Distance in the Ellipticity Graph

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Introduction

Motivation

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Results

I created algorithms for determing when two vertices of $\mathcal{Z}(F)$ are adjacent to a common vertex.

Preliminaries

Definition

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Example

If $F = \langle a, b, c \rangle$, then the following are free splittings of F:

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- Notrivial cyclic words of F.
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Remark

The inner (conjugation) automorphisms of F fix $\mathcal{Z}(F)$, so $\mathrm{Out}(F)$, the group of outer automorphisms of F, acts on $\mathcal{Z}(F)$.

Distance Two in $\mathcal{Z}(F)$

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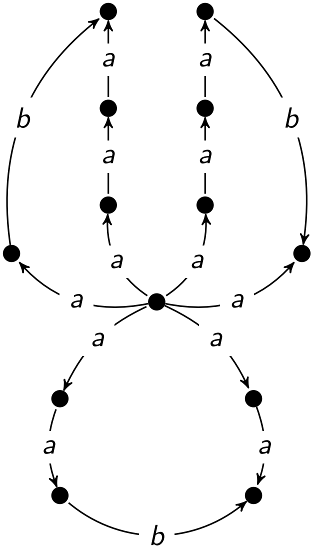
Two free splittings with common elliptic word

Given finitely generated subgroups H and K of F, is there a notrivial conjugacy class that intersects both?

Stallings Folding

$H = \langle aba^{-3}, a^2ba^{-2}, a^3ba^{-1} \rangle < F = \langle a, b \rangle$



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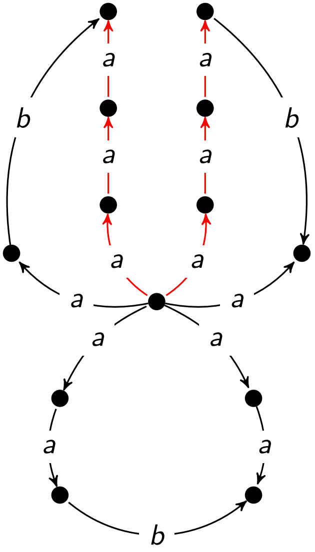

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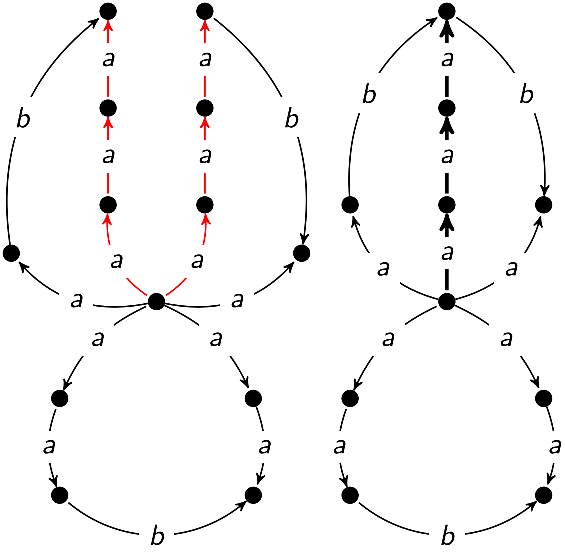

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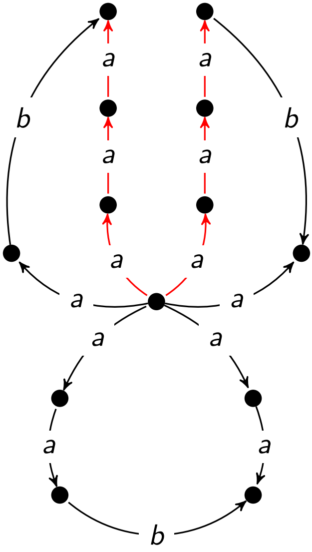

