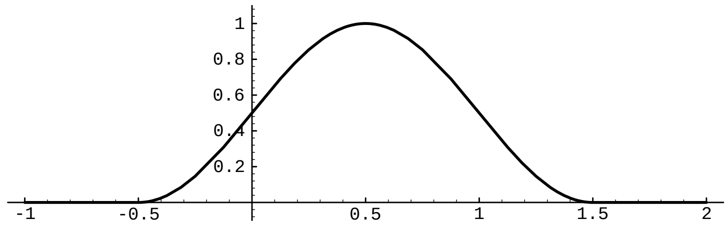

Figure 3.2: Wide Hanning window function.

Since $u(0-) = u'(0-) = u''(0-) = 0$ and $u(1+) = u'(1+) = u''(1+) = 0$, continuity at 0 and 1 of u and its first two derivatives requires $A - B + C = 0$ and $B - 4C = 0$. It is also convenient for u to have maximum value 1, but this must occur at the critical point $t = \frac{1}{2}$ where $u(t) = A + B + C$. The resulting system of three linear equations has a unique solution $A = 0.375$, $B = 0.500$, and $C = 0.125$, which gives the window illustrated in the right half of Figure 3.1. It has an additional nice property since $\cos 4\pi t = 2\cos^2 2\pi t - 1$:

$$u(t) = \frac{1}{8}(3 - 4\cos 2\pi t + \cos 4\pi t) = \frac{1}{4}(1 - \cos 2\pi t)^2 \geq 0,$$

for all $t \in [0, 1]$. Evidently, this window is the square of the Hanning window, suggesting a generalization: let $u^n(t) = [(1 - \cos 2\pi t)/2]^n$ for $t \in [0, 1]$, with $u^n(t) = 0$ elsewhere. Then u^n will be continuous and will have n continuous derivatives, as well as being nonnegative with maximum value 1 at $t = \frac{1}{2}$. We may expand u^n as a sum of cosines; it will have $n + 1$ terms since

$$\left[\frac{1 - \cos 2\pi t}{2}\right]^n = \frac{a(0)}{2} + \sum_{j=0}^{n} a(j)\cos 2\pi jt; \qquad a(j) \stackrel{\text{def}}{=} 2\sum_{k=j}^{n} \binom{n}{k}\binom{2k}{k-j}\left(\frac{-1}{4}\right)^k.$$

This expansion is called the *Fourier series* for u^n.

Rather than pinch off the signal within the interval of interest to get a smooth function, we can mix in parts of the signal just outside the interval by using a wider window. For example, let $I = [0, 1]$ and define $w = w(t)$ by

$$w(t) = \begin{cases} 0, & \text{if } t < -\frac{1}{2} \text{ or } t > \frac{3}{2}; \\ (1 + \sin \pi t)/2, & \text{if } -\frac{1}{2} \leq t \leq \frac{3}{2}. \end{cases} \tag{3.6}$$

Its graph is plotted in Figure 3.2. This is just the Hanning window composed with the substitution $t \leftarrow \frac{1}{2}(t + \frac{1}{2})$, and has the same smoothness: continuity and one continuous derivative.

Periodic extension

A function localized to a bounded interval I can always be periodized since the sum in Equation 3.2 will be finite for any $T > 0$. A natural choice of period is $T = |I|$,