?

a) $54\pi$  b) $60\pi$  c) $64\pi$  d) $72\pi$  e) $80\pi$

f) $84\pi$  g) $94\pi$  h) $102\pi$  i) $109\pi$  j) $115\pi$

Solution: i

> f := x -> 3*sqrt(x);
6. Let \( R \) be the trapezoidal region in the first quadrant that is bounded above by the graph of \( y = 2x \), below by the \( x \)-axis, on the left by the vertical line \( x = 1 \), and on the right by the vertical line \( x = 2 \). What is the moment of \( R \) about the vertical line \( x = -1 \)? (Assume that \( R \) has a constant density equal to 1.)

\[
\begin{array}{cccccc}
a) & \frac{20}{3} & b) & 7 & c) & \frac{23}{3} & d) & 8 & e) & \frac{25}{3} \\
f) & 9 & g) & \frac{32}{3} & h) & 11 & i) & \frac{35}{3} & j) & 12 \\
\end{array}
\]

Solution:  \( c \)
7. Let \( R \) be the region in the first quadrant that is bounded above by the graph of \( y = 4 - x^2 \) and below by the x-axis. What is the x-coordinate of the center of mass of \( R \)?

\[
\begin{align*}
a) \frac{1}{3} & \quad b) \frac{3}{8} & \quad c) \frac{2}{5} & \quad d) \frac{1}{2} & \quad e) \frac{3}{5} \\
f) \frac{5}{8} & \quad g) \frac{2}{3} & \quad h) \frac{3}{4} & \quad i) \frac{5}{4} & \quad j) \frac{3}{2}
\end{align*}
\]

Solution: \( h \)

\[
> f := x -> 4 - x^2;
\]

\( f := x \rightarrow 4 - x^2 \)

\[
> M := \text{Int}(\text{delta}\cdot f(x), x = 0..2); \quad \# \text{The mass in terms of the mass density}
\]

\[
M := \int_0^2 \delta (4 - x^2) \, dx
\]

\[
> M := \text{value}(M);
\]

\[
M := \frac{16 \delta}{3}
\]

\[
> \text{Moment}[`x=0`] := \text{Int}(\text{delta}\cdot x\cdot f(x), x = 0..2); \quad \# \text{Moment about y-axis}
\]

\[
\text{Moment}_{x=0} := \int_0^2 \delta x (4 - x^2) \, dx
\]

\[
> \text{Moment}[`x=0`] := \text{value}(\text{Moment}[`x=0`]);
\]

\[
\text{Moment}_{x=0} := 4 \delta
\]

\[
> x_{\text{bar}} := \text{Moment}[`x=0`]/M; \quad \# x\text{-coordinate of center of mass}
\]

\[
x_{\text{bar}} := \frac{3}{4}
\]
8. What is the y-coordinate of the center of mass of the region \( R \) of the preceding question?

\[
\begin{align*}
a) & \quad \frac{5}{3} & b) & \quad \frac{6}{5} & c) & \quad \frac{11}{8} & d) & \quad \frac{4}{3} & e) & \quad \frac{7}{5} \\
f) & \quad \frac{13}{8} & g) & \quad \frac{8}{5} & h) & \quad \frac{7}{4} & i) & \quad \frac{9}{5} & j) & \quad \frac{15}{8}
\end{align*}
\]

Solution: g

\[> \text{Moment[`}y=0`\] := Int(delta*f(x)^2/2, x = 0 .. 2); \# Moment about x-axis\]
9. Suppose that \( f(x) = x^2 \). If \( f(7) \) is equal to the average value of \( f(x) \) over the interval \([2, b]\), then what is \( b \)?
Solution: g

```maple
> a := 2:
f := x -> x^2;
f := x ↦ x^2
> eqn := f(7) = simplify(int(f(x), x = a .. b)/(b-a));
eqn := 49 = \frac{b^2}{3} + \frac{2}{3} + \frac{4}{3}
> solve(eqn, b);
-13, 11
```

10. Suppose that \( f(x) = \frac{c}{1 + x^2} \) is the probability density function of a random variable that has values in the interval \( [0, \sqrt{3}] \). What is \( c \)?

