The Apical Angle: A Mathematical Analysis of the Ellipse

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BACKGROUND. The elliptical excision is a common surgical procedure. The dermatologic literature predominantly describes an excisional geometry with a 3:1 length:width ratio and an apical angle of 30°.

OBJECTIVE. To analyze the elliptical excision by applying mathematical principles and define the apical angle and its relationship to the length:width ratio.

METHODS. We examined numerous examples of elliptical excisions as presented in the dermatologic literature. We analyzed the geometry of the excisions and defined it mathematically.

RESULTS. The apical angle of a 3:1 elliptical excision is not 30°. The true apical angle varies from 37° to 74° depending on excisional geometry.

CONCLUSION. The commonly presented apical angle of 30° is incorrect and does not reflect the true apical angle of elliptical excisions.

THE APICAL ANGLE is the angle created at the ends of an elliptical excision. The elliptical excision, more properly termed a fusiform excision, and its variants are commonly utilized for excisional surgery. Many texts have described the geometry of the ellipse.[1–7] We present a detailed mathematical analysis of the apical angle and define the relationship between the apical angle and the length:width ratio of the elliptical excision. The literature predominantly advocates an apical angle of 30°. Our analysis demonstrates that 30° is not generated.

Methods

Certain assumptions must be made with regards to how the surgeon designs the excision. We analyzed examples of elliptical excisions as presented in the literature.[1–7] The relationship between the apical angle (α) and the length:width ratio (l/w, ρ) depends on the geometry the surgeon creates in performing the excision. Bennett,[1] in his text on cutaneous surgery, provided a complete discussion of the construction of the ellipse. In his and other author's examples, the arc of a circle is employed as the incision. However, other assumptions may be made, such as using the arc of a parabola or employing straight-line incisions at the angles, creating a rhomboidally derived shape.

Tissue distensibility, while important to the surgeon, is not relevant to the mathematical definition of a geometric figure. For example, the circumference of a circle drawn in sand, clay, or stone is always defined by the equation $C = 2\pi r$, despite radically different properties of the substances. Therefore, for the purposes of this study, tissue characteristics may be discounted entirely.

Using the assumption that the surgeon creates the excision by incising two arcs from a circle of radius r, we derived the relationship between the apical angle and the length:width ratio. Figure 1 demonstrates the arc from a circle of radius r. To define the relationship of the apical angle (α) and l/w, one must establish the relationship between $\alpha/2$ and l and w. This can be done by employing the Pythagorean theorem:

$$r^2 = l^2 + (r - w)^2 \tag{1}$$

$$r = (l^2 + w^2)/2w \tag{2}$$

$$\tan \alpha/2 = l/(r - w) = 2wl/(l^2 - w^2) \tag{3}$$

$$\alpha = 2\tan^{-1}[2wl/(l^2 - w^2)]. \tag{4}$$

In terms of the length:width ratio ($\rho = l/w$),

$$\alpha = 2\tan^{-1}[2\rho/(\rho^2 - 1)]. \tag{5}$$

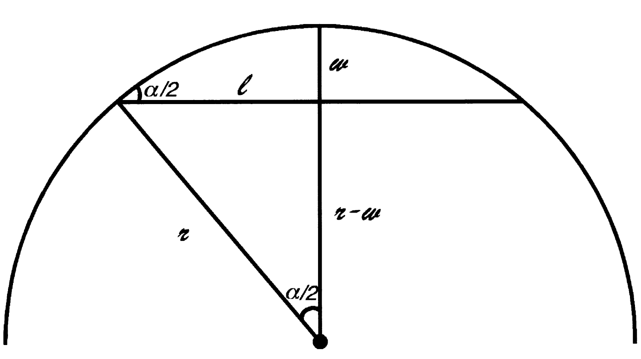

Figure 1. Circular arc used to define the mathematical relationship of length:width ratio and apical angle for a fusiform excision derived from the arcs of a circle. Similar mathematical reasoning is employed to define the relationships for parabolic and rhomboidally derived excisions.