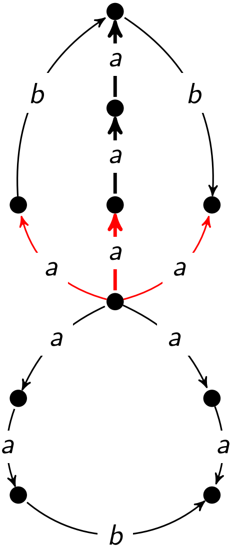

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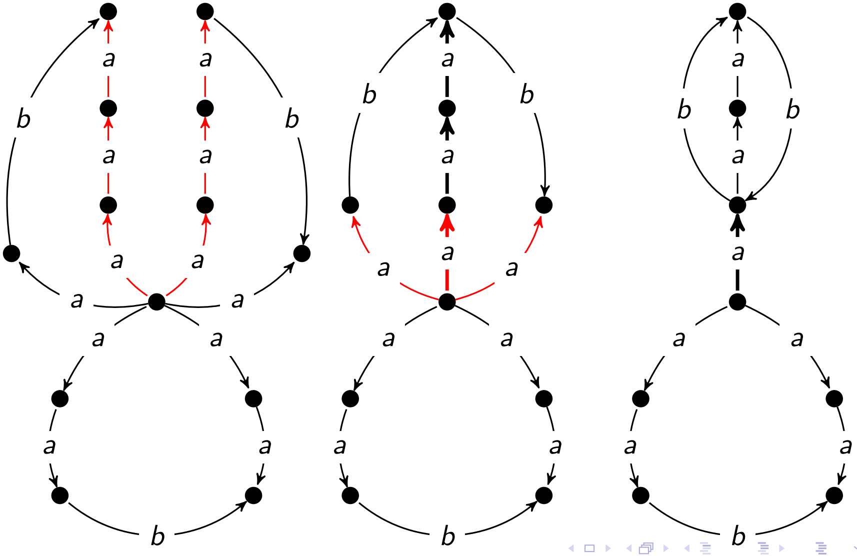

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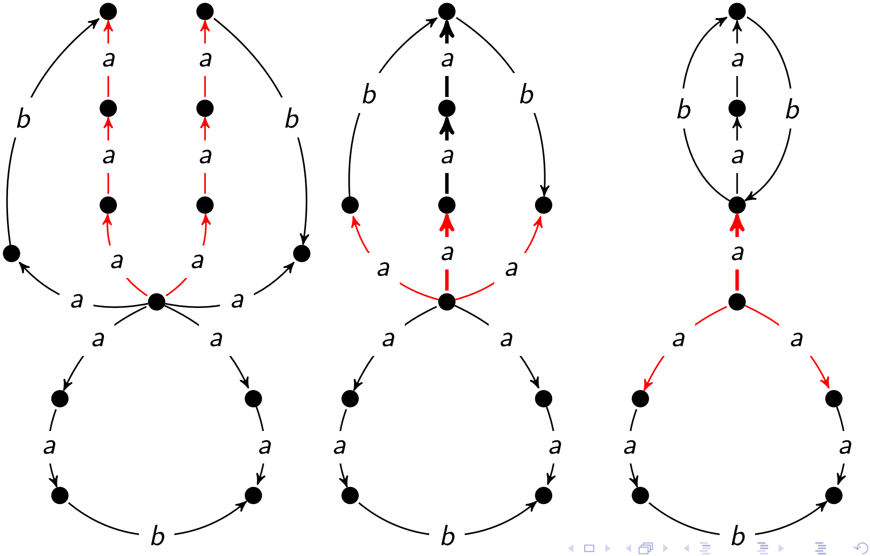

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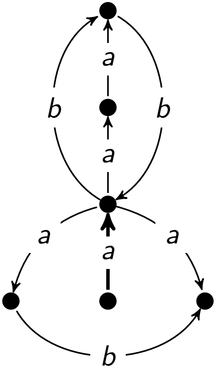

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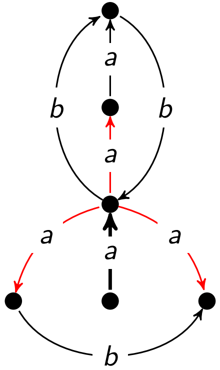

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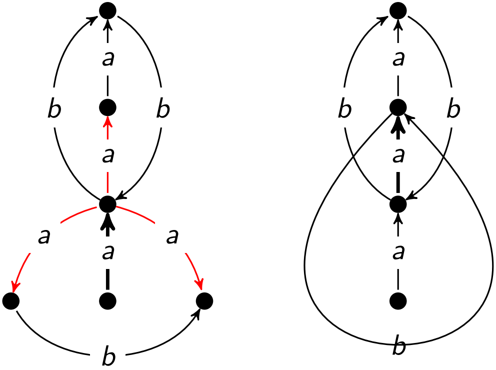

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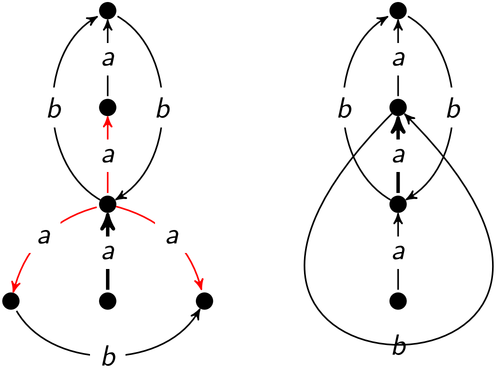

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A word is in the subgroup if and only if it is the label of a path from the base vertex to the base vertex.

