Research Statement

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1 Background

Consider H^2, the Hardy space of the unit circle $\mathbb{T}$, and the forward shift operator $S : f(z) \mapsto zf(z)$. A classical theorem of Beurling's [2] classifies every invariant subspace of S:

Theorem 1.1. *If u is an inner function, the set uH^2 is an S-invariant subspace of H^2. Conversely, every proper non-trivial S-invariant subspace of H^2 equals uH^2 for some inner function u.*

A consequence of this is that every proper non-trivial subspace of H^2 invariant under the backwards shift S^* is equal to $H^2 \ominus uH^2$ for some inner function u. Let $K_u^2 = H^2 \ominus uH^2$. K_u^2 is a Hilbert space with inner product equal to the Hardy space inner product

$$\langle f, g \rangle = \int_{\mathbb{T}} f(w)\overline{g(w)}\, dm(w)$$

where m is normalized arc length measure on $\mathbb{T}$. Further, these spaces are reproducing kernel Hilbert spaces with reproducing kernel K_z^u. The motivation for studying these model spaces was originally to understand the behavior of S^* on the Hardy space [5] and because of their central role in model theory [12].

A truncated Toeplitz operator (TTO) is an operator A_Φ, where Φ (called a *symbol*) is in $L^2(\mathbb{T})$, and $A_\Phi f = P_u(\Phi f)$ for $f \in K_u^2$, where P_u is the orthogonal projection from $L^2(\mathbb{T})$ to K_u^2. A special example of a TTO is the truncated shift $S_u = A_z = P_u S$ and its adjoint $S_u^* = S^*$. In fact, given A_Φ a TTO, its adjoint is $A_{\overline{\Phi}}$, another TTO. For any $\Phi \in L^2(\mathbb{T})$ this is densely defined on the bounded functions in K_u^2. In general, a bounded TTO does not have a bounded symbol [1]. However, in [10], Sarason proved the following:

Theorem 1.2. *If T is an operator on K_u^2 that commutes with S_u, then there is a function $\phi \in H^\infty$ such that $\|\phi\|_\infty = \|T\|$ and $T = \phi(S_u)$.*

I concern myself with the cases where the resulting operator is bounded.

A TTO has a certain symmetry called C-symmetry. Let C be the antilinear involutive isomorphism that sends f in $L^2(\mathbb{T})$ to $u\overline{zf}$. The notation $\tilde{f}$ is sometimes used in place of Cf. This operator fixes K_u^2, and can naturally be thought of as an operator on K_u^2. An operator A is said to be C-symmetric if $A^* = CAC$. C-symmetry is a broad topic; much of the early basic work is due to Garcia and Putinar [7, 8, 9].

The study of TTOs is motivated in part by the fact that a surprisingly large number of operators are unitarily equivalent to TTOs (for example, any normal operator), and an even larger number of operators are similar to TTOs (for example, any operator on a finite dimensional Hilbert space) [4]. In addition, there are connections between symbols of TTOs and solutions to interpolation problems on the disc that allow one to solve interpolation problems via operator theory [10].

Brown and Halmos [3] studied the algebraic properties of Toeplitz operators. Recently, Sarason [11] has published an overview of TTOs, including some recent results which mirror those of Brown and Halmos. One of the most useful results is an operator-theoretic characterization of TTOs.

Theorem 1.3. *A bounded operator A on K_u^2 is a TTO if and only if $A - S_u A S_u^* = \phi \otimes K_0^u + K_0^u \otimes \psi$ for some $\phi, \psi \in K_u^2$. In this case, $A = A_{\phi+\overline{\psi}}$.*

$f \otimes g$ is the operator that maps h to $f \langle h, g \rangle$.

2 Current work

My current work was inspired by the following result of Brown and Halmos: The product of two Toeplitz operators is a Toeplitz operator if and only if the first Toeplitz operator has an antiholomorphic symbol, or the second Toeplitz operator has a holomorphic symbol. In either case the symbol of the product operator is the product of the symbols of the two original Toeplitz operators. The same result is clearly not true for TTOs.

Example 2.1. Let $A_z = S_u$ be the compressed forward shift. Then $S_u^* S_u = I - \widetilde{K_0^u} \otimes \widetilde{K_0^u}$, which by Theorem 1.3 is not a TTO.

Using Sarason's condition (Theorem 1.3), we can prove the following result.

Theorem 2.2. *Let $\phi_1, \phi_2, \psi_1, \psi_2 \in K_u^2$ such that $A_{\phi_1+\overline{\phi_2}}, A_{\psi_1+\overline{\psi_2}}$ are both bounded. Then $A_{\phi_1+\overline{\phi_2}} A_{\psi_1+\overline{\psi_2}}$ is a TTO if and only if there exist Φ, Ψ such that $\phi_1 \otimes \psi_2 - S_u\widetilde{\phi_2} \otimes S_u\widetilde{\psi_1} = \Phi \otimes K_0^u + K_0^u \otimes \Psi$.*

It turns out that this is true if both $A_{\phi_1+\overline{\phi_2}}$ and $A_{\psi_1+\overline{\psi_2}}$ are of the form $B_{\varphi_i}^{\alpha} := A_{\varphi_i+\alpha\overline{S_u\widetilde{\varphi_i}}}$ for some $\varphi_i \in K_u^2$ and $\alpha \in \mathbb{C}^*$, where the case $\alpha = \infty$ is taken to mean that the symbols are the complex conjugates of functions in K_u^2. I call TTOs of this form TTOs of type α. The product of two TTOs of type α is another TTO of type α. In fact, excluding the trivial case, this condition is also necessary.

