Differential equations

Math 217 — Spring 2008

In-term exam February 5th. — Solutions

This exam contains fourteen problems numbered 1 through 14. Problems 1 – 13 are multiple choice problems, which each count 6% of your total score. Problem 14 will be hand-graded and counts 22% of your total score.

Problem 1

Solve

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x}, \qquad y(1) = 2.$$

What is $y(2)$?

A) 0 **B)** 1 **C)** 2 **D)** 3 $\boxed{\textbf{E)}\ 4}$ **F)** 5

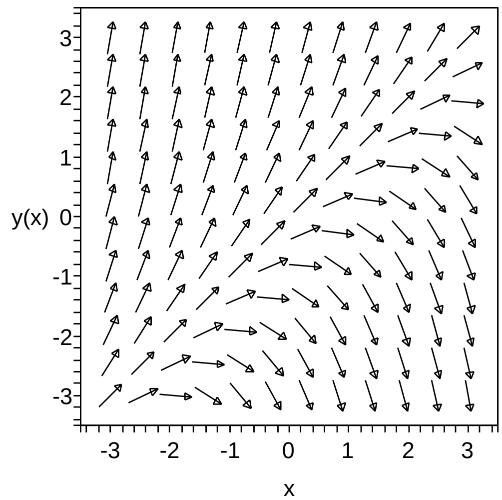
This is a separable equation:

$$\int \frac{1}{y}\,\mathrm{d}y = \int \frac{1}{x}\,\mathrm{d}x.$$

So $\ln y = \ln x + C_1$, which after exponentiation gives $y = C_2 x$. Inserting $Y(1) = 2$ gives $C_2 = 2$, so $y(x) = 2x$.

Problem 2

Identify the differential equation whose slope field looks like this:



A) $y' = -y - \sin x$ **B)** $y' = x + y$ **C)** $y' = x - y$ $\boxed{\textbf{D)}\ y' = y - x + 1}$
E) $y' = x - y + 1$ **F)** $y' = \sin x + \sin y$

There are several ways to do this. One way is to look at each alternative, and see if it fits with the diagram. For instance can A) and F) fairly quickly be ruled out, since there is no oscillation that would appear if a sin was present.

We can also note that the horisontal lines (corresponding to $y' = 0$) seem to lie on the line $y = x - 1 \Leftrightarrow y - x + 1 = 0$. Of the alternatives, this only happens for D.

Problem 3

How many solutions does the equation

$$y' = \frac{\sin t}{t + y^2}$$

have satisfying the initial value $y(0) = 1$?

A) None $\boxed{\textbf{B)}\ 1}$ **C)** 2 **D)** 3 **E)** 4 **F)** ∞

At and close to the point $(t, y) = (0, 1)$, both the function $f(t, y) = \frac{\sin t}{t+y^2}$ and the partial derivative $\frac{\partial f}{\partial y} = -\frac{2y \sin t}{(t+y^2)^2}$ are continuous, so the equation has one unique solution there.

Problem 4

A population $P(t)$ of sparrows is known to satisfy the natural growth equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP.$$

Find k if $P(0) = 217$ and $P(2.5) = 2008$. Pick the closest answer.

A) 0.8 $\boxed{\textbf{B)}\ 0.9}$ **C)** 1.0 **D)** 1.1 **E)** 1.2 **F)** 1.3

The natural growth equation has solution $P(t) = Ce^{kt}$ (found by separation if it is not already known). Here $C = P(0) = 217$. To find k we use the second condition,

$$P(2.5) = 2008 = 217e^{2.5k} \qquad \Leftrightarrow \qquad k = \tfrac{1}{2.5}\ln\tfrac{2008}{217} \approx 0.9.$$

Problem 5

Find the solution of the initial value problem

$$y' + \frac{2}{t}y = \frac{\cos t}{t^2}, \qquad y(\pi) = 0,\ t > 0.$$

A) $\frac{\sin t}{t}$ **B)** $\frac{\sin(t^2)}{t}$ $\boxed{\textbf{C)}\ \frac{\sin t}{t^2}}$ **D)** $\frac{\sin(t^2)}{t^2}$ **E)** $\frac{\cos(t^2)}{t}$ **F)** $\frac{\cos t}{t^2}$

This is linear with integrating factor $e^{\int \frac{2}{t}\,\mathrm{d}t} = e^{\ln t^2} = t^2$. Therefore

$$t^2 y = \int t^2 \frac{\cos t}{t^2}\,\mathrm{d}t = \int \cos t\,\mathrm{d}t = \sin t + C,$$

and

$$y(t) = \frac{\sin t + C}{t^2}.$$

Inserting $y(\pi) = 0$ yields $C = 0$.

