Distance in the Ellipticity Graph

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Introduction

Motivation

- For a free group F, we wish to gain insight into the group of automorphisms $\mathrm{Aut}(F)$ by studying spaces on which $\mathrm{Aut}(F)$ acts.
- One such space is the ellipticity graph $\mathcal{Z}(F)$, defined by I. Kapovich and M. Lustig in a 2009 paper, which contains free splittings and conjugacy classes of F.

Results

I created algorithms for determing when two vertices of $\mathcal{Z}(F)$ are adjacent to a common vertex.

Preliminaries

Definition

A free splitting $A * B$ is a decomposition of F into two subgroups A and B that generate F but do not have relations between them.

Example

If $F = \langle a, b, c \rangle$, then the following are free splittings of F:

- $\langle a \rangle * \langle b, c \rangle$
- $\langle ab^2 \rangle * \langle ab^2ab^3, c \rangle$

Definition

A cyclic word is a conjugacy class of the free group F.

Definition

A cyclic word is elliptic to a free splitting $A * B$ if it has a representative in either A or B.

The Ellipticity Graph

Definition

For a free group F, the ellipticity graph $\mathcal{Z}(F)$ is bipartite graph, with the following vertex classes.

- Notrivial cyclic words of F.
- Nontrivial free splittings $A * B$ of F, up to the equivalence relation where $A * B$ is equivalent to $(xAx^{-1}) * (xBx^{-1})$ and to $(xBx^{-1}) * (xAx^{-1})$ for all $x \in F$.

A cyclic word w is adjacent to a free splitting $A * B$ if w is elliptic to $A * B$.

Remark

The inner (conjugation) automorphisms of F fix $\mathcal{Z}(F)$, so $\mathrm{Out}(F)$, the group of outer automorphisms of F, acts on $\mathcal{Z}(F)$.

Distance Two in $\mathcal{Z}(F)$

Questions

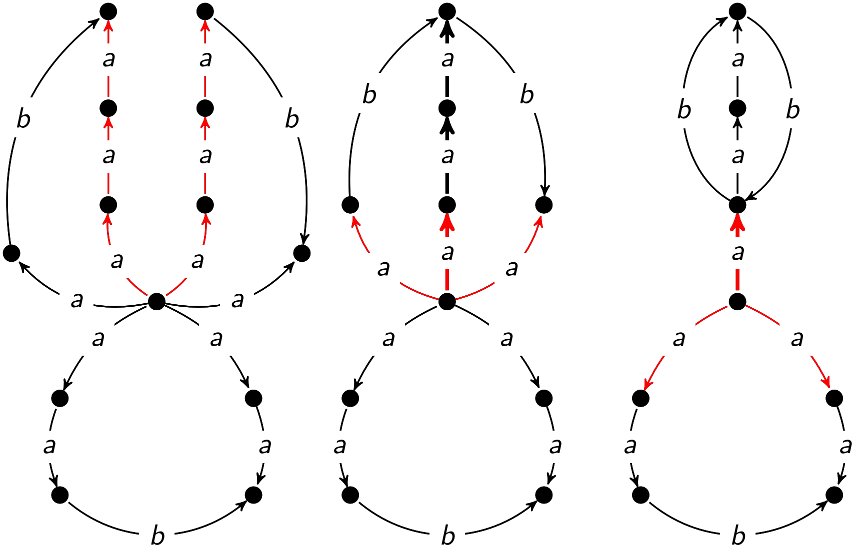
- *Given two free splittings $A * B$ and $C * D$ of F, is there a nontrivial cyclic word of F elliptic to both?*
- Given two cyclic words v and w of F, is there a nontrivial free splitting of F to which they are both elliptic?

Two free splittings with common elliptic word

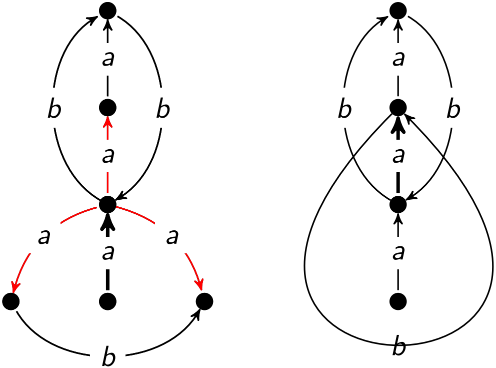
Given finitely generated subgroups H and K of F, is there a notrivial conjugacy class that intersects both?

