

BOUNDARY COMPONENTS OF MUMFORD-TATE DOMAINS

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ABSTRACT. We study certain spaces of nilpotent orbits in the Hodge domains introduced by [GGK1], and treat a number of examples. More precisely, we compute the Mumford-Tate group of the limit mixed Hodge structure of a generic such orbit. The result is used to present these spaces as iteratively fibered algebraic-group orbits in a minimal way.

1. INTRODUCTION

Mumford-Tate groups are the natural \mathbb{Q} -algebraic symmetry groups of Hodge structures, in the sense of stabilizing the Hodge substructures of all tensor powers of a Hodge structure and its dual. The modern definition, for Hodge structures of arbitrary weight, is due to Deligne [De3], with a generalization to mixed Hodge structures by André [An]. They were originally introduced by Mumford [Mu] to provide a Hodge-theoretic characterization of certain families of abelian varieties studied by Kuga and Shimura. In that “classical” context of weight one Hodge structures, they led to Deligne’s reformulation of Shimura varieties in terms of reductive groups and his subsequent proof that Hodge cycles on abelian varieties are absolute (cf. [De1], [Mi, Ke]). Hence they already have deep and longstanding ties with Hodge theory and algebraic geometry on the one hand, and with the theory of automorphic representations on the other. Amplifying the latter point, it turns out that (up to isogeny) the semisimple real Lie groups admitting discrete series representations are precisely those which arise as the group of real points $G(\mathbb{R})$ of a Mumford-Tate group of a pure Hodge structure.

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By the work of Griffiths [Gr], the cohomology of a family of smooth complex projective varieties $\mathcal{X} \rightarrow \mathcal{S}$ determines a period map

$$\Phi : \mathcal{S} \rightarrow \Gamma \backslash D$$

to a discrete quotient of a *period domain* D – that is, a classifying space for pure polarized Hodge structures with given Hodge numbers. As in the theory of Shimura varieties, it is more natural to allow D to be a general *Mumford-Tate domain*, parametrizing only those Hodge structures with the same “symmetries” as those in Φ . (This is a homogeneous space for the corresponding $G(\mathbb{R})$.) Whenever D is Hermitian symmetric, the quotient $\Gamma \backslash D$ admits the toroidal compactifications of [AMRT]. In the special case of weight 1 period domains (Siegel spaces), this compactification was reinterpreted by [CCK] in terms of the adjunction of Hodge-theoretic boundary components $B(\sigma)$. The latter parametrize the possible limiting mixed Hodge structures (LMHS) arising from the degeneration of Φ at a boundary point $x \in \bar{\mathcal{S}} \setminus \mathcal{S}$, where the family of varieties undergoes a degeneration. Kato and Usui recently extended this approach in [KU] to obtain partial compactifications $\Gamma \backslash D_\sigma$ for higher weight period domains.

The purpose of the present paper is to study degenerations of Hodge structure in the more general context of arbitrary Mumford-Tate domains, with an emphasis on the higher weight setting. More precisely, we shall introduce boundary components classifying the possible nilpotent orbits, and present them group-theoretically (as double-coset spaces) in a “minimal” way. In marked contrast to the Shimura case, when D is non-Hermitian the quotients $\Gamma \backslash D$ are usually not algebraic varieties [GRT]. Nevertheless, it has become clear in recent years that these quotients support a rich geometry – including a dense honeycomb of Shimura subvarieties and CM points – which has begun to be explored in [C3, C1, C2], [FL], [FHW], [GGK1], and [Ro1, Ro2].

There are several compelling reasons to introduce and study their boundary components. One is to make predictions about the form (and arithmetic nature) of the LMHS for variations of Hodge structure with given symmetries, and use this information in turn to classify

the possible period maps. Indeed, a variation with fixed underlying local system is determined by a single LMHS, and in keeping with the general simplifying effect of degenerating in algebraic geometry, the latter is often more accessible to computation. Furthermore, while $\Gamma \backslash D$ may be non-algebraic, the corresponding discrete quotient *of a boundary component* may not only be algebraic, but may be *as simple as a CM abelian variety or elliptic modular surface*.

From the standpoint of automorphic representation theory, this is significant: it raises the possibility of putting a $\bar{\mathbb{Q}}$ -algebraic structure on the coherent cohomology of a non-classical $\Gamma \backslash D$ (see the remarks on Fourier coefficients below). One case which surely warrants further investigation is the split- G_2 -domain parametrizing Hodge structures of weight 6 with Hodge numbers $h^{6,0} = h^{5,1} = \dots = h^{0,6} = 1$. It has three types of boundary component quotients, two of which – isomorphic to \mathbb{C}^* resp. a (noncompact) elliptic modular surface – are algebraic. Moreover, the totally degenerate limit of discrete series for G_2 contributes to its automorphic cohomology.

Beyond clarifying the relationship between LMHS and Hodge representations, our construction of boundary components in this paper has several applications to Hodge theory and complex geometry, some of which will be treated in future works. In particular, in [KP2] we will use it to interpret Hodge-theoretically “most” of the $G(\mathbb{R})$ -orbits in the topological boundary of D in its compact dual \check{D} . Meanwhile, [GGK3] will provide a general introductory discussion of $G(\mathbb{R})$ -orbits in \check{D} and the cycle space $\check{\mathcal{U}}$, combining the root-weight analysis of such in [FHW] with the Hodge-theoretic interpretation.

SUMMARY OF RESULTS

Let $D = G(\mathbb{R})/H$ be a Mumford-Tate domain and $\Gamma \leq G(\mathbb{Q})$ be a torsion-free, non-co-compact congruence subgroup. Then the boundary components $B(\sigma)$ we consider parametrize all nilpotent orbits subordinate to a given nilpotent cone σ . Computing the Mumford-Tate group of the generic limit mixed Hodge structure parametrized by a boundary component, in particular, is a natural and unexpectedly subtle problem

which arose during the writing of [GGK1] (cf. §I.C). Our solution is given in Theorem 2.8 for $\dim(\sigma) = 1$ ($\sigma = \mathbb{R}_{\geq 0}\langle N \rangle$, $N \in \mathfrak{g}_{\mathbb{Q}}$), and may be interpreted as follows:

Theorem. *Let $D = G(\mathbb{R})/H$ be a M-T domain, and assume the set $B(N)$ of N -nilpotent orbits $e^{\tau N}F^{\bullet}$ in \check{D} is nonempty. Then the M-T group $M_{B(N)}$ of the LMHS $(F^{\bullet}, W(N)_{\bullet})$ of a general such orbit is a subgroup of the centralizer of N in G , admitting a filtration by normal subgroups $M_{B(N)} = W_0 M_{B(N)} \supseteq W_{-1} M_{B(N)} \supseteq \cdots$ with*

(i) $Gr_0^W M_{B(N)} =: G_{B(N)}$ reductive, and equal to the M-T group of a general $\varphi_{split} := \oplus_i Gr_i^{W(N)} F^{\bullet}$;

(ii) $Gr_k^W M_{B(N)}$ abelian for $k < 0$; and

(iii) $W_{-1} M_{B(N)} = M_N := \exp \{im(adN) \cap \ker(adN)\}$ its unipotent radical.

Moreover, $M_N(\mathbb{C}) \rtimes G_{B(N)}(\mathbb{R})$ acts transitively on the corresponding set $\tilde{B}(N)$ of LMHS.

Theorem 5.2 generalizes this result beyond the rank one case.

One motivation for pursuing this is to give a precise relation between the $B(\sigma)$'s and classifying spaces for polarized MHS of the type considered by Hertling [He], and use this to understand their structure. This is carried out below in the first half of §7, and leads to the following for $\sigma = \mathbb{R}_{\geq 0}\langle N \rangle$:

Theorem. *If Γ is neat, then $\overline{B(N)} := (\Gamma \cap stab\langle N \rangle) \backslash B(N)$ admits a C^{∞} fibration tower*

$$\overline{B(N)} \twoheadrightarrow \cdots \twoheadrightarrow \overline{B(N)}_{(k)} \xrightarrow{\bar{\rho}_{(k)}} \cdots \twoheadrightarrow \overline{B(N)}_{(1)} \xrightarrow{\bar{\rho}_{(1)}} \overline{D(N)}$$

with (for $k > 1$, generalized) intermediate Jacobian fibers at each stage, and base $\overline{D(N)}$ a discrete quotient of the M-T domain $G_{B(N)}(\mathbb{R}) \cdot \varphi_{split}$.

Another piece of motivation comes from the study of automorphic cohomology in [C3] and [GGK2]. Let D be a nonclassical Mumford-Tate domain, where “nonclassical” is meant in the strong sense of possessing no nonconstant holomorphic or antiholomorphic fibration over a Hermitian symmetric domain. Then with Γ as above, the quotient

$\Gamma \backslash D$ is non-algebraic [GRT].¹ In particular, one cannot use sections of nontrivial holomorphic homogeneous line bundles \mathcal{L} (such as $K_D^{\otimes m}$) to embed $\Gamma \backslash D$ in projective space; in fact, it is strongly suspected that all spaces $H^0(D, \mathcal{O}(\mathcal{L}))^\Gamma$ of holomorphic automorphic forms are zero. Instead, one can consider the higher automorphic cohomology groups $H^p(D, \mathcal{O}(\mathcal{L}))^\Gamma$ introduced in [GS]. In the special case studied by Carayol [C3] (also see Example 8.4 below), in spite of the non-algebraicity of $\Gamma \backslash G(\mathbb{R})/H$, the boundary component quotients turn out to be isomorphic either to \mathbb{C}^* , a CM elliptic curve E , or its conjugate \bar{E} . For certain spaces of automorphic cohomology classes, he defines *generalized Fourier coefficients* in $\{H^1(E, \mathcal{O}(-n))\}_{n \in \mathbb{N}}$, and proves that the space is spanned over \mathbb{C} by classes for which all of these coefficients are defined over $\bar{\mathbb{Q}}$. Such an arithmetic structure is potentially very useful from the standpoint of proving algebraicity of the Hecke eigenvalues of automorphic representations not appearing in the coherent cohomology of any Shimura variety.

Consequently one desires, as a first step, some simple conditions under which $\Gamma_\sigma \backslash B(\sigma)$ has a canonical model over $\bar{\mathbb{Q}}$. Such conditions are laid out in Proposition 7.4 and Theorem 7.9, whose corollaries include the

Proposition. *Let M be an adjoint \mathbb{Q} -algebraic group, $D = M(\mathbb{R})/H$ a Hodge domain parametrizing level 2ℓ Hodge structures on $\mathfrak{m} = \text{Lie}(M)$, and $B(N) \neq \emptyset$. If*

(i) $\ker(adN) \cap im\{(adN)^2\} = \langle N \rangle$ and

(ii) $[\ker(adN), \ker(adN)] \subseteq im(adN)$

hold, then $\overline{B(N)}$ is a CM abelian variety of dimension $\frac{1}{2} \dim(Gr_{-1}^{W(N)} \mathfrak{m})$ over \mathbb{C} .

At least for G of Hermitian type, we expect that (as in Carayol's example) one can use such boundary components to layer a notion of arithmeticity on to a subspace of automorphic cohomology by asking for those classes whose generalized Fourier coefficients are defined over

¹In the case where Γ is instead co-compact, this has been known in certain cases for some time [CT].

$\bar{\mathbb{Q}}$. Though we won't pursue this application here, it has influenced our decision to focus on CM abelian varieties, cf. Remark 7.12.

Our results on the structure of $\overline{B(N)}$ are put to work on examples in §8. There, we first consider cases where D is a period domain for Hodge structures of weight 1, 2, and 3 (with $G = Sp_4$ or $SO(4, 1)$), the last of which was the focus of [GGK4]. Turning to subdomains, we then treat Carayol's weight 3 example (with $G = U(2, 1)$ and Hodge numbers $(1, 2, 2, 1)$) and an "exceptional" domain for weight 2 Hodge structures with Mumford-Tate group G_2 and Hodge numbers $(2, 3, 2)$. The boundary components for two other G_2 -related examples, including the one mentioned earlier, will be worked out from an alternative viewpoint in [KP2].

It is natural to expect that the new technology of Mumford-Tate domains and their boundary components should have implications for variations of Hodge structures, say over a fixed curve \mathcal{S} . In §10 we obtain some refined rigidity and Arakelov-type finiteness results for VHS on \mathcal{S} with fixed local system, Hodge numbers and M-T group. We make the elementary observation that they might be classified by their values in a single boundary component, which already in the rigid case should be of arithmetic interest. To a Hodge theorist, this application should legitimize our construction all on its own! A key example is the geometric G_2 -VHS (with Hodge numbers $(1, 1, 1, 1, 1, 1)$) arising in the work of Dettweiler and Reiter [DR] (partly based on [Ka]), which is the subject of §9.

HODGE DOMAINS

Let G be an algebraic group defined over \mathbb{Q} and $k \subseteq \mathbb{C}$ a field; then the group of k -rational points will be denoted $G(k)$.² When k is \mathbb{R} (resp. \mathbb{C}), this will be regarded as a real (resp. complex) Lie group with Lie algebra \mathfrak{g} (resp. $\mathfrak{g}_{\mathbb{C}}$).³ Two examples which play a distinguished role

²A concise review of algebraic groups may be found in [Ke, sec. I.A].

³We will write $\mathfrak{g}_{\mathbb{Q}}$ for the underlying \mathbb{Q} -vector space, and $\mathfrak{g}_k := \mathfrak{g}_{\mathbb{Q}} \otimes_{\mathbb{Q}} k$.

below are the algebraic tori $\mathbb{U} \subset \mathbb{S}$ inside GL_2 with k -rational points

$$\mathbb{U}(k) = \left\{ \left(\begin{array}{cc} a & b \\ -b & a \end{array} \right) \middle| \begin{array}{l} a^2 + b^2 = 1 \\ a, b \in k \end{array} \right\}$$

$$\mathbb{S}(k) = \left\{ \left(\begin{array}{cc} a & b \\ -b & a \end{array} \right) \middle| \begin{array}{l} a^2 + b^2 \neq 0 \\ a, b \in k \end{array} \right\}.$$

Next recall where a Mumford-Tate domain comes from: let

- V be a \mathbb{Q} -vector space;
- Q be a $(-1)^n$ -symmetric bilinear form on V ;
- $'\varphi : \mathbb{U} \rightarrow \text{Aut}(V, Q)$ be a morphism of algebraic groups defined over \mathbb{R} , with $'\varphi(-1) = (-1)^n \text{id}_V$ and $Q(\cdot, '\varphi(i)\cdot) > 0$; and
- $'G \leq \text{Aut}(V, Q)$ be the \mathbb{Q} -algebraic group closure of $'\varphi(\mathbb{U}(\mathbb{R}))$.

Equivalently, $'\varphi$ is a Hodge structure on V polarized by Q , and $'G$ is its Mumford-Tate group. The corresponding Mumford-Tate domain is defined to be the orbit

$$'D := 'G(\mathbb{R}).'\varphi,$$

under the action by conjugation. The *compact dual* $'\check{D}$ of $'D$ is the (left-translation) $'G(\mathbb{C})$ -orbit of the Hodge flag $'F^\bullet$ corresponding to $'\varphi$. This is always a complex projective variety defined over a number field, and contains $'D$ as a real-analytic open subset.

Now, $'G$ is reductive, so we write

- $A := 'G/'G^{\text{der}}$ for the maximal abelian quotient,⁴
- $M := 'G^{\text{Ad}} = 'G/Z_{'G}$ for the (\mathbb{Q} -algebraic) adjoint group,

and consider the composition

$$\varphi : \mathbb{U} \xrightarrow{'\varphi} 'G \xrightarrow[\text{quotient}]{\text{finite}} 'G/\{Z_{'G} \cap 'G^{\text{der}}\} = A \times M =: G.$$

Applying the obvious projections, we have $\varphi_A : \mathbb{U} \rightarrow A$ and

$$\varphi_M : \mathbb{U} \xrightarrow{\varphi} G \rightarrow M \subset \text{Aut}(\mathfrak{m}, B),$$

which gives a polarized Hodge structure of weight zero on $\mathfrak{m} := \text{Lie}(M)$. (The polarizing form B is restricted from $Q \otimes Q^\vee$ on $\text{End}(V)$; it need

⁴It follows from [GGK1, IV.A.2] that $A(\mathbb{R})$ is compact.

not be proportional to the Killing form if M is non-simple, but we shall denote it by B anyway.) This allows us to present $'D$ as a *Hodge domain*

$$D := M(\mathbb{R}).\varphi_M \cong G(\mathbb{R}).\varphi \cong 'G(\mathbb{R}).'\varphi (= 'D),$$

cf. [GGK1, sec. IV.F]. The reason for the different notations $D, 'D$ is that we think of the first as a set of (weight zero) polarized Hodge structures on \mathfrak{m} and the second as polarized Hodge structures on V (usually of positive weight). This also illustrates the fact that the same complex manifold may appear in many different ways as a Mumford-Tate domain $G(\mathbb{R})/H$ (with different groups $G(\mathbb{R})$ and H).

\mathfrak{sl}_2 -TRIPLES

Representations of the Lie algebra \mathfrak{sl}_2 , spanned by

$$n_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad n_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

will play a key role in computing the Mumford-Tate groups. Let \mathbb{F} be a subfield of \mathbb{C} . Any triple of elements $e_+, e_-, x \in \mathfrak{m}_{\mathbb{F}}$ satisfying

$$[x, e_+] = 2e_+, \quad [x, e_-] = -2e_-, \quad [e_+, e_-] = x$$

is called an \mathbb{F} - \mathfrak{sl}_2 -triple.⁵ It is equivalent to a homomorphism

$$\rho : SL_2 \rightarrow M$$

with $d\rho : \mathfrak{sl}_2 \rightarrow \mathfrak{m}$ defined over \mathbb{F} (and $x = d\rho(y)$, $e_+ = d\rho(n_+)$, and $e_- = d\rho(n_-)$).

Let a nilpotent element $e_- \in \mathfrak{m}_{\mathbb{C}}$ be given. Define

$$\mathfrak{m}_{e_-} := \text{im}(\text{ade}_-) \cap \ker(\text{ade}_-)$$

and let M_{e_-} denote the corresponding algebraic subgroup of $M_{\mathbb{C}}$. We have the two famous theorems:

Jacobson-Morosov. [Ja, Mo] *There exists an \mathfrak{sl}_2 -triple (in $\mathfrak{m}_{\mathbb{C}}$) with $e_- = d\rho(n_-)$, i.e. as “nil-negative element”.*

⁵If $\mathbb{F} = \mathbb{C}$, we shall just call this an \mathfrak{sl}_2 -triple.

Kostant. [Ko] (a) *Given additionally $x \in \mathfrak{m}_{\mathbb{C}}$ satisfying*

$$(i) [x, e_-] = -2e_- \quad \text{and} \quad (ii) x \in \text{im}(\text{ade}_-),$$

there is an unique choice of $e_+ \in \mathfrak{m}_{\mathbb{C}}$ completing x, e_- to an \mathfrak{sl}_2 -triple.

(b) *The set of all \mathfrak{sl}_2 -triples having e_- as nil-negative element is the M_{e_-} -orbit*

$$\begin{aligned} & \left\{ (\text{Ad}_g.x, \text{Ad}_g.e_+, e_-) \mid g \in M_{e_-}(\mathbb{C}) \right\} = \\ & \left\{ (e^{\text{ad}\gamma}.x, e^{\text{ad}\gamma}.e_+, e_-) \mid \gamma \in \mathfrak{m}_{e_-, \mathbb{C}} \right\}. \end{aligned}$$

(c) *The set of all x occurring in this orbit (or equivalently, all x satisfying the assumption in (a)) is*

$$x + \mathfrak{m}_{e_-, \mathbb{C}}.$$

In the entire statement of Kostant, one can replace \mathbb{C} by \mathbb{R} (when $e_- \in \mathfrak{m}_{\mathbb{R}}$). According to the following observation, we can also refine parts of the above for smaller ground fields:

Schmid. [Sc] *If $e_- \in \mathfrak{m}_{\mathbb{F}}$, then there exists an \mathbb{F} - \mathfrak{sl}_2 -triple containing it as nil-negative element, say (x, e_+, e_-) . The set of all such is in 1-to-1 correspondence with $x + \mathfrak{m}_{e_-, \mathbb{F}}$. Hence, if $x' \in \mathfrak{m}_{\mathbb{C}}$ satisfies (i) and (ii) above, then the affine subset $x' + \mathfrak{m}_{e_-, \mathbb{C}}$ consisting of all such is defined over \mathbb{F} , and contains elements in $\mathfrak{m}_{\mathbb{F}}$ (with corresponding “ e_+ ” in $\mathfrak{m}_{\mathbb{F}}$).*

CONCLUDING REMARKS

We wrap up this introduction with a couple of further comments on notation and terminology. For an element g in a Lie group, $\langle g \rangle$ denotes the subgroup it generates; for γ an element of a Lie algebra, the line it generates is denoted by $\langle \gamma \rangle$, with the field understood from context (or a subscript). For example, given a Hodge domain D as above and a nilpotent element $N \in \mathfrak{m}_{\mathbb{Q}}$, a *nilpotent orbit* is a subset $e^{\langle N \rangle_{\mathbb{C}}}.F^{\bullet} \subseteq \check{D}$ where $N(F^{\bullet}) \subseteq F^{\bullet-1}$ and $e^{\tau N}.F^{\bullet} \in D$ for $\Im(\tau) \gg 0$.