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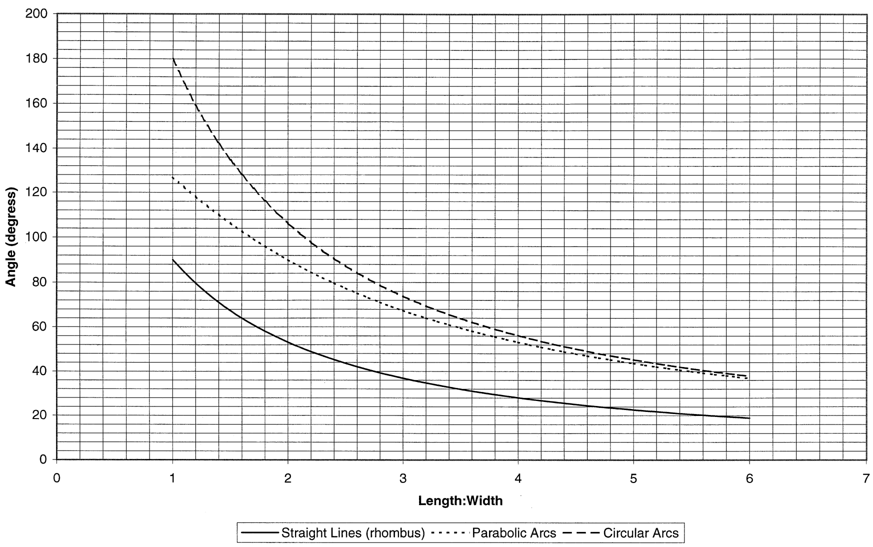

Figure 2. Graphical presentation of the relationships of apical angle and length:width ratio for straight lines, parabolic, and circular arcs.

One can also solve for ρ, thus defining the length:width ratio as a function of the apical angle (α): $\rho = \cot[\alpha/2] + \csc[\alpha/2]$.

Using similar mathematical reasoning, we derived the relationship of α and ρ for parabolically and rhomboidally derived figures.

Rhombus: $\alpha = 2\cot^{-1}\rho$, thus $\rho = \cot[\alpha/2]$.

Parabola: $\alpha = 2\tan^{-1}(2/\rho)$, thus $\rho = 2\cot[\alpha/2]$.

These relationships are presented graphically (Figure 2) and in tabular form (Table 1).

Discussion

Our results demonstrate that the relationship of apical angle to length:width ratio are different from those commonly presented. The literature predominantly advocates an apical angle of 30° and a length:width ratio of ≥3:1.[1–7] To generate a 3–3.5:1 ratio and maintain a 30° apical angle, straight-line cuts, such as in a rhombus, would have to be employed. As demonstrated, with a rhombus or variant (as the case would be if the surgeon