Problem 6

At time $t = 0$, a tank contains 50 lb. of salt dissolved in 100 gallons of water. Assume that water containing $\frac{1}{4}$ lb. of salt per gallon is entering the tank at a rate of 3 gallons per minute and that the well-stirred mixture is draining from the tank at the same rate. Find the limiting amount salt that is present after a very long time.

A) 100 lb. **B)** 50 lb. $\boxed{\textbf{C)}\ 25\text{ lb.}}$ **D)** 0.25 lb. **E)** 3 lb. **F)** 15 lb.

The tank is modeled by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 3 \cdot \tfrac{1}{4} - 3 \cdot \frac{x}{100},$$

which is a linear equation with general solution

$$x(t) = 25 + Ce^{-3t/100}$$

which tends to 25 as $t \to \infty$.

The solution can also be found by observing that after a long time, all the salt and water initially in the tank is gone. Therefore the tank will have the same concentration as the inflow, thus there will be $\frac{1}{4}$ lb./gal $\cdot$ 100 gal $=$ 25 lb.

Problem 7

Which one of the following equations is exact?

A) $(2x + 4y) + (2x - 2y)y' = 0$

B) $\big(x \ln(y) + xy\big)\,\mathrm{d}x + \big(y \ln(x) + xy\big)\,\mathrm{d}y = 0$

$\boxed{\textbf{C)}\ (x + 2y) + (2x + 3y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0}$

D) $(x - 2y) + (2x - 3y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$

E) $(3xy + y^2) + (x^2 + xy)y' = 0$

F) $e^x \sin y - (3x - e^x \sin y)y' = 0$

C is exact because $\frac{\partial}{\partial y}(x + 2y) = 2 = \frac{\partial}{\partial x}(2x + 3y)$.

Problem 8

Find the solution to the initial value problem

$$2xyy' = 4x^2 + 3y^2, \qquad y(1) = -1.$$

A) $y = -5x^3 + 4x$ **B)** $y = -5x^3 + 4x^2$ **C)** $y = -5x^2 + 4x^3$
D) $y^2 = 5x^3 - 4x$ $\boxed{\textbf{E)}\ y^2 = 5x^3 - 4x^2}$ **F)** $y^2 = 5x^2 - 4x^3$

If we divide through by $2xy$, we get the homogeneous equation $y' = 2\frac{x}{y} + \frac{3}{2}\frac{y}{x}$. By substituting $v = \frac{y}{x} \Leftrightarrow y = vx, y' = v + xv'$, we convert this to the separable equation

$$v + xv' = 2\tfrac{1}{v} + \tfrac{3}{2}v \qquad \Leftrightarrow \qquad xv' = \frac{2}{v} + \frac{v}{2} = \frac{4+v^2}{2v}.$$

Separating and integrating gives us that $\ln(4 + v^2) = \ln x + C_1$, so that

$$4 + v^2 = C_2 x.$$

Substituting back $v = \frac{y}{x}$, multiplying through with x^2, and subtracting $4x^2$ on both sides,

$$y^2 = C_2 x^3 - 4x^2.$$

From the condition $y(1) = -1$, we see that $C_2 = 5$.

Problem 9

Consider the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x(x+1)(x-10).$$

Which of the following statements are true?

I) $x = 0$ is a stable critical point.

II) $x = 1$ is an unstable critical point.

III) $x = 10$ is a stable critical point.

$\boxed{\textbf{A)}\text{ Only I}}$ **B)** Only II **C)** Only III **D)** I and II **E)** I and III
F) II and III

Solving $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$ and drawing the phase diagram, we see that

I) $x = 0$ is indeed a stable critical point — TRUE.

II) $x = 1$ is not even a critical point — FALSE.

III) $x = 10$ is an unstable critical point — FALSE.

Problem 10

Suppose that the logistic equation $x' = x(5 - x)$ models a population $x(t)$ of fish in a lake after t months during which no fishing occurs. Now suppose that, because of fishing, fish are removed from the lake at the rate of $3x$ fish per month. What is the limiting population?

A) 0 $\boxed{\textbf{B)}\ 2}$ **C)** 4 **D)** 5 **E)** 6 **F)** 8

With fishing, the situation is modeled by

$$x' = x(5 - x) - 3x \qquad \Leftrightarrow \qquad x' = 5x - x^2 - 3x = 2x - x^2 = x(2 - x),$$

which again is a logistic equation, this time with limiting population 2.