By far the more important point is our intentional (and consistent throughout this paper) *notational disregard for issues of disconnectedness*. For instance, by $M(\mathbb{R})$ we shall always mean the identity

connected component in the sense of real Lie groups, usually written $M(\mathbb{R})^+$; likewise, D_M is always connected (and really an orbit of $M(\mathbb{R})^+$). In addition, there are three further kinds of disconnectedness pertaining specifically to boundary components:⁶

- In $M(\mathbb{R})^+$, $Z(N)$ can fail to be connected. (In fact, the centralizer of N in $M(\mathbb{C})$ can even fail to be connected. On the other hand, M_N – real or complex – is always connected: indeed, since it is unipotent, one can connect each element to the identity by a one-parameter subgroup.)
- $B(N)$ can break up into different components with different $I^{p,q}$ -dimensions. These cannot be mapped one to another, even by $Z(N)(\mathbb{C})$ (which respects $W(N)$ • hence must preserve $I^{p,q}$ -dimensions).
- Even within the subset of $B(N)$ comprising LMHS with fixed $I^{p,q}$ -dimensions, we can have different connected components since $Gr_0^W M_{B(N)}(\mathbb{R})$ may not be connected.

Instead of adding a “+” to every real Lie group and a “connected component of” to every boundary component (and M-T domain), we have simply left these things tacit throughout. There are, beyond readability, two very good reasons for focusing on connected components: the first is that it eases passing back and forth between the Lie-algebra/tangent-space point of view, and the Lie-group/orbit perspective. The second is that invariants (Mumford-Tate groups, fibration structure, etc.) of the connected components in “one” boundary component are likely to be different.⁷

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⁶These are stated for the rank-one case, cf. §2 (where $B(N)$ and the $\{I^{p,q}\}$ are defined); they have obvious generalizations replacing N with σ .

⁷In [GGK1] (cf. Example VI.B.15), it was discovered that (the finitely many) connected components of the subset of a period domain comprising Hodge structures with Mumford-Tate group contained in a given (Mumford-Tate) subgroup of $Aut(V, Q)$, need not have the same generic Mumford-Tate group.

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2. RANK-ONE BOUNDARY COMPONENTS

We continue with the setup of §1. Postponing a fuller discussion of Kato-Usui compactifications to §6 so as to get straight to a statement of the core result, write

- $M = \text{adjoint Mumford-Tate group}$,
- $D_M = \text{Hodge domain for } M$

as above, and also:

- $N \in \mathfrak{m}_{\mathbb{Q}}$ a nilpotent element;
- $W(N)_{\bullet} := \text{the corresponding "weight" filtration on } \mathfrak{m}, \text{ centered about } 0; \text{ and}$
- $\mathfrak{m}_N := \text{im}(\text{ad}N) \cap \ker(\text{ad}N) = \text{Lie}(M_N)$.

The boundary component associated to N is defined by

$$B_M(N) := \tilde{B}_M(N)/e^{\langle N \rangle_{\mathbb{C}}} (= \text{set of nilpotent orbits})$$

where

$$\tilde{B}_M(N) := \left\{ F^{\bullet} \in \check{D}_M \mid \text{Ade}^{\tau N} F^{\bullet} \text{ is a nilpotent orbit} \right\}.$$

For any $F^{\bullet} \in \tilde{B}_M(N)$, we can consider the VHS

$$\Phi_{F^{\bullet}} : \Delta^* \rightarrow \langle T \rangle \backslash D_M$$

given by

- the \mathbb{Q} -local system $\underline{\mathfrak{m}} \rightarrow \Delta^*$ given by the monodromy operator $T := e^{\text{ad}N}$ on $\mathfrak{m}_{\mathbb{Q}}$, and
- the family of filtrations given with respect to the multivalued basis of $\underline{\mathfrak{m}}_{\mathbb{C}}$ by $e^{\frac{1}{2\pi i} \log(q)N} F^{\bullet}$.

This has LMHS $(\mathfrak{m}, F^{\bullet}, W(N)_{\bullet})$ (henceforth abbreviated $(F^{\bullet}, W(N)_{\bullet})$), which means in particular that we can think of $\tilde{B}_M(N)$ as a set of LMHS; accordingly define

$$\tilde{B}_M^{\mathbb{R}}(N) := \left\{ F^{\bullet} \in \tilde{B}_M(N) \mid (F^{\bullet}, W(N)_{\bullet}) \text{ is an } \mathbb{R}\text{-split MHS} \right\},$$

and

$$B_M^{\mathbb{R}}(N) := \tilde{B}_M^{\mathbb{R}}(N)/e^{\langle N \rangle_{\mathbb{R}}} \subset B_M(N).$$

To understand the relationship between M_N and the weight filtration $W(N)_{\bullet}$, introduce:

- the centralizer

$$Z_M(N) := \{g \in M \mid \text{Ad}(g)N = N\} \supseteq M_N,$$

with Lie algebra $\mathfrak{z}_M(N) := \ker(\text{ad}N)$; and

- the “weight filtration”

$$W_{\bullet}^N M := \{g \in M \mid (\text{Ad}(g) - \text{id})W(N)_{i\mathfrak{m}} \subset W(N)_{i+\bullet\mathfrak{m}} \ (\forall i)\}$$

on M (and its subgroups).

We shall prove the following later.

Lemma 2.1. (i) $\text{Lie}(W_k^N M) = W(N)_{k\mathfrak{m}}$,

(ii) $Z_M(N) \subset W_0^N M$, and

(iii) $M_N = Z_M(N) \cap W_{-1}^N M$.

Remark 2.2. (a) In fact, by the construction in §1 of Hodge structures on $\mathfrak{m} \subset \text{End}(V)$ (from HS on V), and the compatibility of taking limit MHS with tensor and dual operations, it is clear that

$$(\text{ad}I^{p,q}(\mathfrak{m}_{\mathbb{C}}))I^{p',q'}(\mathfrak{m}_{\mathbb{C}}) \subset I^{p+p',q+q'}(\mathfrak{m}_{\mathbb{C}})$$

(cf. Definition 2.3 below). Though (i) in Lemma 2.1 is clear from this, we shall also give a more instructive proof in §3.

(b) Hodge tensors in a nilpotent orbit⁸ remain Hodge in the limit; in particular, the polarizing form $B \in (\mathfrak{m}^{\vee})^{\otimes 2}$ gives perfect pairings

$$\begin{cases} Gr_k^{W(N)}\mathfrak{m} \times Gr_{-k}^{W(N)}\mathfrak{m} \rightarrow \mathbb{Q}, \\ I^{p,q}(\mathfrak{m}_{\mathbb{C}}) \times I^{-p,-q}(\mathfrak{m}_{\mathbb{C}}) \rightarrow \mathbb{C}. \end{cases}$$

(c) Furthermore, it is well-known (cf. [Ca1]) that the formulas

$$B_k(v, w) := B(v, N^k w)$$

⁸or, more generally, in a polarized VHS, cf. [GGK1, I.C.1]

yield polarizations on $Gr_k^{W(N)} \mathfrak{m}$ for $k \geq 0$. (If $k < 0$, one defines B_k via B_{-k} and the Hard Lefschetz isomorphism $N^k : Gr_{-k}^{W(N)} \mathfrak{m} \xrightarrow{\cong} Gr_k^{W(N)} \mathfrak{m}$.)

Henceforth all subscript M 's will be dropped.

Next, we recall some material from [De4], [An], [GGK1].

Definition 2.3. Let $(V, F^\bullet, W_\bullet)$ be a MHS.

(i) [De4, CKS] The *Deligne bigrading* $I^{p,q}(V_\mathbb{C})$ associated to this MHS is the unique bigrading of $V_\mathbb{C}$ such that $\bigoplus_{p+q \leq n} I^{p,q} = W_n$, $\bigoplus_{p \geq p_0; q} I^{p,q} = F^{p_0}$, and $\overline{I^{q,p_0}} \equiv I^{p_0,q_0} \pmod{\bigoplus_{p < p_0; q < q_0} I^{p,q}}$. (One has $\overline{I^{q,p}} = I^{p,q}$ ($\forall p, q$) $\iff (V, F, W)$ is \mathbb{R} -split.) It is compatible with tensors and dual in the obvious way.

(ii) [GGK1, I.C.3] Define $\tilde{\varphi} : \mathbb{S}(\mathbb{C}) (\cong \mathbb{C}^* \times \mathbb{C}^*) \rightarrow \text{Aut}(V_\mathbb{C})$ by \mathbb{C} -linear extension of the rule “ $\tilde{\varphi}(z, w)|_{I^{p,q}} = \text{multiplication by } z^p w^q$ ”.

(iii) [An] The *Mumford-Tate group* $M_{(F,W)}$ [resp. *full MTG* $\tilde{M}_{(F,W)}$] of $(V, F^\bullet, W_\bullet)$ is the largest \mathbb{Q} -algebraic subgroup of $GL(V)$ fixing [resp. scaling] all Hodge tensors.

We remind the reader that for $(F^\bullet, W_\bullet) \in \tilde{B}(N)$, N belongs to $I^{-1,-1}(\mathfrak{m}_\mathbb{C})$. To generalize (iii) to a *set* of MHS, require that the group fix [resp. scale] Hodge tensors common to the whole set.

Lemma 2.4. (a) [GGK1, I.C.6] *In the situation of Definition 2.3, the \mathbb{Q} -closure of $\tilde{\varphi}$ is $\tilde{M}_{(F,W)}$. (There is no corresponding result for $M_{(F,W)}$.)*

(b) *For a set of MHS, the full MTG equals the \mathbb{Q} -closure of the corresponding set of $\tilde{\varphi}$'s.*

As above, one defines a weight filtration W_\bullet on $M_{(F,W)}$. The following properties are standard (cf. [A-K],[An]):

- (1) $W_0 M_{(F,W)} = M_{(F,W)}$;
- (2) $W_{-1} M_{(F,W)}$ is its unipotent radical;
- (3) Writing M_{split} for the (reductive) MTG of the Hodge structure $\bigoplus_i (Gr_i^W V, Gr_i^W F^\bullet)$, one has a split short-exact sequence

$$0 \rightarrow W_{-1} M_{(F,W)} \rightarrow M_{(F,W)} \rightarrow M_{\text{split}} \rightarrow 0.$$

This also holds with the two right-hand groups replaced by the respective full Mumford-Tate groups.

We denote the representation of $\mathbb{U}(\mathbb{C})$ [resp. $\mathbb{S}(\mathbb{C})$] corresponding to $\oplus_i (Gr_i^W V, Gr_i^W F^\bullet)$ by φ_{split} [resp. $\tilde{\varphi}_{\text{split}}$]. Now for the central object of study:

Definition 2.5. The *Mumford-Tate group* $M_{B(N)}$ of the boundary component $B(N)$ is the largest algebraic subgroup of M fixing the Hodge tensors common to all $(F^\bullet, W(N)_\bullet) \in \tilde{B}(N)$. (That is, it is the MTG of $\tilde{B}(N)$ viewed as a set of MHS. Similarly one defines $\tilde{M}_{B(N)}$, $M_{B^\mathbb{R}(N)}$, $\tilde{M}_{B^\mathbb{R}(N)}$.)

Since $M_{B(N)}$ is the MTG of a set of LMHS having N as a Hodge tensor, it is contained in $Z(N)$; likewise, $\tilde{M}_{B(N)}$ is contained in

$$\tilde{Z}(N) := \{g \in M \mid \text{Ad}(g)\langle N \rangle \subseteq \langle N \rangle\}.$$

So for any (limit) MHS $\tilde{\varphi} \in \tilde{B}(N)$ we have a diagram

$$(2.1) \quad \begin{array}{ccccc} \mathbb{S} & \xrightarrow{\tilde{\varphi}} & \tilde{Z}(N) & \xrightarrow{\text{Ad}} & \text{Aut}(\mathfrak{m}, B, W(N)_\bullet) \\ \parallel & & \downarrow & & \downarrow \\ \mathbb{S} & \xrightarrow{\tilde{\varphi}_{\text{split}}} & \tilde{Z}(N)/M_N & \xrightarrow{\overline{\text{Ad}}} & \times_i GL(Gr_i^{W(N)} \mathfrak{m}) \\ \uparrow & & \uparrow & & \uparrow \\ \mathbb{U} & \xrightarrow{\varphi_{\text{split}}} & Z(N)/M_N & \xrightarrow{\overline{\text{Ad}}} & \times_i \text{Aut}(Gr_i^{W(N)} \mathfrak{m}, B_i). \end{array}$$

It will be convenient to write $Gr_i^{W(N)}\varphi$ for the composition of $\overline{\text{Ad}} \circ \varphi_{\text{split}}$ with the projection to $\text{Aut}(Gr_i^{W(N)} \mathfrak{m}, B_i)$.⁹

Remark 2.6. The three groups

- (a) $\text{Aut}(Gr_0^{W(N)} \mathfrak{m}, B_0) \times \text{Aut}(Gr_{-1}^{W(N)}, B_{-1})$,
- (b) $\times_{k \geq 0} \text{Aut}(P_k, Q_k)$, where $P_k := \ker(N^{k+1}) \subseteq Gr_k^{W(N)} \mathfrak{m}$ and $Q_k := B_k|_{P_k}$, and

⁹In [GGK1], φ denotes the restriction of $\tilde{\varphi}$ to $\mathbb{U}(\mathbb{C}) \cong \mathbb{C}^*$ (i.e. $\varphi(z) := \tilde{\varphi}(z, z^{-1})$). Since the \mathbb{Q} -closure of its image is not $M_{(F,W)}$ in the mixed case, we will use only “ $Gr_i^{W(N)}\varphi$ ”, and not “ φ ” as such, in this paper.

(c) $\times_{k \geq 0} \text{Aut}(Gr_{-k}^{W(N)} \mathfrak{z}(N), B_{-k} | \dots)$

are embedded “diagonally” in $\times_i \text{Aut}(Gr_i^{W(N)} \mathfrak{m}, B_i)$, using the maps

$$\left\{ \begin{array}{l} Gr_j^{W(N)} \mathfrak{m} \xrightarrow{N^{j/2}} Gr_0^{W(N)} \mathfrak{m} \xrightarrow{N^{j/2}} Gr_{-j}^{W(N)} \mathfrak{m} \quad (j \text{ even}) \\ Gr_j^{W(N)} \mathfrak{m} \xrightarrow{N^{(j+1)/2}} Gr_{-1}^{W(N)} \mathfrak{m} \xrightarrow{N^{(j-1)/2}} Gr_{-j}^{W(N)} \mathfrak{m} \quad (j \text{ odd}) \end{array} \right.$$

and the isomorphisms

$$\left\{ \begin{array}{l} Gr_j^{W(N)} \mathfrak{m} \cong \bigoplus_{\ell \geq \max\{0, -j\}} N^\ell P_{j+2\ell}, \\ P_k \xrightarrow[N^k]{} Gr_{-k}^{W(N)} \mathfrak{z}(N). \end{array} \right.$$

Since $Z(N)$ commutes with N , one easily verifies that the bottom $\overline{\text{Ad}}$ in (2.1) factors through each of (a)-(c). We conclude that the groups

$$Gr_0^W Z(N) = Z(N)/M_N, \quad Gr_0^W \tilde{Z}(N) = \tilde{Z}(N)/M_N$$

act faithfully on the vector spaces

- (a') $Gr_0^{W(N)} \mathfrak{m} \oplus Gr_{-1}^{W(N)} \mathfrak{m}$,
- (b') $\bigoplus_{k \geq 0} P_k$, and (perhaps most naturally)
- (c') $\bigoplus_{k \geq 0} Gr_{-k}^{W(N)} \mathfrak{z}(N)$.

This reveals that the Hodge structure φ_{split} is completely determined by its restriction to these spaces.

An important related point, which establishes $Gr_0^W Z(N)$ as a possible Mumford-Tate group of φ_{split} , is the

Proposition 2.7. (i) $Gr_0^W Z(N)$ and $Gr_0^W \tilde{Z}(N)$ are reductive; and
(ii) M_N is the unipotent radical of $Z(N)$ and $\tilde{Z}(N)$.

Proof. Given $\tilde{\varphi} \in \tilde{B}(N)$, set $C := \tilde{\varphi}(i, -i) \in Z(N)(\mathbb{C})$. We may consider Ψ_C (conjugation by C) as an automorphism of $Z(N)/M_N$. Since $Z(N)/M_N$ acts faithfully in blocks on $\bigoplus_{\ell \geq 0} P_\ell$ (or $\bigoplus_{\ell \geq 0} Gr_{-\ell}^{W(N)} \mathfrak{z}(N)$), and $C^2 = \tilde{\varphi}(-1, -1)$ acts in scalar blocks on $\bigoplus_{\ell} P_\ell$, they commute. Hence Ψ_C is an involution. Considered as Hodge structures, the P_ℓ are polarized by Q_ℓ ; in particular, the Hermitian forms defined by

$Q_\ell(Cv, \bar{w})$ ($v, w \in P_\ell$) are positive-definite. Since the real form

$$\mathcal{G} := \{g \in (Z(N)/M_N)(\mathbb{C}) \mid \Psi_C(\bar{g}) = g\}$$

of $Z(N)/M_N$ evidently preserves these forms, it is compact. We conclude (by the criterion in the proof of Prop. 3.6 in [De1]) that $Gr_0^W Z(N)$ is reductive, and (ii) follows from this at once. \square

We are now ready to state the main result in the rank-one case:

Theorem 2.8. *Assume $B(N) \neq \emptyset$. Then we have $M_N \subseteq M_{B(N)} \subseteq Z(N)$. More precisely,*

- (A) $W_{-1}M_{B(N)} = M_N$; and
- (B) $Gr_0^W M_{B(N)} \subseteq Gr_0^W Z(N)$ is the Mumford-Tate group of the set $Gr_0^W Z(N)(\mathbb{R}) \cdot \varphi_{split}$ of Hodge structures on (b') [resp. (a'), (c')] above, or equivalently the \mathbb{Q} -algebraic group closure of a generic φ_{split} .

A similar statement holds for $\tilde{M}_{B(N)}$.

3. SOME LEMMAS

Definition 3.1. Given a MHS $(V, F^\bullet, W_\bullet)$, we define its Deligne splitting (a splitting of W_\bullet) to be the element $Y \in \text{End}(V_{\mathbb{C}})$ having $\oplus_{p+q=k} I^{p,q}(V_{\mathbb{C}})$ as k -eigenspace ($\forall k$). Note that $Y \in I^{0,0}$.

We reiterate that the mixed Hodge structures parametrized by boundary components $\tilde{B}(N)$ are limits of weight zero N -nilpotent orbits with $V = \mathfrak{m}$, $W_\bullet = W(N)_\bullet$. Hence the eigenvalues of Y are automatically centered about 0.

We shall work for now under the assumption that $\tilde{B}^{\mathbb{R}}(N) \neq \emptyset$. For simplicity, the notation Ad for M acting on \mathfrak{m} will be dropped (except for occasional emphasis); so nilpotent orbits are now just $e^{\tau N} \cdot F^\bullet$.

Lemma 3.2. $\tilde{B}^{\mathbb{R}}(N)$ contains a \mathbb{Q} -split MHS $(F^\bullet, W(N)_\bullet)$.

Proof. By assumption, we have a nilpotent orbit $F_\tau^\bullet \mathfrak{m}_{\mathbb{C}} := e^{\tau N} \cdot F_0^\bullet \mathfrak{m}_{\mathbb{C}}$ with \mathbb{R} -split LMHS $(F_0^\bullet, W(N)_\bullet)$. The Deligne splitting of the latter kills its Hodge $(0,0)$ tensors, in particular any which survive to Hodge

tensors in the nilpotent orbit. Hence this splitting is of the form $\text{ad}Y_0$ for some $Y_0 \in \mathfrak{m}_{\mathbb{R}}$.

Furthermore, since the LMHS is \mathbb{R} -split we have $F_{\tau}^{\bullet} \in D$ for $\Im(\tau) > 0$, cf. [CKS, Lemma 3.12]. (Though the result as stated in [op. cit.] says *only* that F_{τ}^{\bullet} is in the period domain containing D , our definition of $\tilde{B}^{\mathbb{R}}(N)$ ensures that $F_{\tau}^{\bullet} \in D$ for $\Im(\tau) \gg 0$, which combined with [op. cit.] is clearly sufficient.) In particular, $F_i^{\bullet} \in D$. Decomposing $N = N^{(1,-1)} + N^{(0,0)} + N^{(-1,1)}$ with respect to the pure HS on \mathfrak{m} induced by F_i^{\bullet} , we have that $Y_0 = \text{ad}(N)(2N^{0,0})$ is in the image of $\text{ad}(N)$ *acting on* \mathfrak{m} .

Since $N \in \mathfrak{m}_{\mathbb{Q}}$, the results of Kostant and Schmid in §1 now imply the existence of $g \in M_N(\mathbb{R})$ such that $Y := g.Y_0$ belongs to $\mathfrak{m}_{\mathbb{Q}}$; and then

$$g.e^{\tau N}.F_0^{\bullet} = e^{\tau N}.g.F_0^{\bullet} =: e^{\tau N}.F^{\bullet}$$

is a nilpotent orbit with $\text{ad}Y$ the Deligne splitting of its LMHS $(F^{\bullet}, W(N)_{\bullet})$. \square

Lemma 3.3. $Z(N)(\mathbb{R})$ acts transitively on $\tilde{B}^{\mathbb{R}}(N)$.