Theorem 2.3. *Let A_1, A_2 be bounded TTOs. Then $A_1 A_2$ is a TTO if and only if one of the following is true: (1) One of the A_i is of the form cI, or (2) Both A_is are of type α for some $\alpha \in \mathbb{C}^*$. In the second case, the product is itself a TTO of type α.*

This result has several corollaries:

Corollary. *Let u be an inner function and consider TTOs on K_u^2.*

1. *Every nilpotent and idempotent TTO is of type α for some α.*
2. *If A is an invertible TTO, then A^{-1} is a TTO if and only if A is of type α for some α. In this case, A^{-1} is also of type α.*
3. *Every unitary TTO is of type α for some unimodular α.*

This leads to the question of when the operator B_{φ}^{α} is invertible, for which I have the following partial result.

Theorem 2.4. *Let $|\alpha| \leq 1$, let u be a finite Blaschke product and let $\varphi \in K_u^2$. Then B_{φ}^{α} is invertible if and only if $\varphi(\lambda) \neq 0$ for all $\lambda \in \overline{\mathbb{D}}$ such that $u(\lambda) = \alpha$.*

Theorem 1.2 says that every operator of type 0 is $\phi(S_u)$ for some bounded holomorphic ϕ, or in other words every TTO of type 0 has a bounded symbol. By taking advantage of a correspondence between operators of type α in the space K_u^2 and operators of type zero in a related space $K_{u_\alpha}^2$, we can prove the following for $|\alpha| < 1$ (and thus a similar result for $|\alpha| > 1$ by taking adjoints):

Theorem 2.5. *Let A be a bounded TTO. Then there exists a bounded function Φ such that $A = A_{\Phi}$.*

3 Future Direction

Let $|\alpha| = 1$. Then the set of all TTOs of type α is a commuting algebra of normal operators. It follows that they are all simultaneously diagonalizable. By a result of Clark [6] we know that for a given unimodular α there exists a unitary operator $U_\alpha = S_u + \frac{\alpha}{1-\alpha\overline{u(0)}} K_0^u \otimes \widetilde{K_0^u}$, called the Clark unitary operator, and that U_α is unitarily equivalent by way of a unitary operator $\mathcal{U}_\alpha$ to multiplication by z in the space $L^2(\mathbb{T}, d\mu_\alpha)$ where $d\mu_\alpha$ is the spectral measure of U_α. U_α is in fact of type α, and so there is a $\mathcal{U}_\alpha$-equivalence between TTOs of type α and multiplication operators with holomorphic symbols on $L^2(\mathbb{T}, d\mu_\alpha)$. I have shown the following:

Proposition 3.1. *Let $|\alpha| = 1$.*

1. $U_\alpha = \frac{1}{1-\alpha\overline{u(0)}} B^\alpha_{S_u K_0^u + \left(\alpha\overline{u'(0)}K_0^u\right)/\left(1-\alpha\overline{u(0)}\right)}$.
2. $M_z = \frac{1}{1-\alpha\overline{u(0)}} M_{S_u K_0^u + \left(\alpha\overline{u'(0)}K_0^u\right)/\left(1-\alpha\overline{u(0)}\right)}$ *on* $L^2(\mathbb{T}, d\mu_\alpha)$.

Note that in the special case that $u(0) = u'(0) = 0$ the first expression simplifies to $U_\alpha = B^\alpha_z$. This, plus some similar results and computing some examples, suggests the following.

Conjecture. *Let $\varphi \in K^2_u$. Then B^α_φ is $\mathcal{U}$-equivalent to M_φ on $L^2(\mathbb{T}, d\mu_\alpha)$.*

If $|\alpha| = 1$ by using the $\mathcal{U}_\alpha$ unitary equivalence we can show that if T is of type α then there is a function ϕ in $H^\infty(d\mu_\alpha)$ such that $T = \phi(U_\alpha)$. In this sense we can get a "bounded" symbol for any operator of unimodular type. I hope to determine whether or not Theorem 1.2 can be extended to the case that α is unimodular.

Another question suggested by the Brown-Halmos paper – but not in Sarason's – is what are necessary and sufficient conditions for two TTOs to commute. In the Hardy space, two Toeplitz operators commute if and only if both Toeplitz operators are holomorphic, both are antiholomorphic, or one is a linear transformation of the other.

Suppose A and B are C-symmetric operators. Then $AB = BA$ if and only if AB is C-symmetric. Therefore, if A and B are operators of type α, then $AB = BA$. I have proven the following conjecture for some specific inner functions u, but do not know if it is true in general.

Conjecture. *Suppose A, B are TTOs and $AB = BA$. Then either A and B are of type α for some α, or one of the TTOs is a linear transformation of the other.*

Lastly, the Bergman space of the disc (the space of holomorphic functions on $\mathbb{D}$ square summable with respect to area measure) contains many algebras of Bergman space Toeplitz operators (in this setting defined as the projection down to the Bergman space of multiplication operators), each generated by Toeplitz operators with symbols from certain subalgebras C of $L^\infty(\mathbb{D})$, as laid out in a book by Vasilevski [13]. While the underlying spaces are very different, I have hopes of finding equivalent results in the setting of TTOs on model spaces.

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