1. A ball is dropped from a height of 12 meters. Its elastic properties are such that when it strikes the ground, it bounces up to a height that is \( \frac{2}{3} \) of the height from which it had been dropped. It then falls a distance equal to that rebound height, at which point it strikes the ground again and the process is repeated. The total distance the ball falls and rises can be expressed as

\[
\text{Total Distance} = h + 2 \sum_{n=1}^{\infty} \frac{h \cdot r^n}{1 - r}
\]

where the infinite series is a geometric series that, when multiplied by 2, represents the distance the ball rises and falls after all the ground strikes. After filling in the blanks, calculate the total distance the ball falls and rises.

**Solution** Let \( h = 12 \) be the height from which the ball was dropped. This is the distance of the only downward trajectory that was not preceded by an upward trajectory of the same distance. Thus, this distance is not multiplied by 2. Let \( r = \frac{2}{3} \). The ball rebounds upwards a distance of \( h \cdot r \) and then falls the same distance, after which it rebounds upward to a height of \( (h \cdot r) \cdot r \), or \( h \cdot r^2 \) and falls the same distance, and so on. The total distance the ball falls and rises is

\[
h + 2 \sum_{n=1}^{\infty} h \cdot r^n, \text{ or } h + 2h \frac{r}{1 - r}, \text{ or } \frac{1 + r}{1 - r} h.
\]

2. The Test for Divergence allows us to make a decision about the behavior of a series \( \sum_{n=M}^{\infty} a_n \) provided we can establish a particular property concerning \( \lim_{n \to \infty} a_n \). If we cannot establish that property, then we cannot apply the Test for Divergence to the series. For each of the following two series, calculate \( \lim_{n \to \infty} a_n \) and either state the conclusion of the Test for Divergence (if the test can be applied), or state “inconclusive” (if the test cannot be applied):

(a) \( \sum_{n=0}^{\infty} \frac{4^n}{4^n + 2^n} \)  
(b) \( \sum_{n=0}^{\infty} \frac{4^n}{4^n + 5^n} \)

**Solution** Let \( a = 2, b = 4, c = 5 \). What is significant about these values is that \( 1 < a < b < c \). For any values of \( a, b, c \) that satisfy these inequalities, we have \( 0 < r = a/b < 1 \) and \( 0 < \rho = b/c < 1 \). We divide every term of each of the two fractions in the two given series by the fastest growing term in the fraction. We obtain

\[
\lim_{n \to \infty} \frac{b^n}{b^n + a^n} = \lim_{n \to \infty} \frac{b^n/b^n}{b^n/b^n + a^n/b^n} = \lim_{n \to \infty} \frac{1}{1 + r^n} = \frac{1}{1 + 0} = 1,
\]
and
\[
\lim_{n \to \infty} \frac{b^n}{b^n + c^n} = \lim_{n \to \infty} \frac{b^n/c^n}{b^n/c^n + c^n/c^n} = \lim_{n \to \infty} \frac{\rho^n}{\rho^n + 1} = 0 \div \frac{0 + 1}{1} = 0.
\]

The Test for Divergence states that if \(\lim_{n \to \infty} u_n \neq 0\), then the infinite series \(\sum u_n\) diverges. This test provides no conclusion about \(\sum u_n\) if \(\lim_{n \to \infty} u_n = 0\). The test therefore provides the conclusion “divergent” for \(\sum_{n=0}^{\infty} \frac{b^n}{b^n + a^n}\), \((1 < a < b)\) and is inconclusive for \(\sum_{n=0}^{\infty} \frac{b^n}{b^n + c^n}\), \((1 < b < c)\). (The Limit Comparison Test with \(\sum_{n=0}^{\infty} \rho^n\) as the comparison series is conclusive in showing that the series converges, not that the quiz question called for this information.)