Proof. Clearly $Z(N)(\mathbb{R})$ acts on $\tilde{B}^{\mathbb{R}}(N)$, since for $g \in Z(N)(\mathbb{R})$

$$g.e^{\tau N}.F^{\bullet} = e^{\tau N}.g.F^{\bullet}$$

and $ge^{\tau N}.F^{\bullet}$ is still in D for $\Im(\tau) \gg 0$ (as g is real).

Next we determine the tangent space to $\tilde{B}(N)$ at some point $x = (F^{\bullet}, W(N)_{\bullet})$, in terms of whose Deligne bigrading we consider the subspaces $(\mathfrak{m}^{p,q} := I^{p,q}(\mathfrak{m}_{\mathbb{C}}))$

$$\mathfrak{q} := \bigoplus_{\substack{p,q \\ p < 0}} \mathfrak{m}^{p,q}, \quad \mathfrak{p} := \bigoplus_{\substack{p,q \\ p \geq -1}} \mathfrak{m}^{p,q}$$

of $\mathfrak{m}_{\mathbb{C}}$. There is a natural identification of $T_x \check{D}$ with \mathfrak{q} . Let $\{\alpha(t)\}_{t \in (-\epsilon, \epsilon)}$ be a curve in \mathfrak{q} with $e^{\alpha(t)}.F^{\bullet} \in \tilde{B}(N)$ (and $\alpha(0) = 0$). Then $e^{\tau N}.e^{\alpha(t)}.F^{\bullet}$ must be $(\forall t)$ a nilpotent orbit, hence

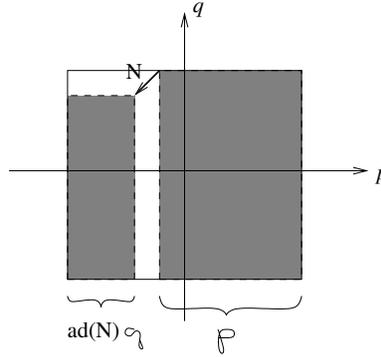
$$Ne^{\alpha(t)}F^p \subset e^{\alpha(t)}F^{p-1}$$

$$\implies \frac{e^{-\alpha(t)} N e^{\alpha(t)}}{e^{-\text{ad}\alpha(t)} N} F^p \subset F^{p-1}.$$

Differentiating and setting $t = 0$ gives

$$\begin{aligned} & [\alpha'(0), N] F^p \subset F^{p-1} \\ \implies & [\alpha'(0), N] \in \mathfrak{p} \cap (\text{ad}N)\mathfrak{q}. \end{aligned}$$

But from the picture



we have $\mathfrak{p} \cap (\text{ad}N)\mathfrak{q} = \{0\}$

$$\implies \alpha'(0) \in \ker(\text{ad}N) = \mathfrak{z}(N)_{\mathbb{C}},$$

so that $\mathfrak{z}(N)_{\mathbb{C}} \cap \mathfrak{q} \cong \mathfrak{z}(N)_{\mathbb{C}}/F^0(\mathfrak{z}(N))$ is identified with $T_x \tilde{B}(N)$.

Now suppose that $\mathfrak{m}^{p,q} = \overline{\mathfrak{m}^{q,p}}$ ($\forall p, q$), i.e. $x \in \tilde{B}^{\mathbb{R}}(N)$, and let $\beta(t)$ be a curve in $\mathfrak{z}(N)_{\mathbb{C}}$ such that $(e^{\beta(t)} F^{\bullet}, W(N)_{\bullet})$ stays in $\tilde{B}^{\mathbb{R}}(N)$ (and $\beta(0) = 0$). Then $\overline{e^{\beta(t)} F^q} \cap e^{\beta(t)} F^p \cap W_{p+q}$, or equivalently $\overline{F^q} \cap e^{-\beta(t)} e^{\beta(t)} F^p \cap W_{p+q}$, must remain of the same dimension as at $t = 0$ (for all p, q). By a short calculation, this implies that $\mathfrak{S}(\beta'(0)) \in (F^0 + \overline{F^0}) \cap \mathfrak{z}(N)_{\mathbb{R}}$, so that

$$\beta'(0) \in \left((F^0 + \overline{F^0}) \cap \mathfrak{z}(N) \right)_{\mathbb{C}} \oplus ((\mathfrak{q} \cap \overline{\mathfrak{q}}) \cap \mathfrak{z}(N))_{\mathbb{R}}.$$

Hence under the projection $\mathfrak{m}_{\mathbb{C}} \xrightarrow{\rho} \mathfrak{m}_{\mathbb{C}}/F^0 \cong \mathfrak{q}$,

$$\rho(\beta'(0)) \in \rho(\mathfrak{z}(N)_{\mathbb{R}}) \cong \mathfrak{z}(N)_{\mathbb{R}}/\{\mathfrak{m}^{0,0} \cap \mathfrak{z}(N)_{\mathbb{R}}\}.$$

It follows that $Z(N)(\mathbb{R})$ acts *locally* transitively on $\tilde{B}^{\mathbb{R}}(N)$, i.e. that the orbit of a point yields a real-analytic open subset; and a connected set cannot be a (disjoint) union of such orbits. \square

Now fix (by Lemma 3.2 and §1)

- $(F^\bullet, W(N)_\bullet) \in \tilde{B}^{\mathbb{R}}(N)$ \mathbb{Q} -split (L)MHS,
- $Y \in \mathfrak{m}_{\mathbb{Q}}$ associated Deligne grading, and
- $N_+ \in \mathfrak{m}_{\mathbb{Q}}$ such that (Y, N_+, N) is a \mathbb{Q} - \mathfrak{sl}_2 -triple.

Consider the decomposition

$$\mathfrak{m} = \bigoplus_{\ell \in \mathbb{Z}_{\geq 0}} \mathfrak{m}(\ell) = \bigoplus_{\ell \in \mathbb{Z}_{\geq 0}} V(\ell)^{\oplus m_\ell}$$

into isotypical components, where $V(\ell)$ is a copy of the $(\ell+1)$ -dimensional irrep of \mathfrak{sl}_2 . More precisely, writing $V(\ell) = \text{span} \{v_{-\ell}^{(\ell)}, v_{-\ell+2}^{(\ell)}, \dots, v_\ell^{(\ell)}\}$ we have on each copy

- $(\text{ad}Y)v_k^{(\ell)} = kv_k^{(\ell)}$,
- $(\text{ad}N_+)v_k^{(\ell)} = \begin{cases} 0, & k = \ell \\ v_{k+2}^{(\ell)}, & \text{otherwise} \end{cases}$,
- $(\text{ad}N)v_k^{(\ell)} = \begin{cases} 0, & k = -\ell \\ v_{k-2}^{(\ell)}, & \text{otherwise} \end{cases}$.

Note that the \mathfrak{sl}_2 -triple itself is a $V(2) \subset \mathfrak{m}$.

Setting

$$E(k) := \{v \in \mathfrak{m} \mid (\text{ad}Y)v = kv\} \subset \mathfrak{m}$$

the definition of $W(N)_\bullet$ is

$$W(N)_m \mathfrak{m} := \bigoplus_{k \leq m} E(k).$$

(This is independent of the choice of Y, N_+ though the individual $E(k)$ are not.)

Proof of Lemma 2.1. (i) Let $v \in E(k)$ be given. For $w \in E(j)$, the Jacobi identity gives

$$\begin{aligned} [Y, [v, w]] &= [v, [Y, w]] + [[Y, v], w] = (j+k)[v, w] \\ &\implies (\text{ad}v)w \in E(j+k). \end{aligned}$$

Moreover, there is *some* $j \in \mathbb{Z}$ and $w \in E(j)$ such that $(\text{ad}v)w \neq 0$: namely, $j = 0$ and $w = Y$. Since

$$\text{Lie}(W_k^N M) = \{\gamma \in \mathfrak{m} \mid (\text{ad}\gamma)W(N)_i \subset W(N)_{i+k} \ (\forall i)\},$$

the statement follows.

(ii) From the above characterization of isotypical components, any $\gamma \in \ker(\text{ad}N)$ is a sum of $v_{-\ell}^{(\ell)}$'s. That is,

$$\mathfrak{z}(N) = \bigoplus_{\ell \geq 0} \mathfrak{m}(\ell) \cap E(-\ell) \subseteq W(N)_0 \mathfrak{m}.$$

(iii) We have

$$\begin{aligned} \mathfrak{z}(N) \cap \text{im}(\text{ad}N) &= \bigoplus_{\ell \geq 0} ((\text{ad}N)\mathfrak{m}(\ell)) \cap E(-\ell) \\ &= \bigoplus_{\ell \geq 1} \mathfrak{m}(\ell) \cap E(-\ell) \\ &= \mathfrak{z}(N) \cap W(N)_{-1} \mathfrak{m}. \end{aligned}$$

□

Emphasizing that we are working with a \mathbb{Q} -split MHS, we turn to

Lemma 3.4. *The $\mathfrak{m}(\ell) \subset \mathfrak{m}$ (a) are sub-MHS, and (b) admit a further decomposition*

$$\mathfrak{m}(\ell)_{\mathbb{C}} = \bigoplus_{\alpha \in \mathbb{Z}} \mathfrak{m}(\ell, \alpha)$$

where

$$\mathfrak{m}(\ell, \alpha) := \mathfrak{m}(\ell) \cap \left\{ \bigoplus_{p-q=\alpha} I^{p,q}(\mathfrak{m}_{\mathbb{C}}) \right\}$$

are themselves sums of copies of $V(\ell)_{\mathbb{C}}$.

Proof. For (a), simply write

$$\mathfrak{m}(\ell) = \bigoplus_{k=0}^{\ell} \left\{ \begin{array}{l} \ker\{(\text{ad}N)^{k+1}\} \cap \text{im}\{(\text{ad}N_+)^k\} \\ \cap \ker\{(\text{ad}N_+)^{\ell-k+1}\} \cap \text{im}\{(\text{ad}N)^{\ell-k}\} \end{array} \right\}$$

and note that $\text{ad}N$, $\text{ad}N_+$ are morphisms of MHS of respective types $(-1, -1)$, $(+1, +1)$. (Of course, for N_+ this is *only* by virtue of $(F^\bullet, W(N)_\bullet)$ being \mathbb{Q} -split.) To produce (b), one takes the $(\text{ad}N)$ -orbit of the Hodge decomposition of the pure Hodge structure $\mathfrak{m}(\ell) \cap E(\ell) = \mathfrak{m}(\ell) \cap \ker(\text{ad}N_+)$. □

Consequently we can read off the full MHS on \mathfrak{m} from pure Hodge structures \tilde{P}_ℓ of weight ℓ

$$\tilde{P}_\ell^{\frac{\ell+\alpha}{2}, \frac{\ell-\alpha}{2}} := \mathfrak{m}(\ell, \alpha) \cap E(\ell).$$

(Note that $E(\ell) \subset W(N)_\ell \mathfrak{m} \rightarrow Gr_\ell^{W(N)} \mathfrak{m}$ sends $\tilde{P}_\ell \xrightarrow{\cong} P_\ell$.) Writing

$$V(\ell) := \bigoplus_{k=0}^{\ell} \mathbb{Q}(k)$$

and endowing \tilde{P}_ℓ with the trivial \mathfrak{sl}_2 -representation,

$$(3.1) \quad \mathfrak{m} \cong \bigoplus_{\ell \geq 0} \tilde{P}_\ell \otimes V(\ell)$$

as MHS and \mathfrak{sl}_2 -representations.

4. PROOF OF THEOREM 2.8

Express the \mathbb{Q} -split MHS $(F^\bullet, W(N)_\bullet)$ and \mathfrak{sl}_2 -representation (Y, N_+, N) of §3 by $(\tilde{\varphi}, d\rho)$. The idea of the proof is to let $Z(N)(\mathbb{R})$ act on this pair to yield (by Lemma 3.3) all of $\tilde{B}^{\mathbb{R}}(N)$. Then take (as per Lemma 2.4(b)) the \mathbb{Q} -closure of the union of the images of all $\tilde{\varphi}' \in Z(N)(\mathbb{R}) \cdot \tilde{\varphi}$, to yield $\tilde{M}_{B^{\mathbb{R}}(N)}$. (At the end of the proof we shall check that this coincides with $\tilde{M}_{B(N)}$.) In fact, all we really need to determine is $\tilde{\mathfrak{m}}_{B^{\mathbb{R}}(N)}$, which we can do by acting infinitesimally (to arbitrary order) on the tangent space to $\tilde{\varphi}$ – that is, by repeated application of $\text{ad}_{\mathfrak{z}}(N)_{\mathbb{R}}$.

Proof. We start with an explicit formula for $\tilde{\varphi}$, using Definition 2.3. The splitting of F^\bullet induced by the $\{\oplus_q I^{p,q}(\mathfrak{m}_{\mathbb{C}})\}_{p \in \mathbb{Z}}$ kills Hodge $(0,0)$ -tensors, hence any Hodge tensors in the nilpotent orbit, and scales N ($\in I^{-1,-1}(\mathfrak{m}_{\mathbb{C}})$). Thus it may be written ad_ξ for some $\xi \in \tilde{\mathfrak{z}}(N)_{\mathbb{C}} = (\mathfrak{z}(N) + \langle Y \rangle)_{\mathbb{C}} \subseteq \mathfrak{m}_{\mathbb{C}}$. Since $\tilde{\varphi}$ is \mathbb{R} -split, ad_ξ not only splits $\overline{F^\bullet}$ but has eigenspaces $\{\oplus_p I^{p,q}(\mathfrak{m}_{\mathbb{C}})\}_{q \in \mathbb{Z}}$. In fact, from $N \in I^{-1,-1}(\mathfrak{m}_{\mathbb{C}})$ it is clear that $[N, \xi] = N = [N, \bar{\xi}]$. Further, ξ and $\bar{\xi}$ commute with each other and with Y since the three are simultaneously diagonalizable. By Definition 2.3(ii), we have immediately that

$$\tilde{\varphi}(z, w) = \exp(\log(z)\text{ad}_\xi + \log(w)\text{ad}_{\bar{\xi}})$$

Now $\tilde{\varphi}(z^2, w^2)/\exp(\log(zw)\text{ad}_Y)$ is clearly of pure weight zero, so is of the form $\exp(\log(z/w)\text{ad}(i\phi))$ where we have

$$Y = \xi + \bar{\xi}, \quad i\phi = \xi - \bar{\xi}.$$

In particular, ϕ commutes with N and Y and so

$$\phi \in (\mathfrak{z}(N) \cap E(0))_{\mathbb{R}} = \mathfrak{m}(0)_{\mathbb{R}}.$$

Recalling (3.1), we see that ϕ acts on \mathfrak{m} through the $\{\tilde{P}_\ell\}$ while Y acts through the $\{V(\ell)\}$. The formula becomes

$$(4.1) \quad \tilde{\varphi}(z, w) = \exp\left(\frac{1}{2} \log(zw) \operatorname{ad} Y + \frac{i}{2} \log(z/w) \operatorname{ad} \phi\right),$$

and ϕ determines φ_{split} through

$$\left(Gr_k^{W(N)} \varphi\right)(z) = \tilde{\varphi}(z, z^{-1})\Big|_{E(k)} = \exp\left(i \log(z) (\operatorname{ad} \phi)\Big|_{E(k)}\right).$$

For later use, we note that $\mathfrak{m}(0)$ is defined over \mathbb{Q} and closed under the Lie bracket.

We wish to take the \mathbb{Q} -algebraic group closure of the set of $Z(N)(\mathbb{R})$ -conjugates of

$$\tilde{\varphi}(\mathbb{C}^* \times \mathbb{C}^*) = \{\exp(\alpha Y + \beta \phi) \mid \alpha, \beta \in \mathbb{C}\}$$

in $\tilde{Z}(N)(\mathbb{C})$. As mentioned before, to compute $\tilde{\mathfrak{m}}_{B^{\mathbb{R}}(N)}$ it suffices to take the \mathbb{Q} -Lie-algebra closure of $Z(N)(\mathbb{R})$ -conjugates of the tangent space

$$(4.2) \quad T_e \tilde{\varphi}(\mathbb{C}^* \times \mathbb{C}^*) = \{\alpha Y + \beta \phi \mid \alpha, \beta \in \mathbb{C}\}$$

in $\tilde{\mathfrak{z}}(N)_{\mathbb{C}}$. We first consider the effect of an infinitesimal $M_N(\mathbb{R})$ -action on (4.2).

Let $\{\gamma_i\} \subset \mathfrak{m}_N$ be a basis consisting of Y -eigenspaces; that is, $\gamma_i \in E(k_i) \cap \mathfrak{m}_N$ with $k_i < 0$. Then

$$(\operatorname{ad} \gamma_i) Y = -(\operatorname{ad} Y) \gamma_i = -k_i \gamma_i,$$

while $\phi \in (\mathfrak{z}(N) \cap E(0))_{\mathbb{R}}$, $\gamma_i \in \mathfrak{z}(N) \cap E(k_i)$ gives

$$(\operatorname{ad} \gamma_i) \phi \in (\mathfrak{z}(N) \cap E(k_i + 0))_{\mathbb{R}} \subset (\mathfrak{m}_N)_{\mathbb{R}}.$$

To first order in ϵ , the tangent plane to $e^{\epsilon \sum \mu_i \gamma_i} \cdot \tilde{\varphi}(z, w)$ ($\mu_i \in \mathbb{R}$; action by conjugation) is then given by

$$e^{\epsilon \sum \mu_i (\operatorname{ad} \gamma_i)} (\alpha Y + \beta \phi) =$$

$$\alpha Y + \beta \phi - \epsilon \alpha \sum_i \mu_i k_i \gamma_i + \epsilon \beta \sum_i \mu_i (\text{ad} \gamma_i) \phi$$

(where $\alpha, \beta \in \mathbb{C}$). By taking $\beta = 0$ we already get all of $(\mathfrak{m}_N + \langle Y \rangle)_{\mathbb{C}}$ from the effect on Y alone. (In fact, this essentially reproves part (c) of Kostant.)

To go from $M_N(\mathbb{R})$ -action to $Z(N)(\mathbb{R})$ -action, let $\gamma \in (\mathfrak{m}(0))_{\mathbb{R}}$; then

$$\begin{cases} (\text{ad} \gamma)Y = -(\text{ad} Y)\gamma = 0 \\ (\text{ad} \gamma)\phi \in (\mathfrak{m}(0))_{\mathbb{R}} \end{cases},$$

and so

$$\begin{aligned} & e^{\epsilon(\text{ad} \gamma + \sum_i \mu_i (\text{ad} \gamma_i))}(\alpha Y + \beta \phi) \\ & \equiv \beta \left(\phi + \epsilon(\text{ad} \gamma)\phi + \frac{\epsilon^2}{2}(\text{ad} \gamma)^2 \phi + \dots \right) \pmod{(\mathfrak{m}_N + \langle Y \rangle)_{\mathbb{C}}}. \end{aligned}$$

Denote by $\mathfrak{g}_{B^{\mathbb{R}}(N)}$ the \mathbb{Q} -Lie-algebra closure in $\mathfrak{m}(0)$ ($\cong \mathfrak{z}(N)/\mathfrak{m}_N$) of

$$(4.3) \quad \langle \phi \rangle_{\mathbb{C}} + \sum_{\gamma \in \mathfrak{m}(0)_{\mathbb{R}}} (\text{ad} \gamma) \langle \phi \rangle_{\mathbb{C}} + \sum_{\gamma \in \mathfrak{m}(0)_{\mathbb{R}}} (\text{ad} \gamma)^2 \langle \phi \rangle_{\mathbb{C}} + \dots.$$

Then $\langle Y \rangle$ and

$$\tilde{\mathfrak{g}}_{B^{\mathbb{R}}(N)} := \mathfrak{g}_{B^{\mathbb{R}}(N)} + \langle Y \rangle$$

are \mathbb{Q} -Lie algebras in $\tilde{\mathfrak{z}}(N)$ mapping isomorphically to their images in $\tilde{\mathfrak{z}}(N)/\mathfrak{m}_N$, and we have proved the short-exact sequence

$$0 \rightarrow \mathfrak{m}_N \rightarrow \tilde{\mathfrak{m}}_{B^{\mathbb{R}}(N)} \rightarrow \tilde{\mathfrak{g}}_{B^{\mathbb{R}}(N)} \rightarrow 0.$$

Note that we have crucially used the facts that $\mathfrak{m}_N + \langle Y \rangle$ is a \mathbb{Q} -Lie algebra and that $\mathfrak{m}(0)$ and Y commute, so that the only \mathbb{Q} -closure which needs to be taken is that of (4.3). Intersecting with $\mathfrak{z}(N)$ now gives

$$0 \rightarrow \mathfrak{m}_N \rightarrow \mathfrak{m}_{B^{\mathbb{R}}(N)} \rightarrow \mathfrak{g}_{B^{\mathbb{R}}(N)} \rightarrow 0.$$

To finish off the proof we must show that none of these groups “increases” upon passing from $B^{\mathbb{R}}(N)$ to $B(N)$. Given $\tilde{\varphi}' \in \tilde{B}(N)$, there

exists a natural element¹⁰

$$\delta \in \left(\bigoplus_{p,q \leq -1} I^{p,q}(\mathfrak{m}) \right)_{\mathbb{R}}$$

such that:

- (1) $e^{-i\delta}.\tilde{\varphi}'$ is \mathbb{R} -split;
- (2) $\delta \in W_{-2}\mathfrak{m}$; and
- (3) δ commutes with all (r, r) -morphisms of MHS.