a) \( \frac{1}{\pi} \)  
b) \( \frac{2}{\pi} \)  
c) \( \frac{3}{\pi} \)  
d) \( \frac{4}{\pi} \)  
e) \( \frac{5}{\pi} \)

f) \( \pi \)  
g) \( \frac{\pi}{2} \)  
h) \( \frac{\pi}{3} \)  
i) \( \frac{\pi}{4} \)  
j) \( \frac{\pi}{5} \)

Solution: c

```maple
> f := x -> c/(1+x^2);
f := x ↦ \frac{c}{1 + x^2}
> eqn := Int(f(x), x = 0 .. sqrt(3)) = 1;
# A probability density integrates to 1
```
\[
\int_0^3 \frac{c}{1 + x^2} \, dx = 1
\]

\[\text{eqn} := \int_0^3 \frac{c}{1 + x^2} \, dx = 1\]

\[> \text{eqn} := \text{map}(\text{value}, \text{eqn});\]

\[\text{eqn} := \frac{\pi c}{3} = 1\]

\[> c = \text{solve}(\text{eqn}, c);\]

\[c = \frac{3}{\pi}\]

11. A random variable \(X\) that assumes values in the interval \([0, 1]\) has probability density function

\[
f(x) = \frac{8}{(\pi + 2)(1 + x^2)^2} \quad \text{for} \quad 0 \leq x \leq 1.
\]

If \(\mu_X\) is the mean of \(X\), what is \((\pi + 2)\mu_X\)?

\begin{align*}
&\text{a) 2} &\text{b) } \sqrt{2} &\text{c) 3} &\text{d) } \sqrt{3} &\text{e) 5} \\
&\text{f) } \sqrt{5} &\text{g) 6} &\text{h) } \sqrt{6} &\text{i) } 2\sqrt{2} &\text{j) } 2\sqrt{3}
\end{align*}

\textbf{Solution: a}

\[> \text{eqn} := \text{mu}[X] = \text{Int}(8x/(1+x^2)^2/(\pi+2), x = 0 .. 1);\]

\[\text{eqn} := \mu_X = \int_0^1 \frac{8x}{(1 + x^2)^2 (2 + \pi)} \, dx\]

\[> \text{eqn} := \text{map}(\text{value}, \text{eqn});\]

\[\text{eqn} := \mu_X = \frac{2}{2 + \pi}\]

12. A random real number \(X\) is chosen from the interval \([1, 9]\). If the p.d.f. of \(X\) is

\[
f(x) = \frac{3\sqrt{x}}{52},\]

then what is the probability that \(X\) is greater than 4?
Solution: f

\[ P(X>4) = \int_{4}^{9} \frac{3\sqrt{x}}{52} \, dx = \frac{19}{26} \]

13. A random variable \( X \) has pdf given by \( f(x) = \frac{2x + 1}{108} \) for \( 1 \leq x \leq 10 \).

What is the median \( m \) of \( X \)?

Solution: f

\[ f := x \rightarrow \frac{(1+2x)}{108}; \]

\[ f := x \rightarrow \frac{1}{108} + \frac{x}{54} \]

\[ eqn := \int_{1}^{m} \frac{1}{108} + \frac{x}{54} \, dx = \frac{1}{2} \]

\[ eqn := map(value, eqn); \]

\[ eqn := \frac{m}{108} - \frac{1}{54} + \frac{m^2}{108} = \frac{1}{2} \]

\[ m = solve(eqn, m); \]

\[ m = (7, -8) \]

Of these two values, only 7 is in the interval.
By way of verification, we will check that P(X > 7) = 1/2

\[
> \text{testeq}(\int(f(x), x = 7 .. 10) = 1/2);
\]

\[
text{true}
\]

14. Starting from equilibrium, 60 J of work are expended when a spring is stretched 2 meters. How many meters beyond equilibrium can the spring be stretched with a force of 45 N?

