Rational inner functions in the Schur-Agler class

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Goal is to understand:

- Function theory on the polydisk $\mathbb{D}^n$.
- von Neumann inequalities
- Positive polynomials and sums of squares decompositions.
Rational inner functions

Rational inner functions are generalizations of finite Blaschke products.

\[ \phi(z) = \frac{\prod_{j=1}^{d} (z - a_j)}{\prod_{j=1}^{d} 1 - \overline{a}_j z} = \frac{\tilde{p}(z)}{p(z)} \]

where \( p \in \mathbb{C}[z] \) has no zeros on \( \overline{D} \), degree \( d \), and

\[ \tilde{p}(z) = z^d \overline{p(1/\bar{z})} \]

The same works in several variables:

\[ \phi(z_1, \ldots, z_n) = \frac{\tilde{p}(z_1, \ldots, z_n)}{p(z_1, \ldots, z_n)} \]

where \( p \in \mathbb{C}[z_1, \ldots, z_n] \) has no zeros on \( \overline{D}^n \), multidegree \( d = (d_1, \ldots, d_n) \), and

\[ \tilde{p}(z_1, \ldots, z_n) = z^d \overline{p(1/\bar{z}_1, \ldots, 1/\bar{z}_n)} \]
von Neumann inequalities and the Schur-Agler class

- (von Neumann) For any holomorphic $f : \mathbb{D} \to \mathbb{D}$ and any contractive operator $T$

  $$\|f(T)\| \leq 1$$

- **Schur class**: Holomorphic $f : \mathbb{D}^n \to \mathbb{D}$

- **Schur-Agler class**: $f$ in Schur class satisfying

  $$\|f(T_1, \ldots, T_n)\| \leq 1$$

  for all commuting, contractive $n$-tuples $(T_1, \ldots, T_n)$.

- Schur-Agler class $\subset$ Schur class, with equality for $n = 1, 2$. 
Sums of squares

Let \( p \in \mathbb{C}[z_1, \ldots, z_n] \) have no zeros in \( \overline{D}^n \), assume \( p \) has multidegree \( d = (d_1, \ldots, d_n) \). Define

\[
\tilde{p}(z) = z^d p(1/\bar{z}_1, 1/\bar{z}_2, \ldots, 1/\bar{z}_n)
\]

\[
\left| \frac{\tilde{p}(z)}{p(z)} \right| = 1 \text{ on } \mathbb{T}^n, \quad \leq 1 \text{ on } \overline{D}^n
\]

So,

\[
|p(z)|^2 - |\tilde{p}(z)|^2 \geq 0 \text{ on } \overline{D}^n
\]

Does the left hand side equal

\[
\sum_{j=1}^{n} (1 - |z_j|^2)SOS_j?
\]
Rational inner functions and the Schur-Agler class

Answer:

\[ \frac{\tilde{p}}{p} \] is in the Schur-Agler class if and only if

\[ |p(z)|^2 - |\tilde{p}(z)|^2 = \sum_{j=1}^{n} (1 - |z_j|^2)SOS_j \]

Why? Possible to plug in \( T = (T_1, \ldots, T_n) \)

\[ I - \frac{\tilde{p}(T)\tilde{p}(T)^*}{p} = \sum A_{j,k}(T)(I - T_j T_j^*)A_{j,k}(T)^* \]
References


Philosophy

- Rational inner functions form a natural (dense) subclass of the Schur class. (More natural than normalized polynomials?)
- Try to understand the Schur-Agler class (and hence von Neumann inequalities) by understanding rational inner functions in the Schur-Agler class.
Questions

- How do you tell if $\tilde{p}/p$ is in the Schur-Agler class?
- If $\tilde{p}/p$ is in the Schur-Agler class, how do you write down a sums of squares decomposition?
- How many squares are required in the sums of squares?
Two variables $z = (z_1, z_2)$

Suppose $p \in \mathbb{C}[z_1, z_2]$ has no zeros on $\mathbb{D}^2$ and degree $(d_1, d_2)$. There exist 2 canonical sums of squares decompositions.

$$|p(z)|^2 - |\tilde{p}(z)|^2 = (1 - |z_1|^2)SOS_1 + (1 - |z_2|^2)SOS_2$$

It is possible to...

