Differential equations

Math 217 — Spring 2008

In-term exam March 4th.

This exam contains fourteen problems numbered 1 through 14. Problems 1 – 13 are multiple choice problems, which each count 6% of your total score. Problem 14 will be hand-graded and counts 22% of your total score.

Problem 1

Use the improved Euler method with step-size $h = 1.0$ to find an approximate value of $y(2)$, where $y(x)$ is the solution of

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{3y^2}{x} = 0, \qquad y(1) = 1.$$

A) -7.00 **B)** -5.25 **C)** -3.50 **D)** -1.75 **E)** 0.00 **F)** 1.75

Here $f(x, y) = -\frac{3y^2}{x}$, while $x_0 = 1$, and $y_0 = 0$. For the improved Euler method, we first find the regular Euler point:

$$u_1 = y_0 + h \cdot f(x_0, y_0) = 1 + 1 \cdot (-3) = -2.$$

We then use this point to find the improved Euler point:

$$y(2) \approx y_1 = y_0 + h \cdot \tfrac{1}{2}\big(f(x_0, y_0) + f(x_1, u_1)\big) = 1 + 1 \cdot \tfrac{1}{2}\big((-3) + (-6)\big) = -3.50.$$

Problem 2

Find the solution to the differential equation

$$y'' - 4y' + 5y = 0$$

satisfying the initial conditions $y(0) = 1$, $y'(0) = 5$.

A) $\cos x$ **B)** $\sin x$ **C)** $e^{2x}\cos x$ **D)** $3e^{2x}\sin x$ **E)** $e^{2x}\big(\cos x + 3\sin x\big)$
F) $e^{2x}\big(\cos x - 3\sin x\big)$

The characteristic equation $r^2 - 4r + 5 = 0$ has solutions $r = 2 \pm i$, so a general solution of the differential equation is

$$y(x) = Ae^{2x}\cos x + Be^{2x}\sin x.$$

The condition $y(0) = 1$ then implies that $A = 1$. Inserting that into $y(x)$ and differentiating gives

$$y'(x) = e^{2x}\big((2 + B)\cos x + (2B - 2)\sin x\big).$$

Then $y'(0) = 2 + B = 5$, so $B = 3$. Thus

$$y(x) = e^{2x}\big(\cos x + 3\sin x\big).$$

Problem 3

Consider the following pairs of functions.

i) $f(x) = x, \quad g(x) = x^2,$

ii) $f(x) = e^x, \quad g(x) = x,$

iii) $f(x) = 3x - 5, \quad g(x) = 9x - 12,$

iv) $f(x) = \cos^2 x, \quad g(x) = 1 - \sin^2 x,$

v) $f(x) = e^x, \quad g(x) = e^{\pi+x},$

vi) $f(x) = x, \quad g(x) = x^{-1}.$

How many of the pairs i) - vi) are linearly independent on the interval $(0, 1)$?

A) 1 **B)** 2 **C)** 3 **D)** 4 **E)** 5 **F)** 6

To check if a pair of functions is linearly independent, we can either compute their Wronskian or we can simple check if one is a multiple of the other by division.

For instance for i) the Wronskian is

$$\begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = x \cdot 2x - x^2 \cdot 1 = x^2.$$

Since x^2 is not identically zero, the functions x and x^2 are linearly independent.

Alternatively, we could see that they are linearly independent by considering the quotient

$$\frac{x}{x^2} = \frac{1}{x}.$$

Since this quotient is not a constant, the functions are linearly independent.

Checking all the pairs, we conclude that i), ii), iii), and vi) are linearly independent, while iv) and v) are linearly dependent.

Problem 4

Find a particular solution of the nonhomogeneous differential equation

$$y'' - 4y = e^{2x}.$$

A) 0 **B)** $-\frac{1}{4}e^{2x}$ **C)** $\frac{1}{4}e^{2x}$ **D)** $-\frac{1}{4}xe^{2x}$ **E)** $\frac{1}{4}xe^{2x}$ **F)** $\frac{1}{4}x^2e^{2x}$

The complementary solution is $y_C = Ae^{2x} + Be^{-2x}$. Since there is a term of the form Ae^{2x} apperaring in the complementary solution, we choose a particular solution of the form $y_P = Cxe^{2x}$. Its derivatives are

$$y_P' = 2Cxe^{2x} + Ce^{2x},$$

and

$$y_P'' = 4Cxe^{2x} + 4Ce^{2x}.$$

Then

$$y_P'' - 4y_p = 4Cxe^{2x} + 4Ce^{2x} - 4Cxe^{2x} = 4Ce^{2x}.$$

It follows that $C = \frac{1}{4}$.

