Differential equations

Math 217 — Spring 2008

Practice exam April

This practice exam contains fourteen problems numbered 1 through 14. Problems 1 – 13 are multiple choice problems. Problem 14 is a free-response question.

Problem 1

Write the system

$$\begin{aligned} x'' &= y - 2x \\ y' &= x + 2y \end{aligned}$$

as a linear differential equation.

A) $x''' - 2x'' + 2x' = 0$ **B)** $x''' + x'' + x' + x = 0$ **C)** $x''' - 2x'' - 5x = 0$
D) $x''' = 0$ **E)** $x''' + 2x' - 5x = 0$ **F)** $x''' - 2x'' + 2x' - 5x = 0$

Problem 2

The functions $x(t)$ and $y(t)$ are solutions to the initial value problem

$$\begin{aligned} x' &= 3x - 2y \qquad & x(0) &= 3 \\ y' &= 5x - 4y, \qquad & y(0) &= 6. \end{aligned}$$

Use the improved Euler method with step-size $h = 0.2$ to find approximations of $x(0.2)$ and $y(0.2)$.

A) $x(0.2) = 2.58$, $y(0.2) = 4.58$ **B)** $x(0.2) = 2.58$, $y(0.2) = 4.60$ **C)** $x(0.2) = 2.58$, $y(0.2) = 4.62$

D) $x(0.2) = 2.60$, $y(0.2) = 4.58$ **E)** $x(0.2) = 2.60$, $y(0.2) = 4.60$ **F)** $x(0.2) = 2.60$, $y(0.2) = 4.62$

Problem 3

It is known that the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ a & 2 \end{bmatrix}$$

has eigenvalues $\lambda = 1$ and $\lambda = 3$. What is a?

A) -1 **B)** 0 **C)** 1 **D)** 2 **E)** 3 **F)** 4

Problem 4

Find an eigenvector of

$$\mathbf{A} = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$$

corresponding to the eigenvalue $\lambda = -3$.

A) $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ **B)** $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ **C)** $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ **D)** $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ **E)** $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ **F)** $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

Problem 5

Let

$$\mathbf{A} = \begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix}.$$

Which of the following statements about $\mathbf{A}$ is correct.

I) Any vector $\left[\begin{smallmatrix} c_1 \\ c_2 \end{smallmatrix}\right] \neq \mathbf{0}$ (c_1 and c_2 are constant) is a generalized eigenvector of $\mathbf{A}$.

II) The vector $\left[\begin{smallmatrix} -e^{4t} \\ e^{4t} \end{smallmatrix}\right]$ is a solution to the equation $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

III) The vector $\left[\begin{smallmatrix} -te^{4t} \\ te^{4t} \end{smallmatrix}\right]$ is a solution to the equation $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

A) Only I **B)** Only II **C)** Only III **D)** I and II **E)** I and III
F) II and III

Problem 6

Find the solution to the initial value problem

$$\mathbf{x}' = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

A) $\frac{1}{7}\begin{bmatrix} 3e^{-2t} + 4e^{5t} \\ -9e^{-2t} + 2e^{5t} \end{bmatrix}$ **B)** $\frac{1}{7}\begin{bmatrix} 4e^{-2t} + 3e^{5t} \\ 2e^{-2t} - 9e^{5t} \end{bmatrix}$ **C)** $\frac{1}{7}\begin{bmatrix} 9e^{-2t} - 2e^{5t} \\ -4e^{-2t} - 3e^{5t} \end{bmatrix}$

D) $\frac{1}{7}\begin{bmatrix} 3e^{2t} + 4e^{-5t} \\ -9e^{2t} + 2e^{-5t} \end{bmatrix}$ **E)** $\frac{1}{7}\begin{bmatrix} 4e^{2t} + 3e^{-5t} \\ 2e^{2t} - 9e^{-5t} \end{bmatrix}$ **F)** $\frac{1}{7}\begin{bmatrix} 9e^{2t} - 2e^{-5t} \\ -4e^{2t} - 3e^{-5t} \end{bmatrix}$

Problem 7

The matrix

$$\mathbf{\Phi}(t) = \begin{bmatrix} e^{4t} & e^{-3t} \\ 0 & 5e^{-3t} \end{bmatrix}$$

is a fundamental matrix of the matrix **A**. What is the solution of the problem

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} -1 \\ 3 \end{bmatrix}?$$

A) $\begin{bmatrix} -\frac{8}{5}e^{4t} + \frac{3}{5}e^{-3t} \\ 3e^{-3t} \end{bmatrix}$ **B)** $\begin{bmatrix} -e^{4t} \\ 3e^{-3t} \end{bmatrix}$ **C)** $\begin{bmatrix} -\frac{8}{5}e^{4t} + \frac{3}{5}e^{-3t} \\ \frac{7}{5}e^{4t} + \frac{8}{5}e^{-3t} \end{bmatrix}$