Clearly (2) gives $e^{-i\delta} \in (W_{-2}M)(\mathbb{C}) \subseteq (W_{-1}M)(\mathbb{C})$, while (3) $\implies [\delta, N] = 0$

$$\implies \delta \in \mathfrak{z}(N)_{\mathbb{R}} \implies e^{-i\delta} \in Z(N)(\mathbb{C}).$$

By Lemma 2.1, we now have $\delta \in M_N(\mathbb{C})$. Now *a priori*, $e^{-i\delta}.\tilde{\varphi}$ is just a MHS polarized by N . The “converse to (2.3)” in [CK] says that this suffices to make $e^{\tau N}e^{-i\delta}F_{\tilde{\varphi}}^{\bullet}$ a nilpotent orbit; hence $e^{-i\delta}.\tilde{\varphi}' \in B^{\mathbb{R}}(N)$. This shows that

$$\tilde{B}(N) \neq \emptyset \implies \tilde{B}^{\mathbb{R}}(N) \neq \emptyset.$$

Provided we act on the Hodge filtration F^{\bullet} (instead of $\tilde{\varphi}$), the “converse theorem” (loc. cit.) also implies that $\tilde{B}(N)$ is closed under the action of $M_N(\mathbb{C})$. So by the last paragraph, the $M_N(\mathbb{C})$ -orbit of $\tilde{B}^{\mathbb{R}}(N)$ is all of $\tilde{B}(N)$. Together with the proof of Lemma 3.3, this means that on an infinitesimal level the $\text{ad}(\mathfrak{q} \cap \bar{\mathfrak{q}} \cap \mathfrak{m}_{N,\mathbb{C}})_{i\mathbb{R}}$ -orbit of our given \mathbb{Q} -split MHS $p = (F^{\bullet}, W(N)_{\bullet})$ surjects onto $T_p\tilde{B}(N)/T_p\tilde{B}^{\mathbb{R}}(N)$. This is consistent with the action of $\text{ad}(\mathfrak{q} \cap \bar{\mathfrak{q}} \cap \mathfrak{m}_{N,\mathbb{C}})$ on $\tilde{\varphi}$. The upshot is that we only need to examine the effect on our original $\alpha Y + \beta\phi$ by a larger subset of $\text{ad}(\mathfrak{m}_{N,\mathbb{C}})$ replacing $\text{ad}(\mathfrak{m}_{N,\mathbb{R}})$ in the above argument, which obviously does not change the \mathbb{Q} -closure. The proof of Theorem 2.8 is now complete. \square

Two new results come out of this proof. For the first, we need

¹⁰This is by a result of Deligne, cf. [KP1, sec. 4]. What is not addressed there is the fact that (in the present setting) δ belongs to the Mumford-Tate Lie algebra \mathfrak{m} . This follows at once from its functoriality with respect to Tannakian operations and property (3) below.

Definition 4.1. $\mathfrak{g}_N := \mathfrak{z}(N) \cap E(0)$, $\tilde{\mathfrak{g}}_N := \tilde{\mathfrak{z}}(N) \cap E(0) (= \mathfrak{g}_N + \langle Y \rangle)$, $\mathfrak{g}_{B(N)} := \mathfrak{g}_{B^{\mathbb{R}}(N)}$; with corresponding Lie groups $G_N, G_{B(N)} \leq M_{B(N)}$ and $\tilde{G}_N, \tilde{G}_{B(N)} \leq \tilde{M}_{B(N)}$.

Proposition 4.2. *We have semi-direct product decompositions*

$$\tilde{M}_{B(N)} = M_N \rtimes \tilde{G}_{B(N)} \subseteq M_N \rtimes \tilde{G}_N$$

$$M_{B(N)} = M_N \rtimes G_{B(N)} \subseteq M_N \rtimes G_N$$

with all groups \mathbb{Q} -algebraic, $\tilde{G}_{B(N)}, \tilde{G}_N, G_{B(N)}, G_N$ reductive, and M_N the unipotent radical in each case.

While $G_{B(N)}$ depends on the choice of \mathbb{Q} -split MHS in Lemma 3.2, its (isomorphic) image under the quotient by M_N is $Gr_0^W M_{B(N)}$ (which does not).

The second result refines Lemma 3.3:

Proposition 4.3. (i) $M_{B(N)}(\mathbb{R})$ acts transitively on $\tilde{B}^{\mathbb{R}}(N)$;
(ii) $M_N(\mathbb{C}) \rtimes G_{B(N)}(\mathbb{R})$ acts transitively on $\tilde{B}(N)$.

5. ARBITRARY RANK

We shall now retell the story of §§2 – 4 in the more general context of a boundary component associated to a (finitely-generated) *rational nilpotent cone*

$$\sigma := \mathbb{Q}_{\geq 0} \langle N_1, \dots, N_r \rangle \subset \mathfrak{m}_{\mathbb{Q}},$$

where $N_1, \dots, N_r \in \mathfrak{m}_{\mathbb{Q}}$ are commuting nilpotent elements. (Without loss of generality these may be assumed to be *convex*, i.e. $\sigma \cap (-\sigma) = \{0\}$.) Write

$$\sigma^\circ := \mathbb{Q}_{> 0} \langle N_1, \dots, N_r \rangle$$

for its interior and $\langle \sigma \rangle$ for its \mathbb{Q} -vector-space closure, with $\sigma_{\mathbb{R}}^\circ, \langle \sigma \rangle_{\mathbb{R}}, \langle \sigma \rangle_{\mathbb{C}}$ all having the obvious meaning.

A σ -nilpotent orbit is a subset of \check{D}_M of the form $e^{\langle \sigma \rangle_{\mathbb{C}}} \cdot F^\bullet$ where F^\bullet satisfies

$$(5.1) \quad \begin{aligned} & \text{(i) } N_j F^p \subset F^{p-1} \quad (\forall j) \\ & \text{(ii) } e^{\sum_j \tau_j N_j} F^\bullet \in D_M \quad \text{if all } \Im(\tau_j) \gg 0. \end{aligned}$$

As before, we set

$$\tilde{B}(\sigma) := \left\{ F^\bullet \in \check{D}_M \mid F^\bullet \text{ satisfies (i) and (ii)} \right\}$$

and then

$$B(\sigma) := \tilde{B}(\sigma) / e^{\langle \sigma \rangle c}$$

is the set of σ -nilpotent orbits. We assume $B(\sigma) \neq \emptyset$. The *rank* of $B(\sigma)$ is just $\dim \langle \sigma \rangle (= r)$.

A fundamental result of Cattani and Kaplan ([CK], Thm. 2.3(ii)) says that the weight filtration $W(N)_\bullet$ of \mathfrak{m} is independent of the choice of $N \in \sigma^\circ$. Without loss of generality we can put $N := N_1 + \cdots + N_r$, and write $W(\sigma)_\bullet \mathfrak{m} := W(N)_\bullet \mathfrak{m}$, $W_\bullet^\sigma M := W_\bullet^N M$. It will also be convenient to take elements $\hat{N}_1, \dots, \hat{N}_r \in \sigma^\circ$ giving a basis for $\langle \sigma \rangle$. We have the Lie algebras

$$\begin{aligned} \mathfrak{z}(\sigma) &:= \bigcap_{j=1}^r \ker(\text{ad} N_j) = \bigcap_{j=1}^r \ker(\text{ad} \hat{N}_j) \\ \mathfrak{m}_\sigma &:= \text{im}(\text{ad} N) \cap \mathfrak{z}(\sigma) \end{aligned}$$

and their corresponding Lie groups $Z(\sigma)$, $M_\sigma \leq M$. Since $\mathfrak{z}(\sigma) \subset \mathfrak{z}(N)$, we have

$$\begin{aligned} \mathfrak{m}_\sigma &= \mathfrak{m}_N \cap \mathfrak{z}(\sigma) \\ &= (W(N)_{-1} \cap \mathfrak{z}(N)) \cap \mathfrak{z}(\sigma) \\ &= W(\sigma)_{-1} \cap \mathfrak{z}(\sigma) \end{aligned}$$

which is clearly independent of the choice of N .

As before we may interpret the elements $F^\bullet \in \tilde{B}(\sigma)$ as LMHS $(F^\bullet, W(\sigma)_\bullet)$ on \mathfrak{m} , and define the \mathbb{R} -split loci¹¹ $\tilde{B}^{\mathbb{R}}(\sigma)$, $B^{\mathbb{R}}(\sigma)$ and the Mumford-Tate group $M_{B(\sigma)}$, $M_{B^{\mathbb{R}}(\sigma)}$. These MHS have a richer structure than in the rank-one setting: according to [CK, Thm. 2.3(iii)], the Hard Lefschetz isomorphisms and polarizations B_k of Remark 2.2(c) are induced by any element of σ° (e.g., all the \hat{N}_j). (However, we shall put $B_k(v, w) := B(v, N^k w)$ with our choice of N above.) Moreover, the $\{N_j\}$ all belong to $I^{-1, -1}(\mathfrak{m}_{\mathbb{C}})$ – meaning that $\langle \sigma \rangle$ consists of Hodge tensors.

¹¹also nonempty by the [CK] “converse result” and $e^{-i\delta}$ process from the end of §4

Accordingly, for a given $\tilde{\varphi} \in \tilde{B}(\sigma)$, there is a reduction in the information required to describe φ_{split} in (2.1), where σ replaces N everywhere and

$$\tilde{Z}(\sigma) := \{g \in M \mid \text{Ad}(g)N_j = \psi(g)N_j \ (\forall j) \text{ for some character } \psi\}.$$

Namely, in order for the primitive spaces in Remark 2.6(b),(b') to match up with $Gr_{-k}^{W(\sigma)}\mathfrak{z}(\sigma)$ under N^k , we must define¹²

$$P_k := \bigcap_{j=1}^r \ker \left(\text{ad}N_j \circ (\text{ad}N)^k \right) \subset Gr_k^{W(\sigma)}\mathfrak{m}.$$

Proposition 5.1. (i) $Z(\sigma)/M_\sigma$ is reductive and acts faithfully on $\bigoplus_{k \geq 0} P_k$.

(ii) The HS on $\bigoplus_k Gr_k^{W(\sigma)}\mathfrak{m}$ can be recovered from that on $\bigoplus_{k \geq 0} P_k$ by

$$(5.2) \quad Gr_k^{W(\sigma)}\mathfrak{m} = \bigoplus_{\ell \geq \max\{0, -k\}} \sum_{1 \leq j_1, \dots, j_\ell \leq r} (\text{ad}N_{j_1}) \circ \dots \circ (\text{ad}N_{j_\ell}) P_{k+2\ell}.$$

Proof. (i) is as before. For (ii), the mixed Lefschetz result of [Ca1] implies that

$$Gr_k^{W(\sigma)}\mathfrak{m} = \ker \left(\text{ad}N_j \circ (\text{ad}N)^k \right) \bigoplus \text{im}(\text{ad}N_j)$$

for each j . By the semisimplicity of polarized Hodge structures this gives

$$Gr_k^{W(\sigma)}\mathfrak{m} = P_k \oplus \left(\sum_{j=1}^r \text{im}(\text{ad}N_j) \right),$$

which inductively leads to (5.2). \square

We are now ready for

Theorem 5.2. For a nonempty Kato-Usui boundary component $B(\sigma)$ of D_M associated to a nilpotent cone $\sigma \subset \mathfrak{m}_\mathbb{Q}$, the Mumford-Tate group $M_{B(\sigma)}$ satisfies the following:

(A) $M_{B(\sigma)}$ is contained in the centralizer $Z(\sigma)$ of the cone;

(B) $W_{-1}M_{B(\sigma)} = M_\sigma$ is its unipotent radical; and

(C) $Gr_0^W M_{B(\sigma)} = M_{B(\sigma)} / M_\sigma \ (\subset Z(\sigma) / M_\sigma)$ is the \mathbb{Q} -algebraic-group closure of the orbit $(Z_\sigma / M_\sigma)(\mathbb{R}) \cdot \varphi_{\text{split}}$, which may be regarded

¹²Note that $k \geq 0$, and $\text{ad}(N_j) \circ (\text{ad}(N))^k$ maps from $Gr_k^{W(\sigma)}\mathfrak{m}$ to $Gr_{-k-2}\mathfrak{m}$.

as the Mumford-Tate group of a set of polarized HS¹³ on $\bigoplus_{k \geq 0} P_k$.

The analogues of Definition 4.1, Proposition 4.2, and Proposition 4.3 all hold, with N replaced by σ everywhere.

Proof. Let $(F^\bullet, W(\sigma)_\bullet) \in \tilde{B}^{\mathbb{R}}(\sigma)$ with corresponding representation $\tilde{\varphi} : \mathbb{S} \rightarrow \tilde{Z}(\sigma)$ (defined over \mathbb{R}) and Deligne splitting $\text{ad}Y$, $Y \in \mathfrak{m}_{\mathbb{R}}$. From the above discussion and [CK, (3.3)], we have for each j

$$\begin{cases} Y \in \text{im}(\text{ad}\hat{N}_j) \\ [Y, \hat{N}_j] = -2\hat{N}_j \end{cases}.$$

From §1 (and noting $\mathfrak{m}_\sigma = \bigcap_j \mathfrak{m}_{\hat{N}_j}$, $M_\sigma = \bigcap_j M_{\hat{N}_j}$), we obtain that the subsets of $\mathfrak{m}_{\mathbb{R}}$ given by

$$(5.3) \quad \left\{ x \in \mathfrak{m}_{\mathbb{R}} \mid [x, \hat{N}_j] = -2\hat{N}_j, x \in \text{im}(\text{ad}\hat{N}_j) (\forall j) \right\}$$

$$(5.4) \quad \bigcap_j \{Y + \mathfrak{m}_{\hat{N}_j, \mathbb{R}}\} = Y + \mathfrak{m}_{\sigma, \mathbb{R}}$$

$$(5.5) \quad \bigcap_j \{(\text{Ad}M_{\hat{N}_j}(\mathbb{R})) \cdot Y\} = (\text{Ad}M_\sigma(\mathbb{R})) \cdot Y$$

are equal. Clearly (5.3) is defined over \mathbb{Q} , so the existence of a \mathbb{Q} -split LMHS goes through as in Lemma 3.2, and we henceforth assume $Y \in \mathfrak{m}_{\mathbb{Q}}$ ($\implies \tilde{\varphi}$ and $(F^\bullet, W(\sigma)_\bullet)$ \mathbb{Q} -split).

The formula (4.1) for $\tilde{\varphi}$ holds as before, with the following difference: since the \hat{N}_j all polarize $(F^\bullet, W(\sigma)_\bullet)$, they produce $(-1, -1)$ -morphisms of HS from each $E(k)$ to $E(k-2)$. Hence they commute with ϕ and we have

$$\phi \in (\mathfrak{z}(\sigma) \cap E(0))_{\mathbb{R}} = \mathfrak{g}_{\sigma, \mathbb{R}} \left(\xrightarrow{\cong} (\mathfrak{z}(\sigma)/\mathfrak{m}_\sigma)_{\mathbb{R}} \right).$$

Using the equality of (5.4) and (5.5) we have

$$\text{Ad}M_\sigma(\mathbb{R}) \cdot \langle Y \rangle = \langle Y \rangle + \mathfrak{m}_{\sigma, \mathbb{R}}$$

$$\text{Ad}M_\sigma(\mathbb{R}) \cdot \langle \phi \rangle \subset \mathfrak{m}_{\sigma, \mathbb{R}}$$

$$\{\text{ad}\mathfrak{g}_\sigma\} \langle Y \rangle = \{0\}$$

¹³note that a “polarized HS” means a *direct sum of pure polarized HS*

whereupon the remainder of the proof of Theorem 2.8 goes through with superficial changes. \square

6. INTERLUDE ON KATO-USUI SPACES

In the remainder of this paper we shall be concerned with the structure of boundary components $B(\sigma)$ and certain quotients thereof. The purpose of this section is to offer the reader a glimpse of how these quotients fit into partial compactifications of quotients $\Gamma \backslash D_M$. The material surveyed here is done only for period domains in [KU], but extends in a straightforward way to the more general Mumford-Tate domain setting.

Let Σ be a *fan* in \mathfrak{m} , that is, a *set* of (finitely generated, convex) rational nilpotent cones in \mathfrak{m} intersecting in faces, which is *closed* under the operation of taking faces. We define

$$\begin{aligned} D_{M,\Sigma} &:= \coprod_{\sigma \in \Sigma} \{ Z \subset \check{D}_M \mid Z \text{ is a } \sigma\text{-nilpotent orbit} \} \\ &= \coprod_{\sigma \in \Sigma} B(\sigma), \end{aligned}$$

noting that this always contains $B(\{0\}) = D_M$. In particular, we shall write

$$D_{M,\sigma} := D_{M, \{\text{faces of } \sigma\}},$$

which in the rank one case $\sigma = \mathbb{Q}_{\geq 0} \langle N \rangle$ is nothing but $D_M \amalg B(N)$.

Next take $\Gamma \subset M(\mathbb{Z})$ to denote a neat¹⁴ subgroup of finite index, and consider the monoid

$$\Gamma(\sigma) := \Gamma \cap \exp(\sigma_{\mathbb{R}})$$

with group-theoretic closure $\Gamma(\sigma)^{\text{gp}}$. We assume that Γ is strongly compatible with Σ , i.e.

- (1) $\text{Ad}(\gamma).\sigma \in \Sigma$ ($\forall \gamma \in \Gamma, \sigma \in \Sigma$)
- (2) $\sigma_{\mathbb{R}} = \mathbb{R}_{\geq 0} \langle \log \Gamma(\sigma) \rangle$.

¹⁴Recall that this means the subgroup of \mathbb{C}^* generated by the eigenvalues of the elements of Γ acting on \mathfrak{m} is torsion-free. Using the Jordan decomposition for M , it follows that the action on any tensor space $T^{a,b}\mathfrak{m}$ (and its subquotients) is also free of torsion eigenvalues. Neat subgroups of finite index always exist in $M(\mathbb{Z})$.

By (1), $Z \mapsto \gamma.Z$ induces maps $B(\sigma) \mapsto B(\text{Ad}(\gamma).\sigma)$ and so the quotient $\Gamma \backslash D_\Sigma$ makes sense on the set-theoretic level. The essence of Theorem A of [KU] can be transcribed as follows:

Theorem 6.1. *$\Gamma \backslash D_{M,\Sigma}$ is a logarithmic manifold, which is Hausdorff in the strong topology. For each $\sigma \in \Sigma$, the map*

$$\Gamma(\sigma)^{\text{gp}} \backslash D_{M,\sigma} \xrightarrow{\Upsilon_\sigma} \Gamma \backslash D_{M,\Sigma}$$

is open and locally an isomorphism.

Let $\Gamma_\sigma \subseteq \Gamma$ denote the largest subgroup stabilizing σ (viz., $\text{Ad}(\gamma).\sigma \subseteq \sigma$). Noting that $\Gamma(\sigma)^{\text{gp}}$ acts trivially on $B(\sigma)$, we have

$$B(\sigma) \subset (\Gamma(\sigma)^{\text{gp}} \backslash D_{M,\sigma})$$

with image

$$(6.1) \quad \overline{B(\sigma)} := \Gamma_\sigma \backslash B(\sigma)$$

under Υ_σ . It is these boundary component quotients which will be of particular interest in §7.

Though we shall not address log analytic spaces, it seems desirable to give the reader some idea of how the MT-domain and boundary-component quotients get “pasted together”. Kato and Usui accomplish this with the aid of $\langle \sigma \rangle_{\mathbb{C}}$ -torsors

$$(6.2) \quad E_\sigma \xrightarrow{\Theta_\sigma} \Gamma(\sigma)^{\text{gp}} \backslash D_{M,\sigma}$$

which we now explain in the rank-one case where $\sigma := \mathbb{Q}_{\geq 0} \langle N \rangle$, $\langle \sigma \rangle_{\mathbb{C}} = \langle N \rangle_{\mathbb{C}}$, and

$$\Gamma(\sigma)^{\text{gp}} \backslash D_{M,\sigma} = (\Gamma(\sigma)^{\text{gp}} \backslash D_M) \amalg B(N).$$

Put

$$E_\sigma := \left\{ (q, F^\bullet) \in \mathbb{C} \times \check{D}_M \left| \begin{array}{l} e^{\frac{1}{2\pi i} \log(q)N} F^\bullet \in D_M \text{ if } q \neq 0; \quad \text{and} \\ e^{\langle N \rangle_{\mathbb{C}}} F^\bullet \text{ is a } (\sigma\text{-})\text{nilpotent orbit if } q = 0 \end{array} \right. \right\},$$

and define Θ_σ by

$$(q, F^\bullet) \longmapsto \begin{cases} e^{\frac{1}{2\pi i} \log(q)N} F^\bullet \in (\Gamma(\sigma)^{\text{gp}} \backslash D_M), & \text{if } q \neq 0 \\ e^{\langle N \rangle_{\mathbb{C}}} F^\bullet \in B(N) & , \quad \text{if } q = 0 \end{cases}.$$

The fibres of this map are the orbits of the action

$$\xi.(q, F^\bullet) := (e^{2\pi i\xi}q, e^{-\xi N}F^\bullet)$$

of $(\langle N \rangle_{\mathbb{C}} \cong) \mathbb{C}$ on E_σ , giving an isomorphism

$$(6.3) \quad E_\sigma / \langle \sigma \rangle_{\mathbb{C}} \xrightarrow{\cong} \Gamma(\sigma)^{\text{gp}} \backslash D_\sigma$$

(6.3) is valid quite generally, and the right-hand side gets topologized by the left.