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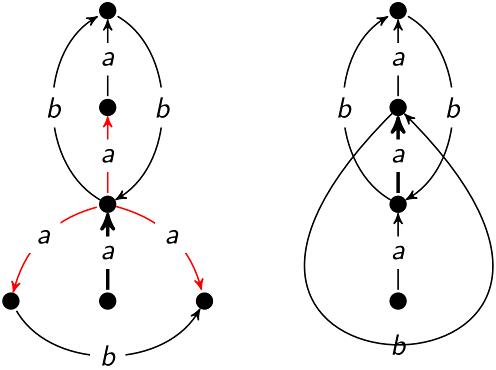

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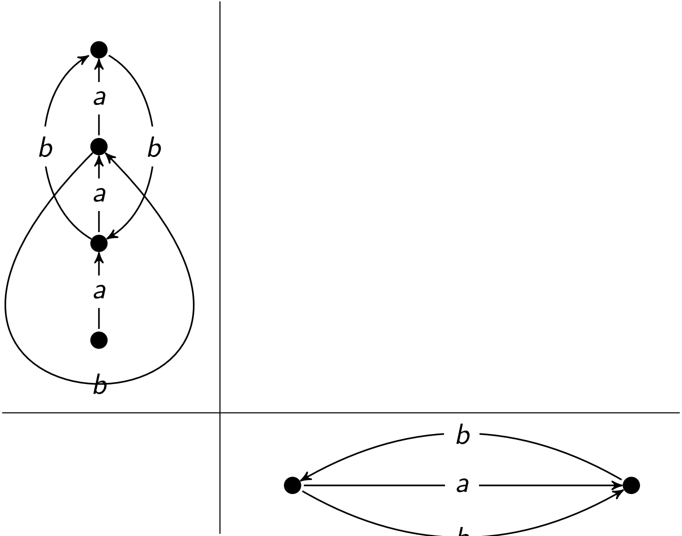
A word is in the subgroup if and only if it is the label of a path from the base vertex to the base vertex.

Example

ab^4a^{-1} is in the subgroup, but aba^{-1} and $ababa^{-1}$ are not.

The product graph



The Product Graph

$V(\Gamma \times \Delta) = V\Gamma \times V\Delta.$

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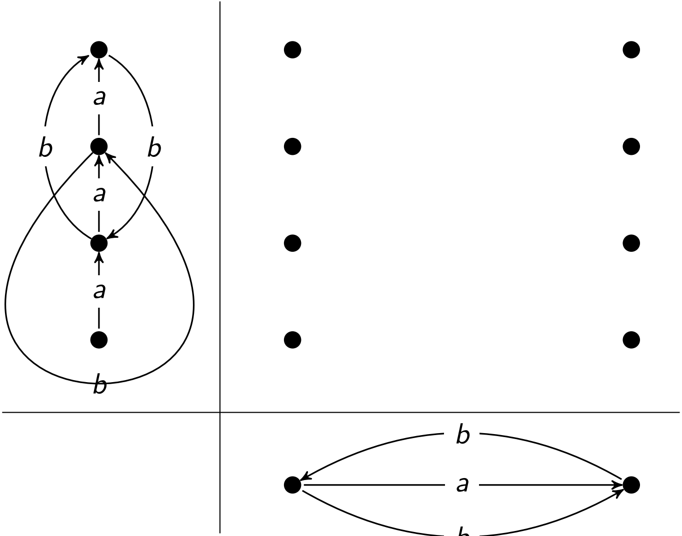

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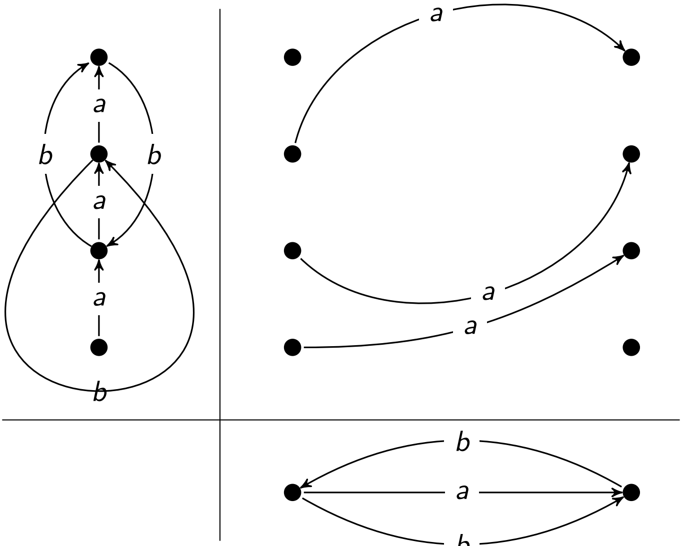