Problem 5

Find the general solution of $y''' - 2y'' + y' = 0$.

A) $Ae^x + Be^x$ **B)** $Ae^x + Bxe^x$ **C)** $Ae^x + Be^x + C$ **D)** $Ae^x + Bxe^x + C$
E) $Ae^x + Be^x + Cx$ **F)** $Ae^x + Bxe^x + Cx$

The characteristic equation is $r^3 - 2r^2 + r = 0 \Leftrightarrow r(r-1)^2 = 0$. The root $r = 0$ gives the constant solution C, while the twice repeated root $r = 1$ corresponds to $(A + Bx)e^x$.

Problem 6

A simple pendulum satisfies

$$x'' + (2.17)^2 x = 0, \qquad x(0) = 3, \quad x'(0) = 4.$$

What is the amplitude of its motion? Pick the closest value.

A) 3.5   **B)** 4.0   **C)** 4.5   **D)** 5.0   **E)** 5.5   **F)** 6.0

The general solution of the equation is

$$x(t) = A\cos(2.17t) + B\sin(2.17t),$$

so

$$x'(t) = -2.17A\sin(2.17t) + 2.17B\cos(2.17t).$$

The initial conditions implies that $A = x(0) = 3$, and then $B = \frac{x'(0)}{2.17} = \frac{4}{2.17}$. We then find the amplitude as

$$C = \sqrt{A^2 + B^2} = \sqrt{3^2 + (\tfrac{4}{2.17})^2} \approx 3.52.$$

Problem 7

The general solution of $y'' - 2y' + y = 0$ is

$$y_C(x) = (C_1 + C_2 x)e^x.$$

Find the form of a particular solution of

$$y'' - 2y' + y = x(1 - \cos x) + e^x.$$

A) $Ax(1 - \cos x) + Be^x$
B) $(A_0 + A_1 x)(1 - \cos x) + Be^x$
C) $(A_0 + A_1 x) + (B_0 + B_1 x)\cos x + Cx^2 e^x$
D) $(A_0 + A_1 x) + (B_0 + B_1 x)\cos x + (C_0 + C_1 x)\sin x + Dx^2 e^x$
E) $(A_0 + A_1 x) + (B_0 + B_1 x)(\cos x + \sin x) + Cx^2 e^x$
F) $(A_0 + A_1 x) + (B_0 + B_1 x)\cos x + (C_0 + C_1 x)\sin x + (D_0 + D_1 x + D_2 x^2)e^x$

To find a particular solution we can consider the three terms x, $x\cos x$, and e^x independently. For x we need a degree one polynomial; $A_0 + A_1 x$. For $x\cos x$, we need $\cos x$ times a degree one polynomial and $\sin x$ times another degree one polynomial; $(B_0 + B_1 x)\cos x + (C_0 + C_1 x)\sin x$. For e^x we ordinarily just needs an exponent, but because both e^x and xe^x appears as solutions of the complementary solution, we need to multiply by x^2. Thus we need $Dx^2 e^x$.

Problem 8

A complementary solution of

$$y'' + y = \frac{1}{\cos x}$$

is $y_C = C_1 \cos x + C_2 \sin x$. Find a particular solution.

Hint: The formula

$$\int \frac{\sin x}{\cos x}\,\mathrm{d}x = -\ln|\cos x| + C$$

may be useful.

A) $\frac{1}{\cos x}$ **B)** $x \sin x$ **C)** $\cos x \cdot \ln|\cos x|$ **D)** $x \sin x + \cos x \cdot \ln|\cos x|$
E) $x \sin x - \cos x \cdot \ln|\cos x|$ **F)** $\cos x + x \sin x$

Because $\frac{1}{\cos x}$ has infinitely many lineraly independent derivatives, we need to use variation of parameters. Taking $y_1(x) = \cos x$ and $y_2(x) = \sin x$ (with Wronskian 1), the formula gives

$$y_P(x) = -\cos x \int \frac{\sin x}{\cos x}\,\mathrm{d}x + \sin x \int \frac{\cos x}{\cos x}\,\mathrm{d}x = \cos x \cdot \ln|\cos x| + x \sin x.$$

Problem 9

Consider the end-point problem

$$y'' + \lambda y = 0, \qquad y(0) = 0, \quad y(1) = 0.$$

For which of the following λ does there exist a non-trivial solution?