Remark 6.2. In [KU], Theorem 6.1 is proved for partial compactifications of $\Gamma \backslash \mathbf{D}$, where $\mathbf{D} = \mathbf{G}(\mathbb{R})/\mathbf{H}$ is a period domain; no finite index assumption is imposed on Γ . To see why the same result holds for a M-T domain D_M , note that it embeds homogeneously in such a \mathbf{D} , and that we may use the corresponding inclusion of M into \mathbf{G} to regard $\Gamma \leq M(\mathbb{Z})$ and $\Sigma \subset \mathfrak{m}$ as belonging also to $G(\mathbb{Q})$ resp. \mathfrak{g} . With these definitions, $\Gamma \backslash D_{M,\Sigma}$ embeds continuously in $\Gamma \backslash \mathbf{D}_\Sigma$, since E_σ is a connected component of the intersection of $\mathbb{C} \times \check{D}_M$ with the torsor \mathbf{E}_σ used to topologize $\Gamma \backslash \mathbf{D}_\Sigma$; it follows that $\Gamma \backslash D_{M,\Sigma}$ is Hausdorff.

Remark 6.3. The proof (see Chapter 5 of [KU]) of Theorem A depends on the properties of a fundamental diagram which connects various enlargements of $\mathbf{D} = \mathbf{G}(\mathbb{R})/\mathbf{H}$ to the Borel-Serre space attached to the set of all maximal compact subgroups of $G(\mathbb{R})$. The corresponding statement for Mumford-Tate domains, and in particular the fact that $\Gamma \backslash D_{M,\Sigma}$ is a logarithmic manifold carries over *mutatis mutandis* since the underlying formal theory is identical. Indeed, the key ingredients for the proofs in [KU] are the Borel-Serre theory for semisimple Lie groups and the form of the SL_2 -orbit theorem of [CKS] for period domains. More precisely, to a nilpotent orbit of pure Hodge structure $e^{iyN}F$, the SL_2 -orbit theory [Sc] attaches an associated SL_2 -orbit $e^{iyN}\hat{F}$ and a function $g(y)$ of the form

$$g(y) = 1 + \sum_{k>0} g_k y^{-k},$$

such that

$$e^{iyN}F = g(y)e^{iyN}\hat{F}$$

for $y \gg 0$. In Chapter IV of [GGK1], it is shown that Schmid's arguments work in the setting of nilpotent orbits into Mumford-Tate domains. To obtain the several variable version of the SL_2 -orbit theorem of [CKS] for Mumford-Tate domains, one needs to know that $g(y)$ retains the same dependence on the pair (N, F) . As the underlying system of differential equations governing $g(y)$ is the same in both cases, this must be true. More formally, by Proposition 3.28 of [CKS], the coefficients g_k are given by universal non-commutative polynomials in the Hodge components of δ and the \mathfrak{sl}_2 -representation attached to $e^{iyN}\hat{F}$. Alternatively, instead of reconstructing the fundamental diagram and proofs of [KU], one can probably also proceed by comparison of $\Gamma \backslash D_{M,\Sigma}$ to $\Gamma \backslash \mathbf{D}_\Sigma$ and the corresponding E_σ 's.

Remark 6.4. Kato and Usui do not offer any magic prescription for how to choose Σ . (One of their key examples is simply to take all rays generated by rational nilpotent elements of \mathfrak{m} .) Indeed, it can be tedious to check that a given choice of σ (or N) satisfies criterion (5.1)(ii) in the definition of a (σ) -nilpotent orbit. We have worked around this issue by assuming $B(\sigma) \neq \emptyset$ above, but obviously must face it for the examples in §8. By [CK, (2.3)], in order for $B(\sigma)$ to be nonempty, the *necessary* conditions (on σ) are:

- (a) $(\text{ad}N_j)^{k+1} = 0$ ($\forall j$), where k is the level¹⁵ of the Hodge structures (on \mathfrak{m}) in D_M ; and
- (b) all $N \in \sigma^\circ$ define the same $W(N)_\bullet$.

It turns out (by the “converse to (2.3)” [op. cit.]) that a given F^\bullet satisfying (5.1)(i) gives a σ -nilpotent orbit (i.e. (5.1)(ii) holds) if one has (a),(b), and

¹⁵Recall that for a weight n HS with Hodge numbers $\{h^{p,n-p}\}_{p \in \mathbb{Z}}$, the *level* is the difference $\max\{p | h^{p,n-p} \neq 0\} - \min\{p | h^{p,n-p} \neq 0\}$; in our (adjoint HS) setting, $n = 0$ and the level is even.

(ii') the pure Hodge structures induced by F^\bullet on the¹⁶

$$\hat{P}_k := \ker \left((\text{ad} N)^{k+1} \right) \subset Gr_k^{W(N)} \mathfrak{m}$$

are polarized by the restriction of $B \left((\cdot), (\text{ad} N)^k(\cdot) \right)$ (for some $N \in \sigma^\circ$).

We expect, but have not proved in full, that this only needs to be checked on the $\{P_k\}$.

7. BOUNDARY COMPONENT STRUCTURE

We are interested in when $\Gamma_\sigma \backslash B(\sigma)$ has an arithmetic algebraic structure – i.e., a canonical model over $\bar{\mathbb{Q}}$ – especially in those cases when $\Gamma \backslash D_M$ does *not*. In order to uncover this we have to understand the fibration structure, starting with the prequotient $B(\sigma)$. We make the slight notational change of letting $\tilde{\varphi}_0$ (or $(F_0^\bullet, W(\sigma)_\bullet)$) denote our \mathbb{Q} -split point, and $\tilde{\varphi}$ (or $(F^\bullet, W(\sigma)_\bullet)$) an arbitrary point of $\tilde{B}(\sigma)$. We shall write $I_{(0)}^{p,q}(\mathfrak{m}_{\mathbb{C}})$ for the Deligne subspaces corresponding to $\tilde{P}_{(0)}$, and $\Lambda_{(0)}^{-1,-1}(\mathfrak{m}_{\mathbb{C}})$ for $\bigoplus_{p<0, q<0} I_{(0)}^{p,q}(\mathfrak{m}_{\mathbb{C}})$. Let n denote the level of the Hodge structures parametrized by D_M .

By Proposition 4.3, together with the faithfulness of the action of $M_N(\mathbb{R})$ on Y_0 , we obtain:

$$\tilde{B}^{\mathbb{R}}(\sigma) = M_{B(\sigma)}(\mathbb{R}) \cdot \tilde{\varphi}_0 = \left(M_\sigma(\mathbb{R}) \rtimes G_{B(\sigma)}(\mathbb{R}) \right) / K_{\tilde{\varphi}_0}$$

where $K_{\tilde{\varphi}_0} := \left(G_{B(\sigma)}(\mathbb{R}) \right)^{\tilde{\varphi}_0}$; and

$$\begin{aligned} \tilde{B}(\sigma) &= \left(M_\sigma(\mathbb{C}) \rtimes G_{B(\sigma)}(\mathbb{R}) \right) \cdot (F_0^\bullet, W(\sigma)_\bullet) \\ &= \left(M_\sigma(\mathbb{C}) \rtimes G_{B(\sigma)}(\mathbb{R}) \right) / K_{F_0^\bullet} \end{aligned}$$

where $K_{F_0^\bullet} = H_{F_0^\bullet} \rtimes K_{\tilde{\varphi}}$ and $H_{F_0^\bullet} = (M_\sigma(\mathbb{C}))_{F_0^\bullet}$ (stabilizer of F_0^\bullet). (The action on $W(\sigma)_\bullet$ is of course trivial.) Passing to nilpotent orbits, we have

$$B(\sigma) = e^{\langle \sigma \rangle_{\mathbb{C}}} \backslash \tilde{B}(\sigma), \quad B^{\mathbb{R}}(\sigma) = e^{\langle \sigma \rangle_{\mathbb{R}}} \backslash M_{B(\sigma)}(\mathbb{R}) / K_{\tilde{\varphi}_0}.$$

¹⁶These \hat{P}_k (which contain our P_k) are the more conventional primitive spaces.

The Mumford-Tate *domain* of a generic φ_{split} , viz.

$$D(\sigma) := G_{B(\sigma)}(\mathbb{R}) \cdot (\varphi_0)_{\text{split}} \cong G_{B(\sigma)}(\mathbb{R}) / K_{\tilde{\varphi}},$$

is contained in the product of period domains¹⁷ for the (P_k, Q_k) . Sending $\tilde{\varphi}$ resp. $e^{(\sigma)c}\tilde{\varphi}$ to φ_{split} gives bundles

$$(7.1) \quad \begin{array}{ccc} \tilde{B}(\sigma) & \longrightarrow & B(\sigma) \\ & \searrow \tilde{\rho}_\sigma & \swarrow \rho_\sigma \\ & D(\sigma) & \end{array}$$

with fibers through $\tilde{\varphi}$

$$\tilde{\mathfrak{F}}_{\tilde{\varphi}} = M_\sigma(\mathbb{C}) \cdot (F^\bullet, W(\sigma)_\bullet), \quad \mathfrak{F}_{\tilde{\varphi}} = M_\sigma(\mathbb{C}) \cdot e^{(\sigma)c} F^\bullet.$$

To refine (7.1), recall that we have a filtration of the Lie group

$$\mathcal{M} := M_\sigma(\mathbb{C}) \rtimes G_{B(\sigma)}(\mathbb{R})$$

by normal subgroups $W_{-k}\mathcal{M} = W_{-k}M_\sigma(\mathbb{C})$ ($k \geq 1$). Writing $\mathcal{K} := K_{F_0^\bullet}$, set

$$\tilde{B}(\sigma)_{(k)} := \left(\frac{\mathcal{M}}{W_{-(k+1)}\mathcal{M}} \right) / \left(\frac{\mathcal{K}}{W_{-(k+1)}\mathcal{K}} \right)$$

$$B(\sigma)_{(k)} := \begin{cases} \tilde{B}(\sigma)_{(1)} & , \quad k = 1 \\ e^{(\sigma)c} \backslash \tilde{B}(\sigma)_{(k)} & , \quad k > 1 \end{cases}.$$

The fibration $\tilde{\rho}_\sigma$ then factors

$$\tilde{B}(\sigma) \twoheadrightarrow \cdots \twoheadrightarrow \tilde{B}(\sigma)_{(k)} \xrightarrow{\tilde{\rho}_\sigma^{(k)}} \tilde{B}(\sigma)_{(k-1)} \twoheadrightarrow \cdots \twoheadrightarrow \tilde{B}(\sigma)_{(1)} \xrightarrow{\tilde{\rho}_\sigma^{(1)}} D(\sigma),$$

with fibers (of $\tilde{\rho}_\sigma^{(k)}$, through $\tilde{\varphi}$)

$$(7.2) \quad \tilde{\mathfrak{F}}_{\tilde{\varphi}}^{(k)} = \frac{Gr_{-k}^W \mathcal{M}}{Gr_{-k}^W \mathcal{K}} = \frac{Gr_{-k}^W M_\sigma(\mathbb{C})}{Gr_{-k}^W H_{F^\bullet}} \xleftarrow{\cong} \frac{Gr_{-k}^{W(\sigma)}(\mathfrak{m}_{\sigma, \mathbb{C}})}{F^0 Gr_{-k}^{W(\sigma)}(\mathfrak{m}_{\sigma, \mathbb{C}})}.$$

A key point here is that since the commutator of $W_{-i}\mathcal{M}$ and $W_{-j}\mathcal{M}$ lies in $W_{-(i+j)}\mathcal{M}$, the $Gr_{-k}^W \mathcal{M}$ are abelian ($k \geq 1$) with the resulting additive structure on $\tilde{\mathfrak{F}}_{\tilde{\varphi}}^{(k)}$ visible on the right-hand side of (7.2). The

¹⁷More precisely, for a generic φ_{split} , $D(\sigma)$ surjects onto the MT domains of all the irreducible pure Hodge structures into which φ_{split} decomposes.

version with tildes removed only differs from (7.2) at $k = 2$:

$$(7.3) \quad \mathfrak{F}_{\tilde{\varphi}}^{(2)} = \frac{Gr_{-k}^{W(\sigma)}(\mathbf{m}_{\sigma, \mathbb{C}})}{F^0\{Gr_{-k}^{W(\sigma)}(\mathbf{m}_{\sigma, \mathbb{C}})\} + \langle \sigma \rangle_{\mathbb{C}}}.$$

In (7.2) and (7.3), $Gr_{-k}^{W(\sigma)}(\mathbf{m}_{\sigma})$ is a Hodge structure of weight $-k$ and its F^0 depends on $\varphi_{\text{split}} = \tilde{\rho}_{\sigma}(\tilde{\varphi}) \in D(\sigma)$.

Remark 7.1. To make all of this more concrete in the rank one case, recall that

$$(7.4) \quad \mathbf{m} = \bigoplus_{\ell=0}^n \left(\bigoplus_{j=0}^{\ell} N^j \tilde{P}_{\ell} \right).$$

We may view (for $k \geq 1$)

$$(7.5) \quad \tilde{\mathfrak{F}}_{\tilde{\varphi}}^{(k)} \subseteq \bigoplus_{\ell=0}^n \bigoplus_{j=0}^{\ell} \frac{\text{Hom}_{\mathbb{C}}(P_{\ell}, N^j P_{\ell-k+2j})}{F^0 \text{Hom}_{\mathbb{C}}(P_{\ell}, N^j P_{\ell-k+2j})}$$

as being cut out by (among other equations) polarization conditions. (For higher rank, $\mathbf{m} = \bigoplus_{\ell=0}^n \left\{ \bigoplus_{j=0}^{\ell} \left(\sum_{i_1 \leq \dots \leq i_j} N_{i_1} \cdots N_{i_j} P_{\ell} \right) \right\}$ leads to a slightly more complicated formula.) For the fiber through $\tilde{\varphi}_0$, here is how this works: thinking of $g \in W_{-k}\mathcal{M}$ as an automorphism of (7.4), we write $g = \text{id.} + \tilde{g} + \tilde{g}'$, where

$$\tilde{g} \in \bigoplus_{\ell} \text{Hom}_{\mathbb{C}}(E(\ell), E(\ell - k))$$

$$\tilde{g}' \in \bigoplus_{\ell} \text{Hom}_{\mathbb{C}}(E(\ell), W(N)_{\ell-k-1} \mathbf{m})$$

and \tilde{g} is determined by its “components”

$$\tilde{g}_{(\ell, j)} \in \text{Hom}_{\mathbb{C}}(P_{\ell}, N^j P_{\ell-k+2j}).$$

Given $\alpha \in \tilde{P}_{\ell}$, $\beta \in \tilde{P}_{\ell-k+2j}$, and $q \in \mathbb{Z}_{\geq 0}$ we have

$$\begin{aligned} B(\alpha, N^q \beta) &= B(g\alpha, gN^q \beta) = B(g\alpha, N^q g\beta) \\ &= \{B(\alpha, N^q \beta) + B(\tilde{g}\alpha, N^q \tilde{g}\beta)\} + B(\tilde{g}\alpha, N^q \beta) + B(\alpha, N^q \tilde{g}\beta) \\ &\quad + \{\text{terms involving } \tilde{g}'\}. \end{aligned}$$

Choosing $q = \ell - k + j$ forces $B(\alpha, N^q\beta)$ and both bracketed terms to be zero, so that

$$B(\tilde{g}\alpha, N^q\beta) + B(\alpha, N^q\tilde{g}\beta) = 0.$$

Since \mathfrak{m} is N -polarized (and B pairs $E(k)$ with $E(-k)$), we conclude that $\tilde{g}_{(\ell,j)}$ and $\tilde{g}_{(\ell-k+2j,k-j)}$ determine one another provided the subscripts are distinct. For k even and $j = \frac{k}{2}$ they coincide, and B directly imposes conditions on $\tilde{g}_{(\ell,\frac{k}{2})}$:

$$\begin{aligned} B(\tilde{g}b, N^{\ell-\frac{k}{2}}b') &= -B(b, N^{\ell-\frac{k}{2}}\tilde{g}b') = (-1)^{\ell-\frac{k}{2}+1}B(N^{\ell-\frac{k}{2}}b, \tilde{g}b') \\ &= (-1)^{\ell-\frac{k}{2}+1}B(\tilde{g}b', N^{\ell-\frac{k}{2}}b). \end{aligned}$$

The result is a symmetry (or skew-symmetry, depending on $k - \frac{\ell}{2} + 1$) condition on [the matrix entries of] $\tilde{g}_{(\ell,\frac{k}{2})} \in \text{Hom}_{\mathbb{C}}(P_{\ell}, N^{\frac{k}{2}}P_{\ell})$.

In our situation all these constraints (and others, including those coming from the Lie bracket's status as a Hodge tensor) are implicit in, and computed by, the formula on the right-hand side of (7.2). However, a computation like that above can be valuable for finding dimensions of boundary components of ordinary period domains without entering into Hodge structures on Lie algebras.

We are now ready to consider the (left) quotient by Γ_{σ} in the context of the iterated fibration above.

Lemma 7.2. $\Gamma_{\sigma} \leq M_{B(\sigma)}(\mathbb{Z})$.

Proof. As a subgroup of Γ , Γ_{σ} is integral, unimodular, and neat. Since Γ_{σ} preserves σ and its faces, it acts on $\langle\sigma\rangle$ through a finite group, which by neatness must be trivial. So Γ_{σ} fixes σ (hence N and all P_k) and belongs to $Z(\sigma)(\mathbb{Z})$.

Now the polarizing form B is preserved by the action on \mathfrak{m} of $M(\mathbb{Z})$, *a fortiori* by that of Γ_{σ} . So Γ_{σ} preserves (P_k, Q_k) for each k , and thus acts through the (finite) integer points of an orthogonal group on the spaces of Hodge tensors in each $P_k^{\otimes a} \otimes \check{P}_k^{\otimes b}$. By neatness, these actions are also trivial,¹⁸ and the conclusion follows. \square

¹⁸Lest the reader doubt the assertion that Γ can be chosen neat and of finite index, we remark that by Chevalley's theorem $Gr_0^W M_{B(\sigma)}$ is "cut out" by finitely many

Clearly Γ_σ preserves $W(\sigma)_\bullet$ and commutes with $e^{(\sigma)c}$. We therefore have a fibration tower

$$\overline{B(\sigma)} \twoheadrightarrow \cdots \twoheadrightarrow \overline{B(\sigma)}_{(k)} \xrightarrow{\bar{\rho}_\sigma^{(k)}} \overline{B(\sigma)}_{(k-1)} \twoheadrightarrow \cdots \twoheadrightarrow \overline{B(\sigma)}_{(1)} \xrightarrow{\bar{\rho}_\sigma^{(1)}} \overline{D(\sigma)}$$

with fibers $\tilde{\mathfrak{F}}_\varphi^{(k)}$, where:

- $\overline{D(\sigma)} := Gr_0^W \Gamma_\sigma \backslash D(\sigma)$ is a standard Mumford-Tate domain quotient (i.e. by a neat subgroup of $G_{B(\sigma)}(\mathbb{Z})$)
- $\overline{B(\sigma)}_{(k)} := \left(\frac{\Gamma_\sigma}{W_{-(k+1)} \Gamma_\sigma} \right) \backslash B(\sigma)_{(k)}$
- $\tilde{\mathfrak{F}}_\varphi^{(k)} := Gr_{-k}^W \Gamma_\sigma \backslash \tilde{\mathfrak{F}}_\varphi^{(k)}$ is isogenous to the quotient by $Gr_{-k}^{W(\sigma)} \mathfrak{m}_{\sigma, \mathbb{Z}}$, i.e. to a (generalized if $k > 1$) intermediate Jacobian.

In particular, the $\tilde{\mathfrak{F}}_\varphi^{(k)}$ are complex tori for $k = 1$ and complex semi-tori for $k > 1$, with complex structure depending on $[\varphi_{\text{split}}] \in \overline{D(\sigma)}$.

Remark 7.3. (i) In order that the $Gr_0^W \Gamma_\sigma$ not act on the fibers, we need to know that it acts on $D(\sigma)$ without fixed points. While this follows from the fact (cf. Theorem 6.1) that $\Gamma \backslash D_{M, \Sigma}$ is a log-manifold, a direct argument can be given as follows. If $\varphi_{\text{split}} \in D(\sigma)$ is fixed by $\bar{\gamma} \in Gr_0^W \Gamma_\sigma$, then its \mathbb{Q} -closure \mathcal{M} (as a morphism) commutes with $\bar{\gamma}$. The orbit $\mathcal{M}(\mathbb{R}) \cdot \varphi_{\text{split}}$ is then a MT subdomain of $D(\sigma)$ fixed pointwise by $\bar{\gamma}$. Since every MT domain for Hodge structures contains a CM point, $\bar{\gamma}$ fixes a CM Hodge structure $\hat{\varphi}_{\text{split}}$ and thus its irreducible pure polarized CM HS components. These are all of the form described in [GGK1, sec. V], constructed from a generalized CM type (K, Θ) . The integral points of their MT groups are contained in the elements of \mathcal{O}_K^* with modulus 1 under all complex embeddings. By a theorem of Kronecker, these are roots of unity. The neatness assumption on Γ (hence on $Gr_0^W \Gamma_\sigma$) now implies that $\bar{\gamma}$ acts trivially on $\oplus_\ell Gr_\ell^{W(\sigma)} \mathfrak{m}$ and so is itself trivial.