Problem 11

A baseball that falls towards the ground satifies

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -\rho v - g,$$

where ρ is some positive constant and $g = 32\mathrm{ft/s}^2$. Assume that $v(0) = 0$ and $\lim_{t\to\infty} v(t) = -100$. Find $v(1)$, and pick the closest answer.

A) -21 **B)** -24 $\boxed{\textbf{C)}\ -27}$ **D)** -30 **E)** -33 **F)** -36

The given equation is linear with integrating factor $e^{\rho t}$ and solution

$$v(t) = -\tfrac{g}{\rho} + Ce^{-\rho t}.$$

From the initial condition $v(0) = 0$ we find $C = \frac{g}{\rho}$, and as the limiting velocity is

$$\lim_{t\to\infty} v(t) = -\tfrac{g}{\rho} = -100,$$

we have $\rho = \frac{g}{100}$. Then

$$v(1) = -100 + 100e^{-32/100} \approx -27.$$

Problem 12

Solve

$$y' - 2xy = 2xy^2, \qquad y(0) = 1.$$

A) $2e^{-x^2} - 1$ **B)** $\boxed{\frac{1}{2e^{-x^2}-1}}$ **C)** e^{x^2} **D)** e^{-x^2} **E)** $2 - e^{x^2}$ **F)** $\frac{1}{2-e^{x^2}}$

This is a Bernoulli equation with $n = 2$, so we use the substitution $v = y^{-1} = \frac{1}{y}$ to transform it to the linear equation

$$v' + 2xv = -2x$$

with integrating factor e^{x^2}, and solution (the factor $2x$ on the right-hand side helps us perform the integral by substitution)

$$v(x) = -1 + Ce^{-x^2}.$$

Substituting back,

$$y(x) = \frac{1}{Ce^{-x^2} - 1}.$$

Then $y(0) = 1$ implies that $C = 2$.

Problem 13

Find the limit $\lim_{t\to\infty} y(t)$ where $y(t)$ is the solution to the initial value problem

$$y' + 2y = 4, \qquad y(0) = 1.$$

A) 0 **B)** 1 $\boxed{\textbf{C)}\ 2}$ **D)** 3 **E)** 4 **F)** The limit does not exist

The equation is both linear and separable with solution

$$y(t) = 2 - e^{-2t}.$$

As $t \to \infty$ the term e^{-2t} tends to zero, so $\lim_{t\to\infty} y(t) = 2$.

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The following problem will be hand-graded. To earn full credit you need to justify your answers.

Problem 14

Consider a cascade of two tanks. Tank 1 initially contains 100 gallons of pure ethanol and tank 2 initially contains 100 gallons of pure water. Pure water flows into tank 1 at 10 gallons per minute, and the other two flow rates are also 10 gallons per minute.

a) (8 points) Find the amount $x(t)$ of ethanol in tank 1.

Since there is only pure water flowing in, the equation for the first tank is

$$x' = 0 - 10 \cdot \tfrac{x}{100} = -\tfrac{x}{10}.$$

This is separable and linear, and has solution $x(t) = Ce^{-t/10}$. The initial condition $x(0) = 100$ gives $C = 100$, so finally

$$\underline{\underline{x(t) = 100e^{-t/10}}}.$$

b) (8 points) Find the amount $y(t)$ of ethanol in tank 2.

For tank 2, there is inflow of ethanol from tank 1. The equation in this case becomes

$$y' = 10 \cdot \tfrac{x}{100} - 10 \cdot \tfrac{y}{100} = 10e^{-t/10} - \tfrac{y}{10}.$$

This is a linear equation with integrating factor $e^{t/10}$, so

$$e^{t/10}y = 10 \int e^{t/10}e^{-t/10}\,\mathrm{d}t = 10 \int \mathrm{d}t = 10t + C.$$

Dividing through the integrating factor, and using that $y(0) = 0$ gives

$$\underline{\underline{y(t) = 10te^{-t/10}}}.$$

c) (6 points) Find the maximum amount of ethanol ever in tank 2.

We can find the time at which the maximum occurs by solving $y' = 0$. As

$$y'(t) = 10e^{-t/10} + 10t \cdot (-\tfrac{1}{10})e^{-t/10} = (10 - t)e^{-t/10},$$

the maximum occurs at time $t = 10$, and that maximum is

$$\underline{\underline{y(10) = 100e^{-1}}}.$$