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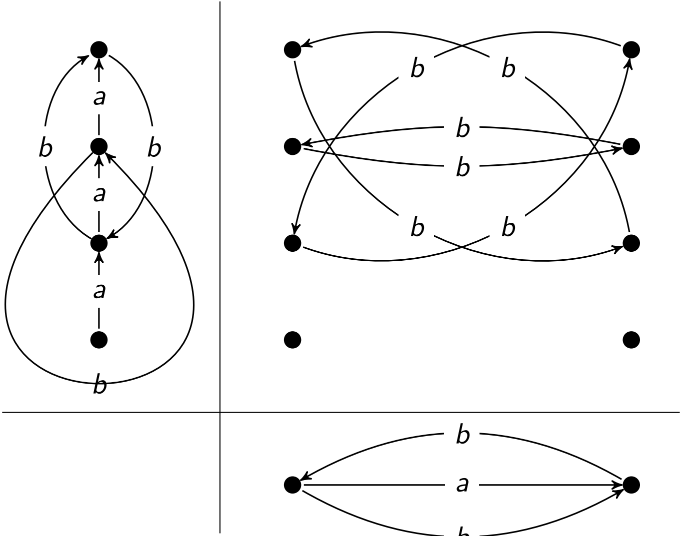

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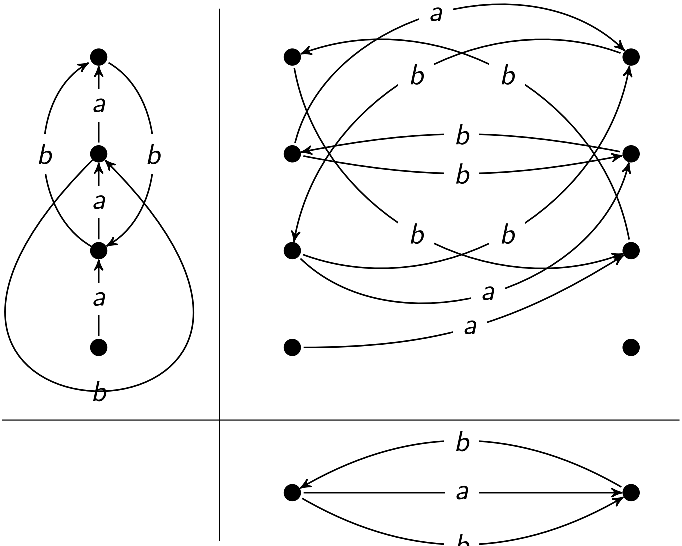

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Free splittings with common elliptic word

Theorem (YBK)

Given two subgroups H and K of a free group F, there is a nontrivial conjugacy class that intersects both of them if and only if the product graph of the Stallings graphs of H and K has a cycle.

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Two free splittings

Given $A * B$ and $C * D$, we test if there is a nontrivial cyclic word elliptic to both by checking the four pairs (A, C), (A, D), (B, C), and (B, D) for having a nontrivial conjugacy class that intersects both.

Two cyclic words elliptic to a common free splitting

Question

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Whitehead Automorphisms

For a free group F generated by a finite set X, the Whitehead automorphisms are a finite set of basic automorphisms that generate all of $\mathrm{Aut}(F)$. Their original use was in the Whitehead algorithm to test if cyclic words are in the same orbit of $\mathrm{Out}(F)$.

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Testing for a common free splitting (YBK)

- Look for a Whitehead automorphism that, when applied to (v, w), creates a pair with smaller total length.

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- $\Lambda(v') \cup \Lambda(w') \neq X$, *or*
- $\Lambda(v')$ *and* $\Lambda(w')$ *are disjoint.*

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Remark

In the first case, then v and w have representatives in the same factor of the free splitting. In the second case, they are in different factors.

Thank You

Acknowledgements

I'd like to thank Ilya Kapovich University and Kim Whittlesey at the University of Illinois at Urbana-Champaign for introducing me to this question, and I'd like to thank Matthew Day for mentoring me further in this subject at Caltech.

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