D) $\begin{bmatrix} -e^{-3t} \\ \frac{7}{5}e^{4t} + \frac{8}{5}e^{3t} \end{bmatrix}$ **E)** $\begin{bmatrix} -e^{4t} \\ 3e^{4t} \end{bmatrix}$ **F)** $\begin{bmatrix} -e^{-3t} \\ 3e^{4t} \end{bmatrix}$

Problem 8

For

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix},$$

which of the following is an eigenvalue of $e^{\mathbf{A}}$?

A) 0 **B)** 1 **C)** 2 **D)** 3 **E)** π **F)** e

Problem 9

What is the form of a particular solution of

$$\mathbf{x}' = \begin{bmatrix} 4 & -3 \\ 2 & -3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 15 \\ 4 \end{bmatrix} e^{-2t} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cos 2t?$$

Below, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$ denote vector coefficients.

A) $\mathbf{a}e^{-2t} + \mathbf{b}\cos 2t$ **B)** $\mathbf{a}te^{-2t} + \mathbf{b}\cos 2t$ **C)** $\mathbf{a}e^{-2t} + \mathbf{b}\cos 2t + \mathbf{c}\sin 2t$
D) $\mathbf{a}te^{-2t} + \mathbf{b}\cos 2t + \mathbf{c}\sin 2t$ **E)** $\mathbf{a}e^{-2t} + \mathbf{b}te^{-2t} + \mathbf{c}\cos 2t + \mathbf{d}\sin 2t$
F) $\mathbf{a}e^{-2t} + \mathbf{b}te^{-2t} + \mathbf{c}t\cos 2t + \mathbf{d}t\sin 2t$

Problem 10

Find a particular solution of

$$\mathbf{x}' = \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 4 \\ e^{2t} \end{bmatrix}.$$

A) $\begin{bmatrix} -2 + t^2 e^{2t} \\ e^{2t} \end{bmatrix}$ **B)** $\begin{bmatrix} t^2 e^{2t} \\ e^{2t} \end{bmatrix}$ **C)** $\begin{bmatrix} 2 + t^2 e^{2t} \\ e^{2t} \end{bmatrix}$ **D)** $\begin{bmatrix} -2 + t^2 e^{2t} \\ te^{2t} \end{bmatrix}$

E) $\begin{bmatrix} t^2 e^{2t} \\ te^{2t} \end{bmatrix}$ **F)** $\begin{bmatrix} 2 + t^2 e^{2t} \\ te^{2t} \end{bmatrix}$

Problem 11

Let $F(s)$ be the Laplace transform of the function

$$f(t) = \begin{cases} 0, & t < 1 \\ t^2, & t \geq 1. \end{cases}$$

Find $F(1)$.

A) e^{-1} **B)** 5 **C)** $5e^{-1}$ **D)** e **E)** -5 **F)** $5e$

Problem 12

Let g be the inverse Laplace transform of

$$G(s) = \frac{1}{s} - \frac{s+2}{s^2+4}.$$

What is $g(1)$? Choose the closest value.

A) -0.9 **B)** -0.2 **C)** 0.5 **D)** 1.2 **E)** 1.9 **F)** 2.6

Problem 13

Find the Laplace transform of the equation

$$x'' - x' - 6x = 0, \qquad x(0) = 2, \quad x'(0) = -1.$$

A) $s^2 - s - 6 = 0$ **B)** $s^2 X - 2s + 1 = 0$ **C)** $(s^2 + s + 6)X - 2s - 1 = 0$
D) $(s^2 - s - 6)X + 2s + 3 = 0$ **E)** $(s^2 - s - 6)X - 2s + 3 = 0$
F) $(s^2 + s + 6)X + 2s - 1 = 0$

The following problem is a free-response question. You should justify your answers.

Problem 14

a) Find the general solution of

$$\mathbf{x}' = \begin{bmatrix} 3 & 4 & 5 \\ 0 & 5 & 4 \\ 0 & 0 & 3 \end{bmatrix} \mathbf{x}.$$

b) Find $e^{\mathbf{A}}$, where

$$\mathbf{A} = \begin{bmatrix} 3 & 4 & 5 \\ 0 & 5 & 4 \\ 0 & 0 & 3 \end{bmatrix}.$$

c) Solve the initial value problem

$$\mathbf{x}' = \begin{bmatrix} 3 & 4 & 5 \\ 0 & 5 & 4 \\ 0 & 0 & 3 \end{bmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$