(ii) That the $Gr_{-k}^W \Gamma_\sigma$ do not have fixed points on $\tilde{\mathfrak{F}}_\varphi^{(k)}$ is a consequence of Lemma 7.2.

We can immediately characterize an important special case.

Hodge tensors. So once triviality is checked for finitely many a and b , it follows for the rest.

Proposition 7.4. (i) If $\mathfrak{m}_\sigma = \langle \sigma \rangle$ and $\mathfrak{g}_{B(\sigma)} \subset I_0^{(-1,1)} + I_0^{(0,0)} + I_0^{(1,-1)}$, then $\overline{B(\sigma)}$ is an irreducible component of a Shimura variety.

(ii) If $W(\sigma)_{-2}\mathfrak{m}_\sigma = \langle \sigma \rangle$, $Gr_{-1}^{W(\sigma)}\mathfrak{m}_\sigma$ is of type $(-1, 0) + (0, -1)$, $\mathfrak{g}_{B(\sigma)}$ is of type $(-1, 1) + (0, 0) + (1, -1)$, and the map $\mathfrak{g}_{B(\sigma)} \rightarrow \text{End}(Gr_{-1}^{W(\sigma)}\mathfrak{m}_\sigma)$ is injective, then $\overline{B(\sigma)}$ is the canonical abelian fibration over (an irreducible component of) a Shimura variety of Hodge type.¹⁹

Proof. That $\mathfrak{g}_{B(\sigma)}$ be of type $(-1, 1) + (0, 0) + (1, -1)$ is the criterion for $D(\sigma)$ to be a Hermitian symmetric space (cf. [Ke, sec. 1.D] or [Mi]), and $Gr_0^W\Gamma_\sigma \subset G_{B(\sigma)}(\mathbb{R})^+$ (being of finite index in $G_{B(\sigma)}(\mathbb{Z})$) is of congruence type. Otherwise, this follows from the theory above. \square

In either case, $B(\sigma)$ is a quasi-projective algebraic variety with a model over $\overline{\mathbb{Q}}$. [Ke, sec. 5.B], [Mi, Ch. 12-14]

Remark 7.5. In order to check the conditions of Proposition 7.4 and later results, it may be helpful to note that

$$W_{-k}\mathfrak{m}_\sigma = (\cap_j \ker(\text{ad}N_j)) \cap \text{im} \left\{ (\text{ad}N)^k \right\}.$$

Now it is natural to expect that the constraints required to make $\overline{B(\sigma)}$ a CM abelian variety are even more stringent than those in Proposition 7.4. As we shall see, this is not the case. We first dig a bit further into the boundary component structure.

Lemma 7.6. *The following are equivalent:*

- (i) the base $\overline{D(\sigma)}$ is a point;
- (ii) the $Gr_{-k}^{W(\sigma)}\mathfrak{z}(\sigma)$ (and thus $P_k, Gr_k^{W(\sigma)}\mathfrak{m}$) are polarized CM Hodge structures, constant in $\tilde{\varphi} \in \tilde{B}(\sigma)$;
- (iii) $G_{B(\sigma)}$ [resp. $\mathfrak{g}_{B(\sigma)}$] is abelian;
- (iv) in the decomposition of G_σ [resp. \mathfrak{g}_σ] into \mathbb{Q} -simple and abelian factors, the projection of $(\varphi_0)_{\text{split}}$ [resp.²⁰ ϕ] onto the simple factors is trivial.

Proof. For (i) \iff (ii) \iff (iii), see [GGK1, Ch. V]. (iii) \implies (iv) is obvious, and is essentially Prop. (IV.A.9) in [GGK1]. \square

¹⁹cf. [Mi, Def. 7.1] for the definition of the canonical abelian fibration.

²⁰Here $\phi \in \mathfrak{g}_{\sigma, \mathbb{R}}$ is the ‘‘tangent’’ to $(\phi_0)_{\text{split}}$ is in §§3 – 4.

Recall the notation $\Lambda^{-1,-1}(\mathbf{m}_{\mathbb{C}}) := \bigoplus_{p < 0, q < 0} \mathbf{m}^{p,q}$.

Lemma 7.7. *The following are equivalent:*

- (i) the fibers $\bar{\mathfrak{F}}_{\bar{\varphi}}$ are compact;
- (ii) $B^{\mathbb{R}}(\sigma) = B(\sigma)$;
- (iii) $\Lambda^{-1,-1}(\mathbf{m}_{\mathbb{C}}) = \langle \sigma \rangle_{\mathbb{C}}$.

Proof. (i) \Leftrightarrow (ii): The point is that $\overline{B^{\mathbb{R}}(\sigma)}$ also admits an iterated fibration, with compact $\bar{\mathfrak{F}}_{\bar{\varphi}}^{(k),\mathbb{R}} \simeq \frac{Gr_{-k}^{W(\sigma)}((\mathbf{m}_{\sigma}/\langle \sigma \rangle)_{\mathbb{R}})}{Gr_{-k}^{W(\sigma)}((\mathbf{m}_{\sigma}/\langle \sigma \rangle)_{\mathbb{Z}})}$. Anything “larger” is not compact.

(ii) \Leftrightarrow (iii): This is because $\bar{\mathfrak{F}}_{\bar{\varphi}}^{(k),\mathbb{R}} = \bar{\mathfrak{F}}_{\bar{\varphi}}^{(k)}$ if and only if

$$Gr_{-k}^{W(\sigma)}((\mathbf{m}_{\sigma}/\langle \sigma \rangle)_{\mathbb{R}}) = Gr_{-k}^{W(\sigma)}((\mathbf{m}_{\sigma}/\langle \sigma \rangle)_{\mathbb{C}}) / F^0.$$

□

Remark 7.8. Under the equivalent conditions of Lemma 7.7, the fibers are complex tori if, in addition, Γ_{σ} is abelian. This is ensured by assuming that the weight filtration on $\mathbf{m}_{\sigma}/\langle \sigma \rangle$ is *short*: for every k , if $Gr_{-k}^{W(\sigma)}(\mathbf{m}_{\sigma}/\langle \sigma \rangle) \neq \{0\}$, then $W(\sigma)_{-2k}(\mathbf{m}_{\sigma}/\langle \sigma \rangle) = \{0\}$.

Now assume $\overline{D(\sigma)}$ is a point, with single fiber $\bar{\mathfrak{F}} (= \overline{B(\sigma)})$ a complex torus. Writing V_k for the weight $(-k)$ Hodge structure $Gr_{-k}^{W(\sigma)}(\mathbf{m}_{\sigma}/\langle \sigma \rangle)$, we know that $\bar{\mathfrak{F}}$ is isogenous to

$$\times_{k>0} \frac{V_{k,\mathbb{C}}}{F^0(V_{k,\mathbb{C}}) + V_{k,\mathbb{Z}}} =: \times_{k>0} J(V_k).$$

The V_k are polarizable CM Hodge structures, and therefore each have a decomposition of the form

$$(7.6) \quad \bigoplus_i \left(V_{(K_i, \Theta_i)}^{-k} \right)^{\oplus m_i}$$

where the $\{K_i\}$ are CM fields and Θ_i $(-k)$ -orientations. Referring to [GGK1, Ch. V] for a fuller discussion, these are partitions of the set of embeddings $\text{Hom}(K_i, \mathbb{C})$ into (p, q) -subsets ($p + q = -k$) in a manner consistent with complex conjugation. This induces a decomposition of $K_i \otimes_{\mathbb{Q}} \mathbb{C}$ into (p, q) -subspaces, putting a (necessarily polarizable, weight $(-k)$) HS on the \mathbb{Q} -vector space K_i .

By Lemma 7.7(iii), in our present case either (p, q) or its “conjugate” (q, p) is always in $\mathbb{Z}_{\geq 0} \times \mathbb{Z}$. For each term in (7.6), define a CM-type Θ'_i by replacing the pairs in $\mathbb{Z}_{\geq 0} \times \mathbb{Z}$ by $(0, -1)$ and their “conjugates” by $(-1, 0)$. The resulting weight (-1) HS $V_{(K_i, \Theta'_i)}^{-1}$ is well-known to be polarizable, with Jacobian a CM abelian variety. So while the HS

$$V'_k := \bigoplus_i \left(V_{(K_i, \Theta'_i)}^{-1} \right)^{\oplus m_i}$$

is of type $(-1, 0) + (0, -1)$, it shares the same underlying integral structure and F^0 as V_k , and we conclude that

$$J(V_k) \cong J(V'_k).$$

This proves that $J(V_k)$ hence $\bar{\mathfrak{F}}$ is a CM abelian variety.²¹

Here, then, is the strongest result that can be stated without explicit knowledge of ϕ :

Theorem 7.9. *Assume $\overline{B(\sigma)} \subset (\Gamma \backslash D_{M, \Sigma})$ is nonempty, and that*

- (a) $W(\sigma) \bullet (\mathfrak{m}_\sigma / \langle \sigma \rangle)$ is short,
- (b) $\Lambda^{-1, -1}(\mathfrak{m}_\sigma / \langle \sigma \rangle) = \{0\}$, and
- (c) $\mathfrak{z}(\sigma) / \mathfrak{m}_\sigma$ is abelian.

Then $\overline{B(\sigma)}$ is a CM abelian variety (and has a model over $\bar{\mathbb{Q}}$).

A useful shortcut is provided by the

Corollary 7.10. *$\overline{B(\sigma)} \neq \emptyset$ is a CM abelian variety if $W(\sigma)_{-2} \mathfrak{m}_\sigma = \langle \sigma \rangle$ and $\mathfrak{z}(\sigma) / \mathfrak{m}_\sigma$ is abelian.*

In the rank one case, using Remark 6.4, this yields the most practical criterion:

Corollary 7.11. *Suppose D_M parametrizes (weight zero, B -polarized) HS on \mathfrak{m} of level 2ℓ . A nilpotent $N \in \mathfrak{m}_{\mathbb{Q}}$ produces a nonempty boundary component $B(N)$ iff:*

- $(\text{ad}N)^{2\ell+1} = 0$;
- $N(F^\bullet) \subset F^{\bullet-1}$, for some $F^\bullet \in \check{D}_M$; and
- $(P_k(N), Gr_k^{W(N)} F^\bullet, B(\cdot, (\text{ad}N)^k \cdot))$ is a polarized HS for each k .

²¹Note that we consider a point to be a zero-dimensional CM abelian variety.

If in addition

- $\ker(\mathrm{ad}N) \cap \mathrm{im}\{(\mathrm{ad}N)^2\} = \langle N \rangle$, and
- $[\ker(\mathrm{ad}N), \ker(\mathrm{ad}N)] \subseteq \mathrm{im}(\mathrm{ad}N)$,

then its quotient $\overline{B(N)}$ is a CM abelian variety, of dimension $\frac{1}{2} \dim \left(\mathrm{Gr}_{-1}^{W(N)} \mathfrak{m} \right)$.

Remark 7.12. For purposes of directly generalizing the arguments in [C3], one requires not only that a boundary component quotient be arithmetic, but to have M of Hermitian type and $(\mathrm{ad}N)^3 = 0$. The reader may have noticed that we ignored the (easy) case when $\overline{B(\sigma)} \cong (\mathbb{C}^*)^m$, although this is obviously defined over $\bar{\mathbb{Q}}$ and we shall encounter such examples in §8. This is because to have a $(\mathbb{C}^*)^m$ boundary component with $(\mathrm{ad}N)^3 = 0$, D_M would have to parametrize HS on \mathfrak{m} of level 2; that is, it would already be classical.

8. EXAMPLES

The authors expect, in a future work, to use Corollary 7.11 (and other results above) to treat systematically the MT domains for simple Lie groups classified in [GGK1, sec. 4]. Here we shall restrict ourselves to a quick analysis of a few examples which motivated this paper. The first is quite classical, and the first three just compute (but very efficiently) the boundary components of some period domains. A slight difference to how the results are stated above, is that we start with a Hodge representation V of M on a smaller vector space than \mathfrak{m} . This can be more convenient for checking that $B(N)$ or $B(\sigma)^{22}$ is nonempty, e.g. by producing an F^\bullet such that $(F^\bullet, W(N)_\bullet)$ is an N -polarized MHS. Having done this, we then content ourselves with $I^{p,q}$ diagrams ($\#$ of dots at the (p, q) -spot = $\dim(I^{p,q})$) for the LMHS on V resp. \mathfrak{m} that are parametrized by each boundary component. These are easy to produce if one knows the $h^{p,q}$'s for \mathfrak{m} (which can be looked up in [GGK1, Ch. IV] for simple Lie algebras) corresponding to those chosen for V , and if one computes the ranks of kernel and image of $(\mathrm{ad}N)^k$ acting on \mathfrak{m}

²²The rank of $B(\sigma)$ ($:= \dim\langle\sigma\rangle$) is one in all but the first example.

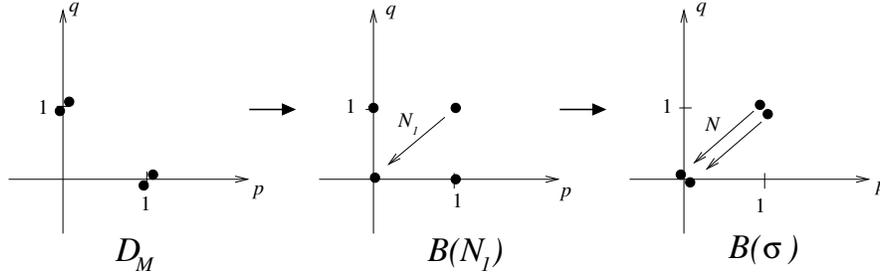
(left to the reader). The structure theory of $M_{B(N)}$ and $\overline{B(N)}$ can then be read off these pictures from the results above.

Example 8.1. $M = Sp_4$, $\dim(V) = 4$, $h^{1,0} = h^{0,1} = 2$,

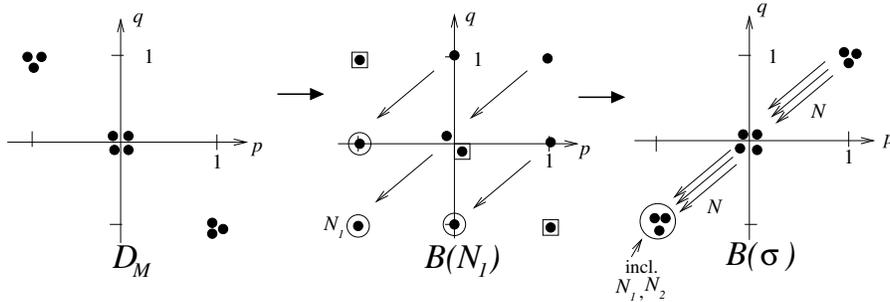
$$Q = \begin{pmatrix} & & 1 \\ -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}, N_1 = \begin{pmatrix} & 0 \\ 1 & \\ & 0 \end{pmatrix}, N_2 = \begin{pmatrix} & 0 \\ 0 & \\ & 1 \end{pmatrix},$$

$$N = N_1 + N_2, \sigma = \mathbb{Q}_{\geq 0}\langle N_1, N_2 \rangle;$$

producing F^\bullet is left to the reader. The standard $I^{p,q}$'s (for V) are then:



while the adjoint $I^{p,q}$'s are depicted in



with generators of \mathfrak{m}_σ circled and generators of $\mathfrak{z}(\sigma)/\mathfrak{m}_\sigma \cong P_0$ boxed. (For $B(\sigma)$, $P_0 = \{0\}$.)

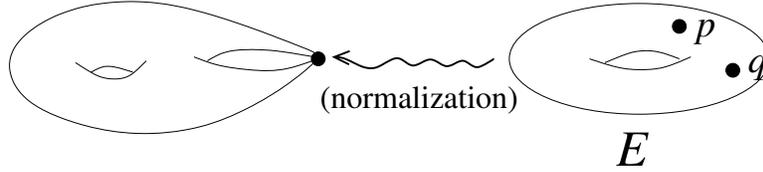
Turning to boundary component structure, for $B(N_1)$ we have

$$Gr_0^W M_{B(N_1)} \cong SL_2$$

over \mathbb{R} because the boxed stuff is \mathfrak{sl}_2 and ϕ cannot be 0 (since the Hodge structures on the P_k are nontrivial); and

$$e^{\langle N_1 \rangle} \backslash W_{-1} M_{B(N_1)} \cong \mathbb{G}_a^{\times 2}.$$

Consequently, $\overline{B(N_1)} \simeq$ a (noncompact) elliptic modular surface. Geometrically, given a degeneration of genus 2 curves

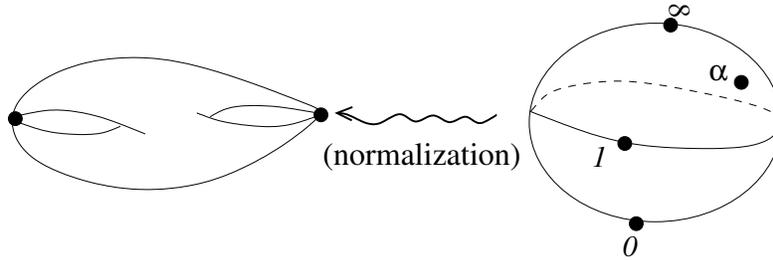


$\overline{B(N_1)}$ records the isomorphism class (with Γ -level structure) of E and $AJ([p] - [q]) \in J(E)$.

For $B(\sigma)$,

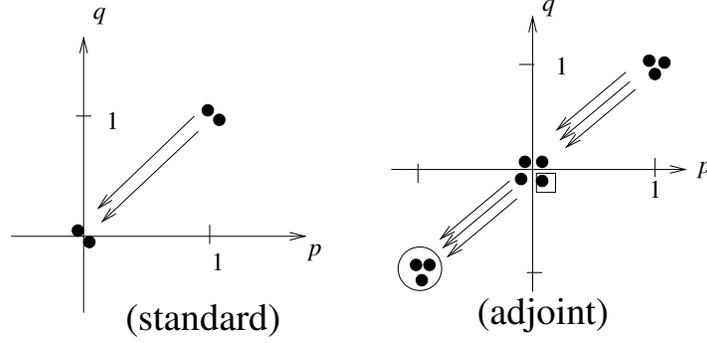
$$\begin{cases} Gr_0^W M_{B(\sigma)} \text{ is trivial} \\ e^{\langle \sigma \rangle} \backslash W_{-1} M_{B(\sigma)} \cong \mathbb{G}_a \end{cases}$$

implies $\overline{B(\sigma)} \cong \mathbb{C}^*$. Given a degeneration of genus 2 curves



the \mathbb{C}^* records α (i.e. the cross-ratio) [CCK, Ca2]. The smooth loci of the Néron n -gons in a minimal smooth compactification of $\overline{B(N_1)}$ can be thought of as unions of $\overline{B(\sigma)}$'s.

We can also consider $B(N)$, which has $I^{p,q}$ diagrams



While $P_0 \neq \{0\}$, ϕ is trivial since the MHS is Hodge-Tate, and so $Gr_0^W M_{B(N)}$ is once again trivial; on the other hand

$$W_{-1}M_{B(N)} \cong \mathbb{G}_a^{\times 3}.$$

So in this example, $M_{B(N)} = M_N \subsetneq Z(N)$, and we have $\overline{B(N)} \cong (\mathbb{C}^*)^{\times 2}$. (In every other example we consider here, it turns out that $M_{B(N)} = Z(N)$ [or $Z(\sigma)$]. This is because all the \mathfrak{g}_N 's which turn up are trivial, 1-dimensional, or [over \mathbb{R}] \mathfrak{sl}_2 .)