- choose $SOS_1$ and $SOS_2$ to have $d_1$ and $d_2$ squares.
- choose $SOS_1$ maximal and $SOS_2$ minimal (or vice versa).
- express $SOS_1, SOS_2$ using orthonormal bases of certain subspaces of polynomials obtained using the measure $\frac{1}{|p|^2}|dz_1||dz_2|$.
- construct $SOS_1, SOS_2$ using the one variable matrix Fejér-Riesz decomposition.
- characterize when $SOS_1$ and $SOS_2$ are unique.
References

Positive extensions, Fejér-Riesz factorization and autoregressive filters in two variables.

Polynomials with no zeros on the bidisk.

Scattering systems with several evolutions and multidimensional input/state/output systems.

Synthesis of two-dimensional lossless $m$-ports with prescribed scattering matrix.
Recent work for more than two variables

- General facts
- Multi-affine symmetric polynomials
- Three variables
Take $p \in \mathbb{C}[z_1, \ldots, z_n]$, degree $d = (d_1, \ldots, d_n)$.

Suppose

$$|p(z)|^2 - |	ilde{p}(z)|^2 = \sum_{j=1}^{n} (1 - |z_j|^2)SOS_j.$$  

- Cannot choose $SOS_j$ to be a sum of $d_j$ squares.
- Example: $p(z) = 3 - z_1 - z_2 - z_3$.
- Can choose $SOS_j$ to be sum of at most $d_j \prod_{k \neq j}(d_k + 1)$ squares.
Multi-affine symmetric case

Take $p \in \mathbb{C}[z_1, \ldots, z_n]$, no zeros on $\mathbb{D}^n$, symmetric, degree $d = (1, \ldots, 1)$.

- Can give a concrete necessary and sufficient condition for

$$|p(z)|^2 - |\tilde{p}(z)|^2 = \sum_{j=1}^{n} (1 - |z_j|^2)SOS_j$$

and can construct $SOS_j$ explicitly.

- Holds for $p_r(z) := p(rz)$ for $0 < r < 1$ small enough.

- Question: is $\tilde{p}/p$ automatically in the Schur-Agler class? Have found no counterexamples!
Three variables

Take $p \in \mathbb{C}[z_1, z_2, z_3]$, no zeros on $\mathbb{D}^3$, degree $d = (d_1, d_2, d_3)$. When is $\tilde{p}/p$ in the Schur-Agler class? i.e.

$$|p(z)|^2 - |\tilde{p}(z)|^2 = \sum_{j=1}^{3} (1 - |z_j|^2)SOS_j?$$

- A. Kummert 1989: if $d = (1, 1, 1)$.
- GK: if $d = (d_1, 1, 1)$.
- GK: if $d = (d_1, d_2, 1)$, for large enough $r, s$

$$z_1^rz_2^s \frac{\tilde{p}(z)}{p(z)}$$

is in the Schur-Agler class.

- Closely related to positive trig polynomials and sums of squares decompositions.
References

- **Kummert, A. (1989).**
  Synthesis of 3-D lossless first-order one ports with lumped elements.

- **Knese, G. (2010).**
  Rational inner functions in the Schur-Agler class of the polydisk.
  to appear in Publicacions Matemàtiques.

- **Knese, G. (2010).**
  Stable symmetric polynomials and the Schur-Agler class.
  preprint.

- **Knese, G. (2010).**
  Schur-Agler class rational inner functions on the tridisk.
  preprint.
Final questions

- Is “the multiplication by a monomial” property on previous page more general?
- The orthogonal polynomials viewpoint is very useful in two variables. Not as useful yet in three or more variables.
- Can one characterize $p$ with $\tilde{p}/p$ Schur-Agler in terms of orthogonality relations in $L^2\left(\frac{1}{|p|^2} d\sigma\right)$?
- If so, can one build “canonical” sums of squares decompositions using subspaces of polynomials in $L^2\left(\frac{1}{|p|^2} d\sigma\right)$?
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