Stallings Folding

$H = \langle aba^{-3}, a^2ba^{-2}, a^3ba^{-1} \rangle < F = \langle a, b \rangle$



Stallings Folding



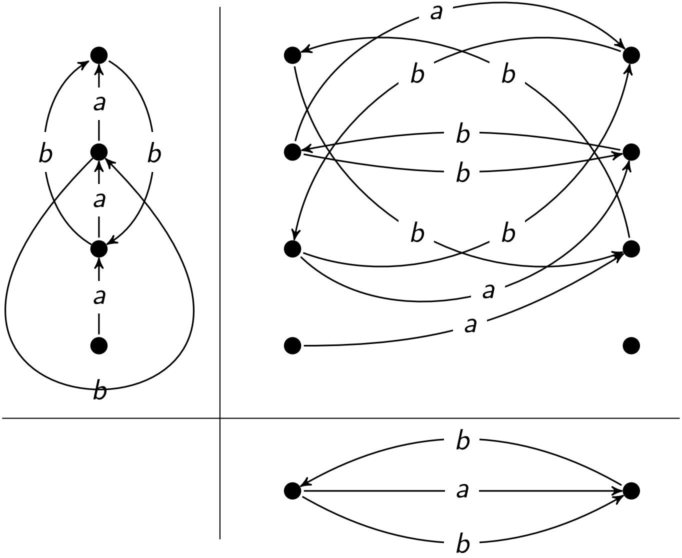
Theorem (Stallings)

A word is in the subgroup if and only if it is the label of a path from the base vertex to the base vertex.

Example

ab^4a^{-1} is in the subgroup, but aba^{-1} and $ababa^{-1}$ are not.

The product graph



The Product Graph

$V(\Gamma \times \Delta) = V\Gamma \times V\Delta.$

$E(\Gamma \times \Delta) = \{(e, f) \in E\Gamma \times E\Delta \mid e \text{ and } f$ have the same label$\}$.

Free splittings with common elliptic word

Theorem (YBK)

Given two subgroups H and K of a free group F, there is a nontrivial conjugacy class that intersects both of them if and only if the product graph of the Stallings graphs of H and K has a cycle.

Two free splittings

Given $A * B$ and $C * D$, we test if there is a nontrivial cyclic word elliptic to both by checking the four pairs (A, C), (A, D), (B, C), and (B, D) for having a nontrivial conjugacy class that intersects both.

Two cyclic words elliptic to a common free splitting

Question

Given two cyclic words v and w of F, is there a nontrivial free splitting of F to which they are both elliptic?

Whitehead Automorphisms

For a free group F generated by a finite set X, the Whitehead automorphisms are a finite set of basic automorphisms that generate all of $\mathrm{Aut}(F)$. Their original use was in the Whitehead algorithm to test if cyclic words are in the same orbit of $\mathrm{Out}(F)$.

Two cyclic words elliptic to a common free splitting

Testing for a common free splitting (YBK)

- Look for a Whitehead automorphism that, when applied to (v, w), creates a pair with smaller total length.
- If such a Whitehead automorphism exists, apply it to (v, w), and repeat with the new pair. If none exist, call the resulting pair (v', w').
- Define $\Lambda(v') = \{x \in X \mid x \text{ or } x^{-1} \text{ appears in } v'\}$, similarly for $\Lambda(w')$.

Theorem (YBK)

The words v and w are elliptic to some splitting $A * B$ if and only if either

- $\Lambda(v') \cup \Lambda(w') \neq X$, *or*
- $\Lambda(v')$ *and* $\Lambda(w')$ *are disjoint.*

Remark

In the first case, then v and w have representatives in the same factor of the free splitting. In the second case, they are in different factors.

Thank You

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References

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