a) \(\frac{1}{5}\)  b) \(\frac{1}{2}\)  c) 1  d) \(\frac{6}{5}\)  e) \(\frac{3}{2}\)

f) \(\frac{8}{5}\)  g) 2  h) \(\frac{12}{5}\)  i) \(\frac{5}{2}\)  j) 3

**Solution:** e

\[
> \text{HookesLaw} := F = k*x;
\]

\[
\text{HookesLaw} := F = k x
\]

\[
> \text{eqn} := 60 = \int(k*x, x = 0 .. 2);
\]

\[
eqn := 60 = \int_0^2 k x \, dx
\]

\[
> \text{eqn} := \text{map(value, eqn)};
\]

\[
eqn := 60 = 2 k
\]

\[
> \text{eqn2} := k = \text{solve(eqn, k)};
\]

\[
eqn2 := k = 30
\]

\[
> \text{eqn3} := \text{subs({eqn2, F = 45}, HookesLaw)};
\]

\[
eqn3 := 45 = 30 x
\]

\[
> \text{solve(eqn3)};
\]

\[
\frac{3}{2}
\]

15. The vertical cross-sections of a tank have the shape \(y = |x| - 12\) for \(-12 \leq x \leq 12\). Horizontal cross-sections of the tank are rectangles that are 5 m long. The widths of these rectangles vary, but can be deduced from the description of the vertical cross-sections. The tank is filled to the top with a fluid that has weight density equal to \(\frac{3}{10} \frac{N}{m^3}\). If after pumping the fluid to the top of the tank, the remaining fluid is 10 m deep, how many Joules of work
have been done?

a) 24  b) 32  c) 36  d) 48  e) 52
f) 56  g) 64  h) 72  i) 80  j) 84

Solution: g

\[
\text{Work} = \int_{0}^{2} \left( \frac{3}{10} w \cdot 5 \cdot 2 \cdot (12-w) \right) \, dw = 64
\]

16. Forty feet of a uniform cable hang over the side of a building. The cable weighs 2 lbs/ft. A 48 pound hook is attached to the dangling end of the cable. The cable is pulled up 30 feet before the hook snags on a balcony rail. How many foot-pounds of work have been done?

a) 2480  b) 2540  c) 2580  d) 2640  e) 2680
f) 2740  g) 2800  h) 2840  i) 2880  j) 2940

Solution: j

\[
\text{Work} = 48 \cdot 30 + (10 \cdot 2) \cdot 30 + \int_{0}^{30} 2 \cdot y \, dy = 2940
\]

17. Calculate

\[
\int_{7}^{8} \frac{1}{(8-x)^{1/3}} \, dx
\]

a) 4/3  b) 3/2  c) 5/3  d) 2  e) 7/3
f) 5/2  g) 8/3  h) 3  i) 10/3  j) 7/2

Solution: b

\[
\int \frac{1}{(8-x)^{1/3}}, x = 7 \ldots 8 = \text{Limit} \left( \int \frac{1}{(8-x)^{1/3}}, x = 7 \right)
\]
.. 8-epsilon), epsilon=0, right);
\[ \int_{7}^{8-\epsilon} \frac{1}{(8-x)^{1/3}} \, dx = \lim_{\epsilon \to 0^+} \int_{7}^{8-\epsilon} \frac{1}{(8-x)^{1/3}} \, dx \]

> \text{Int}(1/(8-x)^{(1/3)}, x = 7 .. 8) = \text{Limit}(\text{Int}(1/(8-x)^{(1/3)}, x = 7 .. 8-\epsilon), \epsilon=0, \text{right});

\[ \int_{7}^{\infty} \frac{1}{(8-x)^{1/3}} \, dx = \lim_{\epsilon \to 0^+} \left( \int_{7}^{8-\epsilon} \frac{1}{(8-x)^{1/3}} \, dx - \frac{3 \epsilon^{2/3}}{2} + \frac{3}{2} \right) \]

> \text{Int}(1/(8-x)^{(1/3)}, x = 7 .. 8) = \text{lim}(\text{Int}(1/(8-x)^{(1/3)}, x = 7 .. 8-\epsilon), \epsilon=0, \text{right});

\[ \int_{7}^{8} \frac{1}{(8-x)^{1/3}} \, dx = \frac{3}{2} \]