A) $\lambda = -\pi^2$ **B)** $\lambda = -1$ **C)** $\lambda = 0$ **D)** $\lambda = 1$ **E)** $\lambda = \pi$
F) $\lambda = \pi^2$

For this problem we can either check the 6 different equations resulting from the 6 different choices of λ in the alternatives, or we can find all the eigenvalues of the problem (that is, all values of λ which give non-trivial solutions). In this case it seems that the latter will take less work!

First, let us check if λ can be negative. For convenience write $\lambda = -\alpha^2$, $\alpha > 0$. Then we are considering $y'' - \alpha^2 y = 0$, with general solution $y(x) = Ae^{-\alpha x} + Be^{\alpha x}$. The two initial conditions then implies that $A + B = 0$ and $Ae^{-\alpha} + Be^{\alpha} = 0$, and it follows that $A = B = 0$.

If $\lambda = 0$, we are looking at $y'' = 0$ with general solution $y(x) = A + Bx$. Again the initial conditions implies that $A = B = 0$.

Therefore the only possible eigenvalues are positive. Let $\lambda = \alpha^2$, $\alpha > 0$. Then $y'' + \alpha^2 y = 0$ has general solution $y(x) = A \cos x + B \sin x$. The condition $y(0) = 0$ implies that $A = 0$. The second condition then implies that $B \sin \alpha = 0$. Since $\sin \alpha$ is zero for α being integer multiples of π, we find non-trivial solutions whenever α is of the form $n\pi$, $n = 1, 2, 3, \ldots$. This implies that λ has the form $n^2\pi^2$, $n = 1, 2, 3, \ldots$.

Problem 10

The functions $y_1(x) = \cos x$, $y_2(x) = \sin x$, $y_3(x) = e^x$, and $y_4(x) = e^{-x}$ are four linearly independent solutions of $y^{(4)} - y = 0$. Which of the following functions are also solutions?

I) $\sin 2x \; (= 2 \sin x \cos x)$

II) $\cos x - \sin x$

III) $e^x \cos x + e^{-x} \cos x - \sin x$

A) Only I **B)** Only II **C)** Only III **D)** I and II **E)** I and III **F)** II and III

The principle of superposition says that linear combinations of y_1, y_2, y_3, and y_4 are also solutions, and furthermore, these are the only solutions. Thus, of the functions given only II is a solution.

Problem 11

Apply Euler's method to approximate the solution of the initial value problem

$$y' = x + \frac{1}{5}y, \qquad y(0) = -3.$$

Find y(2) with step-size $h = 1$. Pick the closest value.

A) -3 **B)** -3.1 **C)** -3.2 **D)** -3.3 **E)** -3.4 **F)** -3.5

We use Euler's method with $h = 1$, $x_0 = 0$, $y_0 = -3$, and $f(x, y) = x + \frac{1}{5}y$. For the first step,

$$y_1 = y_0 + h \cdot f(x_0, y_0) = -3 + 1 \cdot \left(0 + \tfrac{-3}{5}\right) = -3.6.$$

The second step is then,

$$y_2 = y_1 + h \cdot f(x_1, y_1) = -3.6 + 1 \cdot \left(1 + \tfrac{-3.6}{5}\right) = -3.32.$$

Problem 12

Which of the following differential equations describes a spring-mass system which exhibits the phenomenon of resonance?

A) $x'' + 3x' + 2x = 0$
B) $x'' + 3x' + 2x = \cos 2t$
C) $x'' + 3x' + 2x = \sin t$
D) $x'' + x = \frac{1}{2}\sin t$
E) $x'' + x = 0$
F) $x'' + x = \cos 2t$

Resonance happens whenever the "natural frequency" of a system is the same as the frequency of the external force. We must therefore look at what the "natural frequency" is (if there is any) for the two systems

$$x'' + 3x' + 2x = 0 \qquad \text{and} \qquad x'' + x = 0.$$

The first equation has general solution $y(x) = Ae^{-t} + Be^{-2t}$, so it represents an overdamped system, which does not oscillate and therefore has no "natural frequency". Thus, no resonance can occur.

The second equation has general solution $y(x) = A\cos t + B\sin t = C\cos(t - \alpha)$, so it represents a system with natural frequency 1. Therefore resonance occurs when the external force also has frequency 1.

Problem 13

Write down the characteristic equation of $y''' - 7y' + 6y = 2x$.