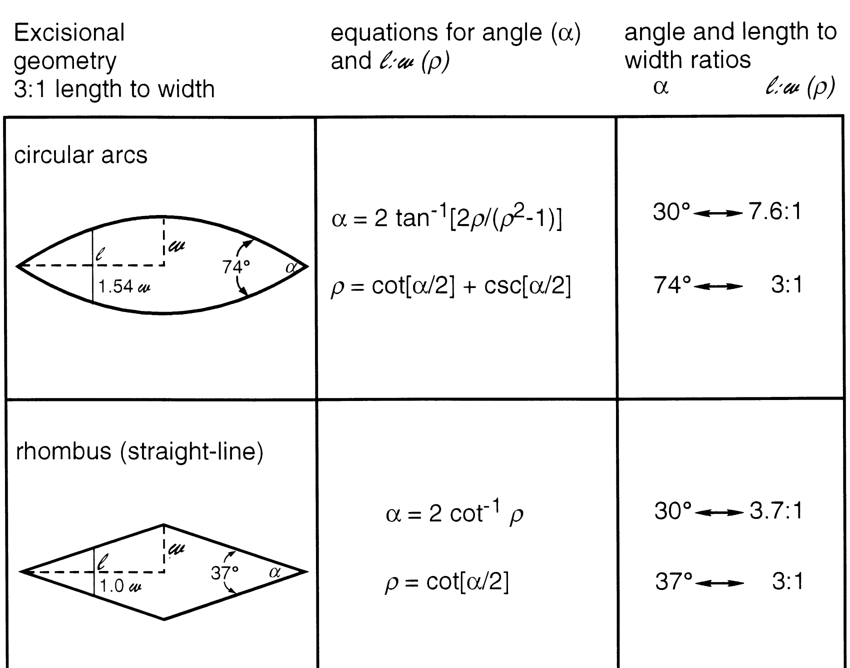

Figure 3. Comparison of excisional geometry, mathematical relationships, and representative angles for circular arcs and rhomboid excisions.

Table 1. Length:Width Ratio and Apical Angles for Straight-Line, Parabolic, and Circular Arc-Derived Fusiform Excisions

	α (degrees)		
Length:Width Ratio	*Straight Lines*	*Parabolic Arcs*	*Circular Arcs*
1	90	127	180
1.1	85	122	169
1.2	80	118	159
1.3	75	114	150
1.4	71	110	142
1.5	67	106	135
1.6	64	103	128
1.7	61	99	122
1.8	58	96	116
1.9	56	93	111
2	53	90	106
2.1	51	87	102
2.2	49	85	98
2.3	47	82	94
2.4	45	80	90
2.5	44	77	87
2.6	42	75	84
2.7	41	73	81
2.8	39	71	79
2.9	38	69	76
3	37	67	74
3.1	36	66	72
3.2	35	64	69
3.3	34	62	67
3.4	33	61	66
3.5	32	59	64
3.6	31	58	62
3.7	30	57	60
3.8	29	56	59
3.9	29	54	58
4	28	53	56
4.1	27	52	55
4.2	27	51	54
4.3	26	50	52
4.4	26	49	51
4.5	25	48	50
4.6	25	47	49
4.7	24	46	48
4.8	24	45	47
4.9	23	44	46
5	23	44	45
5.1	22	43	44
5.2	22	42	44
5.3	21	41	43
5.4	21	41	42
5.5	21	40	41
5.6	20	39	40
5.7	20	39	40
5.8	20	38	39
5.9	19	37	38
6	19	37	38

were to smooth the obtuse angle), a length:width ratio of 3.7:1 will generate apical angles of 30°. To maintain a 30° angle with a smaller length:width ratio would require a nonconvex excision.

Bennett,[1] employing circular arcs, presented similar results to ours, with a 3.5:1 ellipse having an apical angle of 51°. He also demonstrated that straight-line cuts are needed to generate a 30° angle. Our elliptical excision using circular arcs generated an apical angle of 64° when using a length:width ratio of 3.5:1.

A parabolic arc, in comparison to a circular arc, for any given length:width ratio, has a smaller apical angle. However, a parabolic excision has greater curvature in the middle, which could potentially make joining the two sides of the excision more difficult. A rhombus generates a smaller apical angle for any given length:width ratio. Even if the obtuse angle is smoothed and does not meet at a true intersection of lines, it has a high degree of curvature in the middle of the incisions. Figure 3 summarizes the analysis of the circular arc and rhomboid excisional geometry.

Figure 2 also illustrates that at the extremes of length: width ratio, the surgeon gains little benefit in attempts to minimize either scar length or apical angle. The slope of the curves is too great at the low end of the length:width ratio curve to allow for any significant scar shortening. Conversely, the slope is too small at the high end of the length:width ratio curve to significantly decrease the angle by extending scar length.

In conclusion, we have demonstrated that a 30° apical angle, as commonly presented in the literature, is not created with the use of an elliptical excision. If a surgeon uses an arc-derived incision as commonly presented, the apical angles are more likely to be in a range from 53° to 74° for a length:width ratio of 3–4:1, which may result in standing cones at the apices of the fusiform excision. If the surgeon employs straight-line incisions, an angle of 28° to 37° can be generated from a 3–4:1 ratio of closure length to defect width. This scenario diminishes the probability of standing cone formation by minimizing the apical angle.

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