Example 8.2. $M = Sp_4$, $\dim(V) = 4$, $h^{3,0} = h^{2,1} = h^{1,2} = h^{0,3} = 1$, with

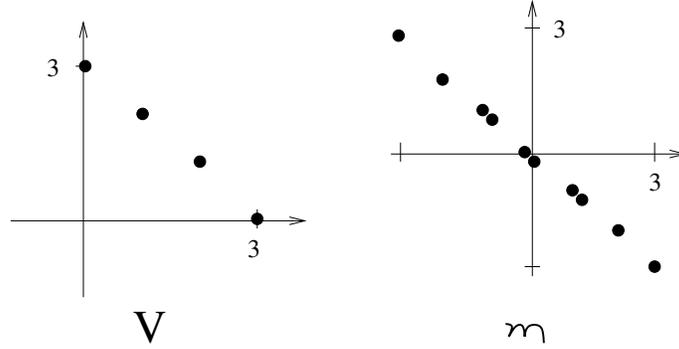
$$Q = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix}.$$

By [GGK4], up to a symplectic change of basis the only possibilities for N are

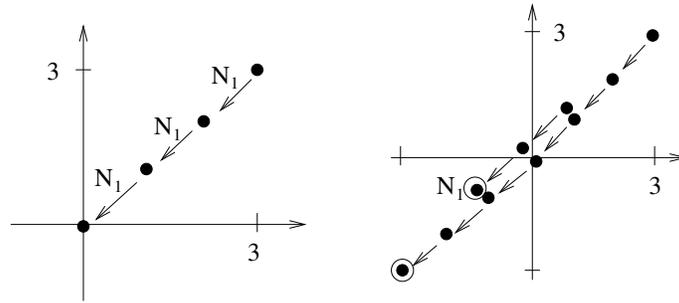
$$N_1 = \begin{pmatrix} a & & & \\ c & b & & \\ d & c & -a & \end{pmatrix}, N_2 = \begin{pmatrix} & & & \\ & & & \\ & & & \\ a & & & \end{pmatrix}, N_3 = \begin{pmatrix} & & & \\ & & & \\ & & & \\ & & & A \end{pmatrix};$$

with ${}^t A = A > 0$

and we refer to [op. cit.] for the F^\bullet 's. For the period domain itself, we have $I^{p,q}$ diagrams

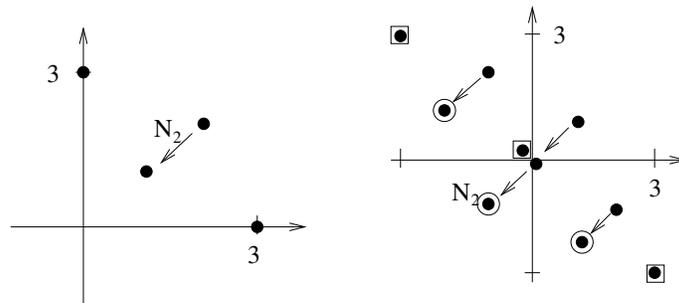


Next, the LMHS in $B(N_1)$ have $I^{p,q}$'s



and $M_{B(N_1)} \cong M_{N_1} \cong \mathbb{G}_a^{\times 2}$, $\overline{B(N_1)} \cong \mathbb{C}^*$.

Turning to $B(N_2)$, one has

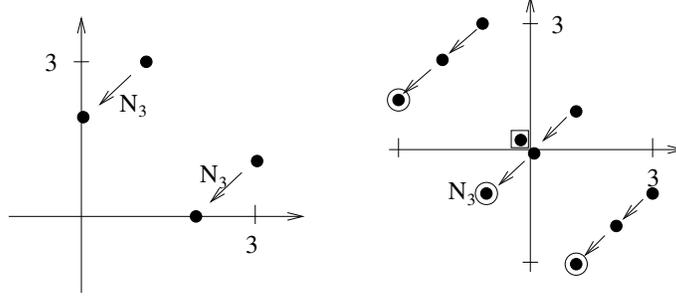


and from

$$\begin{cases} Gr_0^W M_{B(N_2)} \cong SL_2 \text{ (over } \mathbb{R}) \\ e^{\langle N_2 \rangle} \backslash W_{-1} M_{B(N_2)} \cong \mathbb{G}_a^{\times 2} \end{cases}$$

we conclude that $\overline{B(N_2)} \simeq$ a noncompact²³ elliptic modular surface (in spite of the fact that it doesn't fall under Proposition 7.4.

Finally, for $B(N_3)$, the $I^{p,q}$'s are



so

$$\begin{cases} Gr_0^W M_{B(N_3)} \cong \mathbb{G}_m \\ e^{\langle N_2 \rangle} \backslash W_{-1} M_{B(N_3)} \cong \mathbb{G}_a^{\times 3} \end{cases}$$

from which (by Theorem 7.9) $\overline{B(N_3)}$ is a CM elliptic curve.

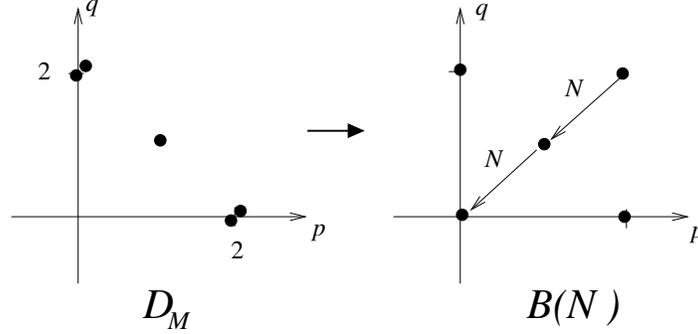
Example 8.3. $M = SO(4, 1)$, $\dim(V) = 5$, $h^{2,0} = h^{0,2} = 2$, $h^{1,1} = 1$,

$$Q = \left(\begin{array}{c|ccc} 1 & & & \\ \hline & -1 & & \\ & & -1 & \\ & & & -1 \\ & & & & -1 \end{array} \right), \quad N = \left(\begin{array}{c|cc} 1 & & \\ \hline 1 & 1 & \\ & -1 & \\ & & 0 \\ & & & 0 \end{array} \right).$$

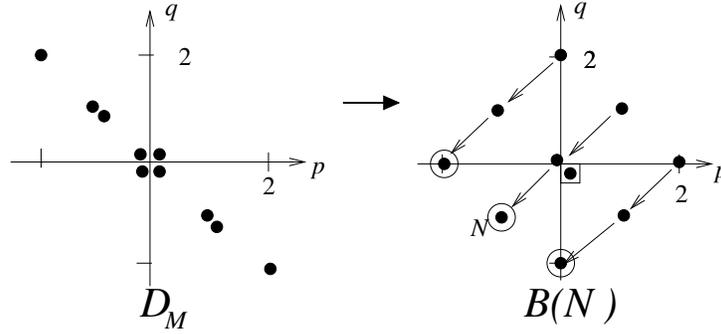
Writing u, v_1, v_2, v_3, v_4 for the basis of V , we have $W_0 V = \langle u - v_2 \rangle$ and $W_2 V = \langle v_1, v_3, v_4, u - v_2 \rangle$. For an $F^\bullet \in \check{D}_M$ for which $e^{\langle N \rangle} F^\bullet$ is a nilpotent orbit, we can take $F^2 = \langle u + v_2, v_3 + iv_4 \rangle$ and $F^1 = \langle v_1, u + v_2, v_3 + iv_4 \rangle$, which satisfies $Q(F^2, F^1) = 0$ and polarizes Gr_4^W and $Gr_2^W \ker N$.

²³Of course, it admits an algebraic compactification, but this time the added points do not correspond to LMHS (essentially by transversality considerations).

The $I^{p,q}$'s for the standard representation are



From the adjoint representation we instead obtain



As far as $I^{p,q}$'s go, $B(N)$ is essentially the only type of boundary component for the period domain D_M . We have

$$\begin{cases} Gr_0^W M_{B(N)} \cong \mathbb{G}_m \\ W_{-1} M_{B(N)} \cong M_N \cong \mathbb{G}_a^{\times 3} \end{cases}$$

and hence that $\overline{B(N)}$ is a CM elliptic curve.

Example 8.4. (Carayol's example [C3])

This takes a bit more setting up, and (unlike the other examples) involves distinct $'G$ and $M = 'G^{\text{ad}}$ — in this case

$$\begin{cases} 'G(\mathbb{R}) \cong U(2, 1) \\ M(\mathbb{R}) \cong SU(2, 1)^{\text{ad}} \end{cases} .$$

Let V be a 6-dimensional vector space, $Q : V \times V \rightarrow \mathbb{Q}$ an alternating nondegenerate bilinear form, and d be a square-free integer. We fix a

ring homomorphism

$$\mu : \mathbb{F} := \mathbb{Q}(\sqrt{-d}) \hookrightarrow \text{End}_{\mathbb{Q}}(V)$$

such that in the decomposition $V_{\mathbb{F}} = V_+ \oplus V_-$ into eigenspaces for the conjugate complex embeddings of \mathbb{F} , V_+ is Q -isotropic. Consider a Q -polarized Hodge structure²⁴

$$' \varphi : \mathbb{U} \rightarrow \text{Aut}(V, Q, \mu)$$

with $h_+^{3,0} = h_+^{2,1} = h_+^{1,2} = 1$ ($\implies h^{3,0} = 1$ and $h^{2,1} = 2$). Finally, set $\mathbf{h}(v, w) := -\sqrt{-d}Q(v, \bar{w})$ and

$$'G := \text{Aut}(V, Q, \mu) \cong \text{Aut}(V_+, \mathbf{h}).$$

Then, as in §1,

$$'G(\mathbb{R}).' \varphi \cong M(\mathbb{R}).(\text{Ad} \circ ' \varphi) = D_M$$

and $'G$ is the Mumford-Tate group of a generic Hodge structure (on V) in the left-hand orbit.

More precisely, we may choose Q and an \mathbb{F} -basis $\{\gamma_1, \gamma_2, \gamma_3 \mid \subseteq V_+$ so that

$$[\mathbf{h}]_{\gamma} = \begin{pmatrix} & & -1 \\ & 1 & \\ -1 & & \end{pmatrix},$$

and then $\mathfrak{m}_{\mathbb{R}}$ identifies with elements of $\text{End}(V_{+, \mathbb{C}})$ of the form

$$\begin{pmatrix} A & B & C \\ D & E & \bar{B} \\ G & \bar{D} & -\bar{A} \end{pmatrix}, \begin{cases} C, E, G \in i\mathbb{R} \\ A, B, D \in \mathbb{C} \end{cases}.$$

(We can define a $' \varphi$ by $F^3 V_+ = \langle \gamma_2 \rangle$, $F^2 V_+ = \langle \gamma_1 + \gamma_3, \gamma_2 \rangle$; this won't necessarily be the F^{\bullet} giving the nilpotent orbits below.) Up to conjugation by $M(\mathbb{R})$, and under this identification, the only nilpotents in

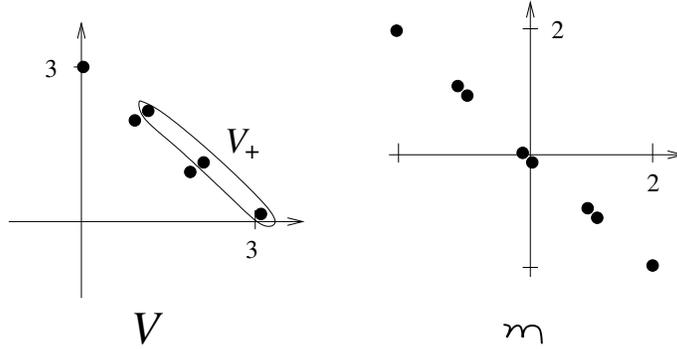
²⁴The notation means in particular that $\mu(\mathbb{F}) \subset \text{End}_{\varphi}(V)$.

\mathfrak{m} giving nilpotent orbits are

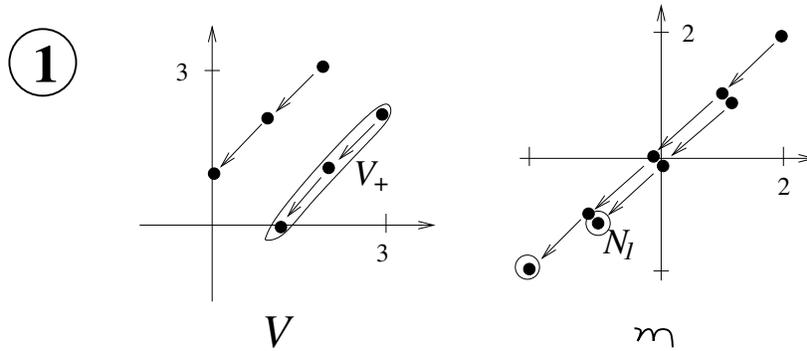
$$N_1 = \begin{pmatrix} 0 & \alpha & ia \\ 0 & 0 & \bar{\alpha} \\ 0 & 0 & 0 \end{pmatrix}, \quad N_2^\pm = \begin{pmatrix} 0 & 0 & \pm ib \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where $\alpha \in \mathbb{F}$, $a \in \mathbb{Q}$, and $b \in \mathbb{Q}_+$. We refer to [C3] for the proof.

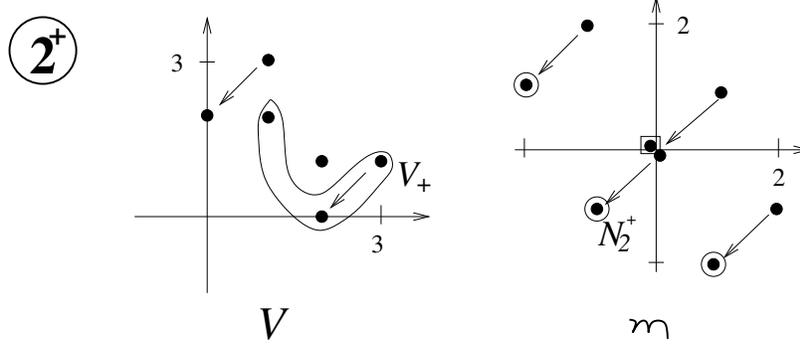
Turning to $I^{p,q}$'s, the generic element of D_M has



For the three types of boundary components, we have



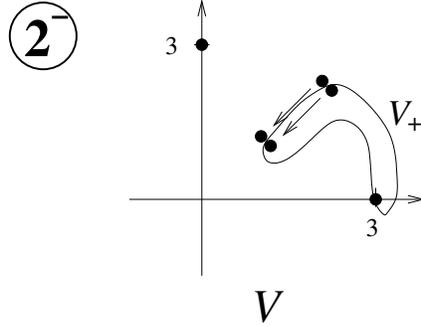
which gives $M_{B(N_1)} \cong M_{N_1} \cong \mathbb{G}_a^{\times 2}$, $\overline{B(N_1)} \cong \mathbb{C}^*$;



which yields

$$\begin{cases} Gr_0^W M_{B(N_2^+)} \cong \mathbb{G}_m \\ e^{(N_2^+)} \setminus W_{-1} M_{B(N_2^+)} \cong \mathbb{G}_a^{\times 2} \end{cases}$$

making $\overline{B(N_2^+)}$ a CM elliptic curve; and



with everything else the same as in the N_2^+ case.

Example 8.5. $M = G_2$, $\dim V = 7$, $h^{2,0} = h^{0,2} = 2$, $h^{1,1} = 3$,

$$Q = \left(\begin{array}{c|c} I_3 & \\ \hline & -I_4 \end{array} \right), \quad N = \left(\begin{array}{cc|cc} & -1 & 1 & & \\ & 0 & & 0 & \\ \hline 1 & & & -1 & 0 \\ 1 & & & -1 & 0 \\ & 0 & & 0 & \\ & -1 & 1 & & \\ & 0 & 0 & & \end{array} \right)$$

(note $N^2 = 0$). Inside $SO(3, 4)$, G_2 is cut out by preserving a certain 3-tensor. In the present $u_1, u_2, u_3, v_1, v_2, v_3, v_4$ basis, \mathfrak{g}_2 consists of matrices of the form

$$\left(\begin{array}{ccc|cccc} 0 & f-a & -b-e & A & B & C & H-F \\ a-f & 0 & d-c & D & E & F & C-G \\ b+e & c-d & 0 & G & H & -A-E & D-B \\ \hline A & D & G & 0 & -a & -b & -d \\ B & E & H & a & 0 & -c & -e \\ C & F & -A-E & b & c & 0 & -f \\ H-F & C-G & D-B & d & e & f & 0 \end{array} \right).$$

Clearly N is of this form.

On V , N induces the weight filtration

$$\left\{ \begin{array}{l} (W_3 = V) \\ W_2 = \langle u_2, v_2, v_4, u_1 + v_3, u_3 + v_1 \rangle \\ W_1 = \langle u_1 + v_3, u_3 + v_1 \rangle \\ (W_0 = \{0\}) \end{array} \right.$$

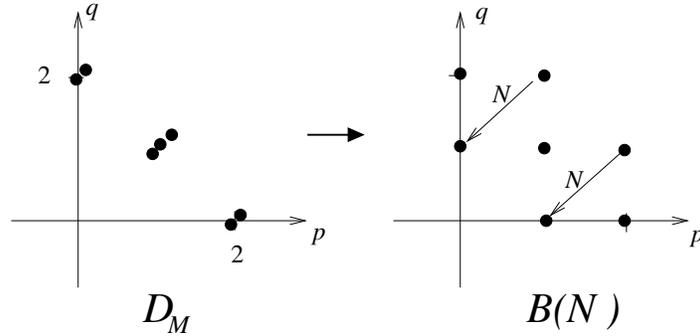
and it is easy to see that

$$\left\{ \begin{array}{l} F^2 := \langle v_1 + iv_3, v_2 - iv_4 \rangle \\ F^1 := \langle u_1, u_2, u_3, v_1 + iv_3, v_2 - iv_4 \rangle \end{array} \right.$$

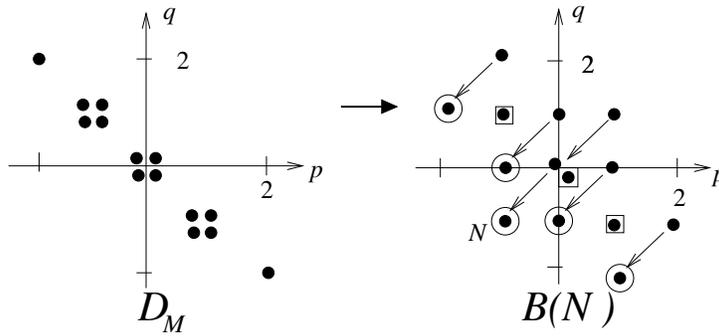
produces an N -polarized MHS and belongs to \check{D}_M . In fact, this F^\bullet defines a HS φ on V polarized by Q , and the differential of φ is a multiple (writing $\Delta = \text{diag}\{1, -1\}$) of

$$\left(\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 0 & \Delta \\ 0 & -\Delta & 0 \end{array} \right)$$

which is clearly in \mathfrak{g}_2 . The $I^{p,q}$ pictures for V are

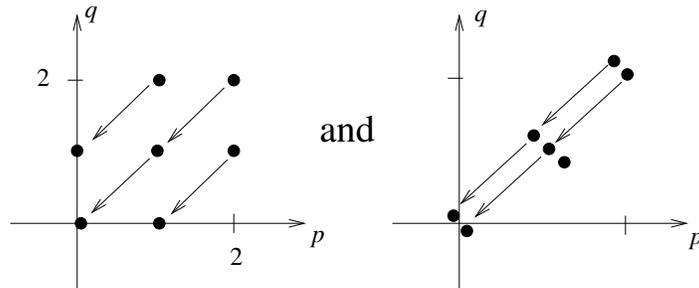


As for the adjoint representation, we have Hodge numbers $h^{-2,2} = 1$, $h^{-1,1} = 4 = h^{0,0}$ and $I^{p,q}$ diagrams



Since ϕ is nontrivial, $Gr_0^W M_{B(N)} \cong SL_2$ (over \mathbb{R}); moreover, $e^{(N)} \setminus W_{-1} M_{B(N)} \cong \mathbb{G}_a^{\times 4}$, and we find that $\overline{B(N)}$ is isomorphic to a family of (compact) complex 2-tori over a modular curve.

As far as $I^{p,q}$ types are concerned, it turns out that there are two further types of rank one boundary components for this G_2 -domain. On V , the corresponding diagrams are



We won't pursue a systematic treatment of this here.

9. A G_2 -VARIATION OF HODGE STRUCTURE

In [GGK1, Ch. IV] it was shown that, up to Tate twists, there were three possible gap-free collections of Hodge numbers for effective rank 7 Hodge structures with Mumford-Tate group G_2 :

- (a) $(2, 3, 2)$ in weight 2 (cf. Example 8.5);
- (b) $(1, 2, 1, 2, 1)$ in weight 4; and
- (c) $(1, 1, 1, 1, 1, 1, 1)$ in weight 6.

For weight 6, the complete list of possibilities is: (c); twists of (a) and (b); and $(2, 0, 0, 3, 0, 0, 2)$.

It was also asked in [*op. cit.*] whether G_2 arises as the Mumford-Tate group of a *motivic* Hodge structure. This is the transcendental question most closely related to Serre's famous problem on motivic Galois groups [Se], and has received less attention than the corresponding one for ℓ -adic monodromy. It turns out that recent work of Dettweiler and Reiter [DR] on Serre's problem produces a family of quasi-projective varieties over $\mathcal{S} := \mathbb{P}^1 \setminus \{0, 1, \infty\}$, the lowest weight part of whose degree-6 cohomology carries a VHS of type (c) and MT group G_2 . By [An, Lemma 4] (also cf. [De3, 7.5]), this is also the MT group of $W_6 H^6$ of a very general fiber.