- 18. Calculate \( \int_{1}^{\infty} \frac{x^2}{(2+x^3)^2} \, dx \).

a) 1 b) \( \frac{1}{2} \) c) \( \frac{1}{3} \) d) \( \frac{1}{4} \) e) \( \frac{1}{5} \)
f) \( \frac{1}{6} \) g) \( \frac{1}{7} \) h) \( \frac{1}{8} \) i) \( \frac{1}{9} \) j) \( \frac{1}{10} \)

Solution: i

> \text{Int}(x^2/((2+x^3)^2), x = 1 .. \infty) = \text{Limit}(\text{Int}(x^2/((2+x^3)^2), x = 1 .. N), N = \infty);

\[ \int_{1}^{\infty} \frac{x^2}{(2+x^3)^2} \, dx = \lim_{N \to \infty} \int_{1}^{N} \frac{x^2}{(2+x^3)^2} \, dx \]

> \text{Int}(x^2/((2+x^3)^2), x = 1 .. \infty) = \text{Limit}(\text{student[changevar]}(u = 2+x^3, \text{Int}(x^2/((2+x^3)^2), x = 1} \]
\[ \int_{1}^{\infty} \frac{x^2}{(2 + x^3)^2} \, dx = \lim_{N \to \infty} \int_{3}^{2 + N^3} \frac{1}{3u^2} \, du \]

\[ \int_{1}^{\infty} \frac{x^2}{(2 + x^3)^2} \, dx = \lim_{N \to \infty} \frac{-1 + N^3}{9 (2 + N^3)} \]

19. Calculate the 100\(^{th}\) partial sum of \[ \sum_{n=1}^{\infty} \frac{1}{(n + 1) \cdot n} \].

   a) 1  b) \( \frac{99}{100} \)  c) \( \frac{100}{99} \)  d) \( \frac{100}{101} \)  e) \( \frac{101}{100} \)
   f) \( \frac{49}{50} \)  g) \( \frac{50}{49} \)  h) \( \frac{98}{101} \)  i) \( \frac{101}{98} \)  j) \( \frac{1}{10100} \)

**Solution: d**

\[ \text{answer} := \text{sum}(1/((n+1)*n), n = 1..100); \]

\[ \text{answer} := \frac{100}{101} \]

\[ 1/1-1/101; \]

\[ \frac{100}{101} \]

20. Consider the series

\[ \ldots N), u), N = \text{infinity}); \]

\# The change of variable \( u = 2 + x^3 \), \( du = 3x^2 \, dx \)

\[ \int_{1}^{\infty} \frac{x^2}{(2 + x^3)^2} \, dx = \lim_{N \to \infty} \int_{3}^{2 + N^3} \frac{1}{3u^2} \, du \]

\[ \int_{1}^{\infty} \frac{x^2}{(2 + x^3)^2} \, dx = \lim_{N \to \infty} \frac{-1 + N^3}{9 (2 + N^3)} \]
I) $\sum_{n=1}^{\infty} \frac{2^n}{n}$  II) $\sum_{n=1}^{\infty} \tan\left(\frac{1}{n}\right)$  III) $\sum_{n=1}^{\infty} \sec\left(\frac{1}{n}\right)$  IV) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

List all given series for which the Divergence Test yields a conclusion.

a) I  b) II  c) III  d) IV  e) I, II  
f) I, III  g) I, IV  h) II, III  i) II, IV  j) III, IV

**Solution: f**

```plaintext
> a := n -> 2^n/n;
  b := n -> tan(1/n);
  c := n -> sec(1/n);
  d := n -> 1/sqrt(n);
```

```plaintext
> testeq( limit(a(n), n = infinity) = 0 ); #Divergence Test applies
false
> testeq( limit(b(n), n = infinity) = 0 );
true
> testeq( limit(c(n), n = infinity) = 0 ); #Divergence Test applies
false
> testeq( limit(d(n), n = infinity) = 0 );
true
```

**Code for Figures**