A) $r^3 - 7r + 6 = 0$ **B)** $r^3 - 7r + 6 = 2$ **C)** $r^3 - 7r = 6$ **D)** $r^4 - 7r^2 + 6r = 0$
E) $r^4 - 7r^2 + 6r = 2$ **F)** $r - 7r + 6r = 0$

Writing down an r for each derivative of y, we find $r^3 - 7r + 6 = 0$.

Name: **Student-ID:** **Section:** ☐ 10 – 11 Tang ☐ 12 – 1 Hjelle

The following problem will be hand-graded. To earn full credit you need to justify your answers.

Problem 14

a) (8 points) Show that $y_1(x) = e^{-3x}\cos 4x$ and $y_2(x) = e^{-3x}\sin 4x$ are solutions of

$$y'' + 6y' + 25y = 0,$$

and compute their Wronskian.

Showing that y_1 and y_2 are solutions can be done either by computing y_i' and y_i'' $(i = 1, 2)$ and plugging them into the equation to see that they produce 0, or by solving the given equation using the characteristic equation.

The Wronskian of y_1 and y_2 is

$$\begin{aligned} W(y_1, y_2) &= \begin{vmatrix} e^{-3x}\cos 4x & e^{-3x}\sin 4x \\ e^{-3x}\big(-3\cos 4x - 4\sin 4x\big) & e^{-3x}\big(-3\sin 4x + 4\cos 4x\big) \end{vmatrix} \\ &= e^{-6x}\big(-3\cos 4x\sin 4x + 4\cos^2 4x + 3\cos 4x \sin 4x + 4\sin^2 4x\big) \\ &= \underline{\underline{4e^{-6x}}}. \end{aligned}$$

In particular, we note that the Wronskian is never zero, so y_1 and y_2 are two linearly independent solutions. Thus a general solution of the equation is

$$y(x) = Ay_1(x) + By_2(x) = e^{-3x}\big(A\cos 4x + B\sin 4x\big).$$

b) (6 points) Find a particular solution $y_P(x)$ of

$$y'' + 6y' + 25y = 195\cos 2x.$$

Since there are no terms of the form $\cos 2x$ or $\sin 2x$ in the general solution, we choose a particular solution

$$y_P(x) = C_1\cos 2x + C_2\sin 2x.$$

Differentiating,

$$\begin{aligned} y_P'(x) &= -2C_1\sin 2x + 2C_2\cos 2x, \\ y_P''(x) &= -4C_1\cos 2x - 4C_2\sin 2x. \end{aligned}$$

Substituting back into the equation,

$$y_P'' + 6y_P' + 25y_P = \big(-4C_1 + 12C_2 + 25C_1\big)\cos 2x + \big(-4C_2 - 12C_1 + 25C_2\big)\sin 2x.$$

So, to determine the coefficients C_1 and C_2 we solve the system

$$\begin{cases} 21C_1 + 12C_2 = 195 \\ -12C_1 + 21C_2 = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} C_1 = 7 \\ C_2 = 4. \end{cases}$$

Thus, a particular solution is

$$\underline{\underline{y_P(x) = 7\cos 2x + 4\sin 2x}}.$$

c) (8 points) Consider a damped mass-and-spring system with mass $m = 2$, damping constant $c = 12$, and spring constant $k = 50$ under the influence of the external force $F_E(t) = 390\cos 2t$. Assume the system starts with $x(0) = 5$ and $x'(0) = 2$.

Set up a differential equation describing the motion $x(t)$ of the mass, and solve it.

A differential equation describing the system is $mx'' + cx' + kx = F_E(t)$. Inserting the values we find

$$2x'' + 12x' + 50x = 390\cos 2t, \qquad x(0) = 5, \quad x'(0) = 2,$$

which after dividing through by the mass $m = 2$ becomes

$$x'' + 6x' + 25x = 195\cos 2t, \qquad x(0) = 5, \quad x'(0) = 2.$$

From a) and b), we know this equation has general solution

$$x(t) = e^{-3t}\big(A\cos 4t + B\sin 4t\big) + 7\cos 2t + 4\sin 2t.$$

We finally use the initial conditions to determine A and B. We have

$$\begin{aligned} 5 &= x(0) = A + 7 & &\Rightarrow & A &= -2, \\ 2 &= x'(0) = -3A + 4B + 8 = 14 + 4B & &\Rightarrow & B &= -3. \end{aligned}$$

So the solution is

$$\underline{\underline{x(t) = e^{-3t}\big(-2\cos 4t - 3\sin 4t\big) + 7\cos 2t + 4\sin 2t}}.$$