Consider the family

$$(9.1) \quad \mathcal{X} \begin{array}{c} \xrightarrow{\pi} \\ \xrightarrow{\quad} \mathcal{Y} \xrightarrow{\quad} \mathcal{S} \end{array}$$

of 6-folds defined by

$$Y_s := \left(\mathbb{P}^1 \setminus \{0, 1, \infty\} \right)^{\times 6} \setminus \left\{ \left(\prod_{i=1}^5 (x_i - x_{i+1}) \right) (x_6 - s) = 0 \right\}$$

$$X_s := \left\{ y^2 = s(x_6 - s) \prod_{i=1}^5 (x_i - x_{i+1}) \prod_{i=1,3,5} x_i \prod_{i=1,2,4,6} (x_i - 1) \right\} \subset \mathbb{C}^* \times Y_s$$

with involution

$$\sigma : \mathcal{X} \rightarrow \mathcal{X}$$

sending $y \mapsto -y$. Denoting by $(\cdot)^-$ the $\sigma(-1)$ -eigenspace, the local system

$$\mathbb{V} := \left(Gr_6^W R^6 \pi_! \mathbb{Q} \right)^-$$

(with stalks V_s) induces a monodromy representation

$$\rho : \pi_1(\mathcal{S}_{\mathbb{C}, s_0}^{an}) \rightarrow Aut(V_{s_0}).$$

with associated (geometric) VHS \mathcal{V} and stalks V_s . The *geometric monodromy group* Π of \mathbb{V} is the identity connected component of the (\mathbb{Q} -)Zariski closure of the image of ρ . Write \mathcal{V} for the associated VHS and $M_{\mathcal{V}}$ for its MT group; we recall that this identifies with the MT groups of fibers \mathcal{V}_s outside a countable union of analytic subvarieties. Denoting unipotent Jordan blocks of length n by $J(n)$, we have the

Theorem 9.1. [DR] (a) *The monodromy types of \mathbb{V} about $0, 1, \infty$ are $(-1)^{\oplus 4} \oplus \mathbf{1}^{\oplus 3}$, $J(2)^{\oplus 2} \oplus J(3)$, and $J(7)$, respectively.*

(b) $\Pi = G_2$.²⁵

Corollary 9.2. (i) $M_{\mathcal{V}} = G_2$; and (ii) \mathcal{V} has Hodge type (c).

Proof. By the Theorem of the Fixed Part [Sc], Π is normalized by $M_{\mathcal{V}}$ (cf. [An]), and so $G_2 \trianglelefteq M_{\mathcal{V}} \leq SL(V_s)$. Using the fact that $N_{SL_7}(G_2) = G_2$, (i) follows at once.

For (ii), one can argue in two ways. On the one hand, the presence of the $J(7)$ block means that \mathcal{V} is not isotrivial ($\implies (2, 0, 0, 3, 0, 0, 2)$ is impossible), and that $N^6 \neq 0$, which is impossible for a VHS of level < 6 . Alternatively, one can show that $\frac{dx_1 \wedge \dots \wedge dx_6}{y}$ extends to an anti-invariant holomorphic 6-form on a σ -compatible good compactification of X_s . Noting that $\mathbb{V}^{\vee} \cong (W_6 R^6 \pi_* \mathbb{Q})^-$, this shows that

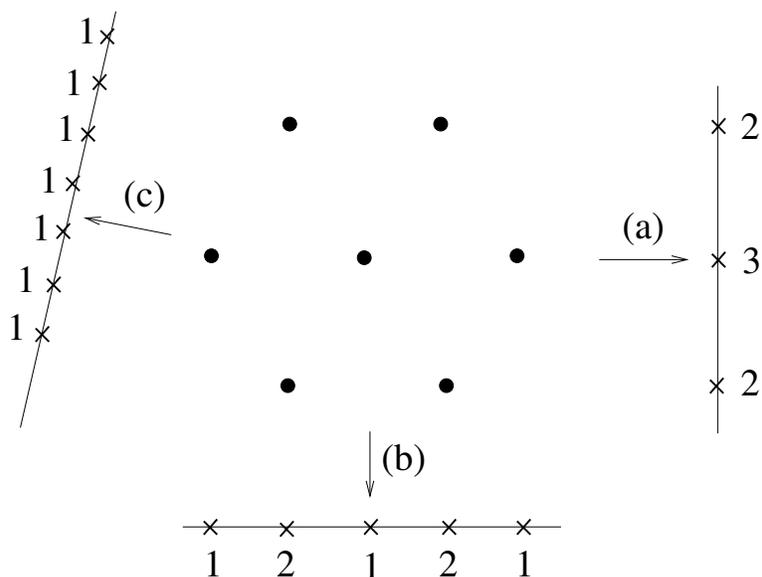
$$\{0\} \neq (\mathcal{V}^{\vee})^{6,0} \cong (\mathcal{V}^{0,6})^{\vee}$$

which at least rules out Hodge types (a) and (b). \square

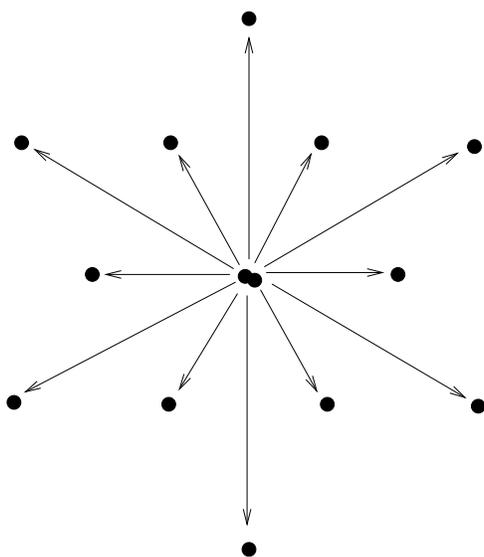
Of course, the Theorem and Corollary are valid for \mathbb{V}^{\vee} and \mathcal{V}^{\vee} as well.

²⁵In fact, they state this for the $\bar{\mathbb{Q}}_l$ -closure of $\text{im}(\rho)$ (cf. the proof of Theorem 3.3.1 in [DR]), but since ρ is defined rationally it makes no difference.

Each of the Hodge types (a)-(c) corresponds to a projection²⁶ on the weight diagram for the standard 7-dimensional irrep of G_2 :



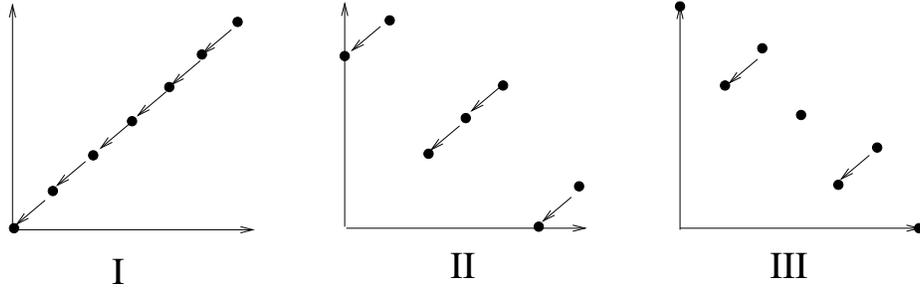
Fix a Hodge type. Reasoning heuristically, from this picture together with the root diagram for \mathfrak{g}_2 ,



one may read off enough information about the possibilities for nilpotent $N \in \mathfrak{g}_2$ satisfying $N(F^\bullet) \subset F^{\bullet-1}$, to classify the $I^{p,q}$ -types of their

²⁶see [GGK1], sections VI.B and IV.F.

associated limit mixed Hodge structures (or boundary components).²⁷
 For type (c), these are



the first two of which match the degenerations of \mathcal{V} (or \mathcal{V}^\vee) at ∞ and 1 (resp.). It would be very interesting to compute the LMHS at these points. In particular, the adjoint $I^{p,q}$ diagram for type I shows that the corresponding $\overline{B(N)}$ is a \mathbb{C}^* which classifies the extension of $\mathbb{Z}(-5)$ by $\mathbb{Z}(0)$ (or dually, $\mathbb{Z}(-6)$ by $\mathbb{Z}(-1)$). One could also try to classify all type (c) VHS over $\mathcal{S}_{\mathbb{C}}^{an}$ with a type I and type II monodromy point, in the spirit of Doran and Morgan [DM]. We put these problems into a broader context in the next section.

Finally, we would be remiss not to point out that the M-T domains for type (b) and (c) both have boundary components isomorphic to a (noncompact) elliptic modular surface. See the forthcoming paper [KP2].

10. RIGIDITY AND BOUNDARY VALUES

To conclude this article, we wish to highlight how MT domains and their boundary components provide a convenient structure in which to think about rigidity and finiteness (Arakelov-type) results for VHS. We fix once and for all a complex algebraic manifold \mathcal{S} and a point $s_0 \in \mathcal{S}$. In view of the result of [De2], that there are only finitely many \mathbb{Q} -local systems $\mathbb{V} \rightarrow \mathcal{S}$ of given rank underlying an integral polarizable variation of Hodge structure, we shall fix \mathbb{V} as well. Writing $V := \mathbb{V}_{s_0}$ ($= \mathbb{Q}$ -vector space), we have as in §9 the monodromy representation ρ with image $\Gamma \subset GL(V)$ and geometric monodromy group $\Pi \subset GL(V)$.

²⁷We shall have more to say about this approach in the future work.

Let $(\mathcal{O}_{\mathcal{S}}, \mathcal{F}^{\bullet}, Q, \mathbb{V})$ be weight n PVHS over \mathcal{S} (abbreviated “ \mathcal{V} ”), with MT group $M_{\mathcal{V}} \leq GL(V)$. As \mathbb{V} is fixed, \mathcal{V} may be recovered from the associated period map²⁸

$$(10.1) \quad \Phi_{\mathcal{V}} : \mathcal{S} \rightarrow \Gamma \backslash D,$$

where D is a (connected component of a) MT domain parametrizing Q -polarized HS on V with the same Hodge numbers \underline{h} as \mathcal{V} and MT group $M_{\mathcal{V}}$. We define the *MT-class* $[\mathcal{V}]$ of \mathcal{V} (or $\Phi_{\mathcal{V}}$) to be the *set* of all (\mathbb{Q}) -VHS on \mathcal{S} with local system \mathbb{V} , polarization Q , Hodge numbers \underline{h} , and MT group $\leq M_{\mathcal{V}} \leq GL(V)$.²⁹

Following [GGK1, sec. III.A], there is an almost-direct-product decomposition $M_{\mathcal{V}} = M_1 \cdots M_{\ell} \cdot A$, with $\{M_i\}_{i=1}^{\ell}$ \mathbb{Q} -simple, A abelian, and $\Pi = M_1 \cdots M_k = M_{var}$ for some $k \leq \ell$, write $M_{fix} = M_{k+1} \cdots M_{\ell} \cdot A$. According to the Structure Theorem for VHS [op.cit.], by passing to a finite cover $\tilde{\mathcal{S}}$ (of \mathcal{S}) we may replace (10.1) by a period map of the form

$$(10.2) \quad \Phi_{\tilde{\mathcal{V}}} : \tilde{\mathcal{S}} \rightarrow \tilde{\Gamma} \backslash D_{var} \times D_{fix} \cong \left(\tilde{\Gamma} \backslash M_{var}(\mathbb{R}) / H_{var} \right) \times M_{fix}(\mathbb{R}) / H_{fix}$$

whose projection to D_{fix} is *constant*. (Here $\tilde{\mathcal{V}}$ is the pullback variation, and $\tilde{\Gamma} \leq \Gamma$.) Hodge-theoretically, (10.2) simply reflects the splitting

$$(10.3) \quad \mathfrak{m}_{\mathcal{V}} = \mathfrak{m}_{var} \oplus \mathfrak{m}_{fix}$$

of the weight-zero variation induced by $\tilde{\mathcal{V}}$. In particular, we note that

$$(10.4) \quad (\mathfrak{m})^{\tilde{\Gamma}} = \mathfrak{m}_{fix}.$$

By a *horizontal local deformation* of $\Phi_{\mathcal{V}}$ in its MT-class, we shall mean an extension of \mathcal{V} (from $\mathcal{S} \times \{0\}$) to a PVHS over $\mathcal{S} \times \Delta$ with MT group $M_{\mathcal{V}}$. (Here Δ is the unit disk, with coordinate t .) Define \mathcal{V} (or $\Phi_{\mathcal{V}}$) to be *MT-rigid* if all such deformations are constant in t .

²⁸For each $s \in \mathcal{S}$, we may consider $\varphi_{\mathcal{V},s} := \Phi_{\mathcal{V}}(s)$ as a morphism $\mathbb{U} \rightarrow Aut(V, Q)$ (defined up to the action of Γ).

²⁹In some ways it is more natural to allow the MT group to be any conjugate $gM_{\mathcal{V}}g^{-1}$ with $g \in Aut(V, Q)^{\Gamma} =: \mathcal{G}_{\mathbb{Q}}$; to get a finiteness result one then has to go modulo the equivalence relation on such VHS induced by the action of $\mathcal{G}_{\mathbb{Q}}$.

Write $\hat{\Phi}$ for the extension of (10.2) induced by such a deformation. Its derivative at $t = 0$ gives a section

$$(10.5) \quad \nu := \left. \frac{\partial \hat{\Phi}}{\partial t} \right|_{\tilde{\mathcal{S}} \times 0} \in \Gamma(\tilde{\mathcal{S}}, (\tilde{\mathcal{V}}^\vee \otimes \tilde{\mathcal{V}})^{-1,1})$$

which, by a curvature argument [Pe, Thm. 3.2(ii)], is *flat*. It is therefore a *constant* section of the fixed part, and determined by its value $\nu(s_0) \in \text{End}(V)^\Pi$. In fact, since our deformation was restricted to the MT domain, using (10.4)

$$(10.6) \quad \nu \equiv \nu(s_0) \in \mathfrak{m}_{fix}^{-1,1}.$$

Here, the Hodge decomposition of \mathfrak{m}_{fix} is constant over \mathcal{S} by the Theorem of the Fixed Part [Sc], and induced by the projection to \mathfrak{m}_{fix} of $\text{Ad}(\varphi_{\mathcal{V},s})$ (for any s , say s_0).

Proposition 10.1. *If $\mathfrak{m}_{fix}^{-1,1} = \{0\}$, then \mathcal{V} is MT-rigid.*

Turning to the cardinality of $[\mathcal{V}]$, instead of complex variations (as in [Pe],[De2]) we shall make use of the Structure Theorem and the following simple uniqueness result.

Lemma 10.2. *Let \mathcal{H} and \mathcal{H}' be two polarizable weight n VHS over \mathcal{S} with the same underlying local system \mathbb{H} . If $\mathbb{H}_\mathbb{C}$ is irreducible then $\mathcal{H} = \mathcal{H}'$.*

Proof. According to the Theorem of the Fixed Part, $\text{End}_\mathcal{S}(\mathbb{H}, \mathbb{H})$ must underlie a subvariation of $\mathcal{H}^\vee \otimes \mathcal{H}'$. The hypothesis implies $\text{End}_\mathcal{S}(\mathbb{H}, \mathbb{H}) \cong \mathbb{Q}\langle \text{id}_\mathbb{H} \rangle$ by Schur's lemma. Therefore $\text{id}_\mathbb{H} \in (\mathcal{H}^\vee \otimes \mathcal{H}')^{(0,0)}$ and is a morphism of VHS. \square

Note that in the Lemma, we need *not* assume that \mathcal{H} and \mathcal{H}' have the same Hodge numbers or polarization.

Example 10.3. If \mathcal{S} is a curve and \mathbb{H} has maximal unipotent monodromy of order equal to its rank about a point of $\bar{\mathcal{S}} \setminus \mathcal{S}$, there is at most one polarizable VHS on \mathbb{H} . This is the case both for the ‘‘Calabi-Yau variations’’ investigated by Doran and Morgan [DM] and for the G_2 -variation in §9.

In the period domain for Q -polarized HS on V with Hodge numbers \underline{h} , the locus $NL(M_{\mathcal{V}})$ of HS with MT group contained in $M_{\mathcal{V}}$ is a finite union of (connected) MT domains. Within each such domain, a HS φ is determined by the weight-zero structure it induces on $\mathfrak{m}_{\mathcal{V}}$, which necessarily splits into $\mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_{\ell} \oplus \mathfrak{a}$ as these summands are closed under the adjoint action of $M_{\mathcal{V}}$. Further, since $\text{Ad} \circ \varphi$ acts trivially on the abelian part \mathfrak{a} , we have $\mathfrak{a}_{\mathbb{C}} = \mathfrak{a}^{0,0}$.

Write $\underline{m} = \underline{m}_1 \oplus \cdots \oplus \underline{m}_{\ell} \oplus \underline{a} \subset \mathbb{V}^{\vee} \otimes \mathbb{V}$ for the \mathbb{Q} -local system arising from $\rho^{\vee} \otimes \rho|_{\mathfrak{m}_{\mathcal{V}}}$. Let $\mathfrak{m}_{\mathcal{V}} = \mathfrak{m}_{\mathcal{V},1} \oplus \cdots \oplus \mathfrak{m}_{\mathcal{V},\ell} \oplus \mathfrak{a}$ and $\mathfrak{m}_{\mathcal{W}} = \mathfrak{m}_{\mathcal{W},1} \oplus \cdots \oplus \mathfrak{m}_{\mathcal{W},\ell} \oplus \mathfrak{a}_{\mathcal{W}}$ denote the VHS on \underline{m} induced by \mathcal{V} and some other variation $\mathcal{W} \in [\mathcal{V}]$ with period mapping $\Phi_{\mathcal{W}} : \mathcal{S} \rightarrow \Gamma \backslash D$ into the same MT domain. Assuming that the $\{M_i\}_{1 \leq i \leq k}$ are \mathbb{C} -simple, the $\{\underline{m}_i\}_{1 \leq i \leq k}$ are absolutely irreducible and Lemma 10.2 immediately implies that $\mathfrak{m}_{\mathcal{V},i} = \mathfrak{m}_{\mathcal{W},i}$ ($1 \leq i \leq k$). Noting that $\mathfrak{a}_{\mathcal{V}} = \mathfrak{a}_{\mathcal{W}}$ (as both are trivial) and $DM := [M, M] = M_1 \cdots M_{\ell}$, we have proved the

Theorem 10.4. *If $\Pi = DM_{\mathcal{V}}$ and the \mathbb{Q} -simple factors M_i of $M_{\mathcal{V}}$ are absolutely simple, then $[\mathcal{V}]$ is finite; more precisely, we have $|[\mathcal{V}]| \leq |\pi_0(NL(M_{\mathcal{V}}))| (< \infty)$.*

Remark 10.5. (i) Hypotheses on \mathcal{V} which ensure $\Pi = DM_{\mathcal{V}}$ are the presence of a CM point [An, sec. 6] or a graded-CM LMHS [KP1, sec. 8].

(ii) An approach to computing $|\pi_0(NL(M_{\mathcal{V}}))|$ is described in [GGK1, Ch. 6].

Beyond their relation to the hypotheses in Example 10.3 and Remark 10.5(i), boundary components provide (together with the choice of $D \in \pi_0(NL(M_{\mathcal{V}}))$) a means of *parametrizing* $[\mathcal{V}]$. It is well-known (cf. [PS, Cor. 12]) that any $\mathcal{W} \in [\mathcal{V}]$ is determined by its restriction to s_0 ; in the same spirit, one has the next

Proposition 10.6. *Suppose \mathcal{S} is a curve, and $x_0 \in \bar{\mathcal{S}} \backslash \mathcal{S}$ a point. Then any $\mathcal{W} \in [\mathcal{V}]$ is determined by its LMHS $\psi_{x_0} \mathcal{W}$ at x_0 .*

Proof. Let $\mathcal{W}, \mathcal{W}' \in [\mathcal{V}]$, and put $\mathcal{E} := \mathcal{W}^{\vee} \otimes \mathcal{W}'$ (with underlying local system $\mathbb{V}^{\vee} \otimes \mathbb{V}$). Since the fixed part $E_{fix} \otimes \mathcal{O}_{\mathcal{S}} \cong \mathcal{E}_{fix} \xrightarrow{\alpha} \mathcal{E}$ is a

(constant) sub-VHS,

$$E_{fix} \cong \psi_{x_0} \mathcal{E}_{fix} \xrightarrow{\alpha_{lim}} \psi_{x_0} \mathcal{E} \cong (\psi_{x_0} \mathcal{W})^\vee \otimes (\psi_{x_0} \mathcal{W}')$$

is a sub-MHS. If $\psi_{x_0} \mathcal{W} \cong \psi_{x_0} \mathcal{W}'$, then $\alpha_{lim}(\text{id}_V)$ is Hodge of type $(0, 0)$ in $\psi_{x_0} \mathcal{E}$. By strictness of morphisms of MHS, id_V is Hodge in E_{fix} . Hence, id_V is Hodge $(0, 0)$ in \mathcal{E}_{fix} *a fortiori* in \mathcal{E} , and $\mathcal{W} = \mathcal{W}'$. \square

When \mathcal{V} is rigid in $[\mathcal{V}]$ (let alone $||[\mathcal{V}]|| < \infty$), it is natural to expect that the LMHS at x_0 has arithmetic significance – particularly if \mathcal{V} is motivic and \mathcal{S} defined over $\bar{\mathbb{Q}}$ (cf. [GGK4, Conj. (III.B.5)]). A classic example of this is the class of the LMHS of the mirror quintic $(1, 1, 1, 1, 1)$ -VHS at the maximal unipotent monodromy point, which is given by $-200\zeta(3) \in \mathbb{C}/\mathbb{Q}(3) \cong Ext_{\text{MHS}}^1(\mathbb{Q}(0), \mathbb{Q}(3))$ [op. cit., sec. III.A]. We expect that the Picard-Fuchs equations identified in [DM] will allow one to examine how this invariant changes with the choice of local system. Moreover, it seems likely that the corresponding invariant (at the type I boundary component) for the G_2 example of §9 is a rational multiple of $\zeta(5)$. These questions will be taken up in a later work.

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