TWISTOR AND GAUSS LIFTS OF SURFACES IN FOUR–MANIFOLDS

Gary R. Jensen & Marco Rigoli*

§1 **Introduction.** Let M be a Riemann surface, (N,h) a Riemannian 4–manifold and let $f:M \to N$ be a conformal immersion with induced metric (i.e., first fundamental form) $g = f^*h$. The area functional has a critical point at f (i.e., f is minimal) if and only if the mean curvature vector H of f vanishes. As a classical reference point, recall that if N is Euclidean 3–space, then f is minimal if and only if its Gauss map $\gamma_f:M \to S^2 \subset \mathbb{R}^3$ is anti–holomorphic. If N is Euclidean n–space, then Chern [Ch] generalized this result to the Gauss map $\gamma_f:M \to \tilde{G}_2(\mathbb{R}^n)$ into the Grassmannian of oriented 2–dimensional subspaces of $\mathbb{R}^n$. This latter space can be identified with the complex hyperquadric $Q_{n-2} \subset \mathbb{C}P^{n-1}$, by a biholomorphic isometry.

In the special case when N is Euclidean 4–space, the hyperquadric Q_2 splits biholomorphically and isometrically into a product of 2–spheres,

$$Q_2 = S^2 \times S^2, \tag{1.1}$$

and projection on each factor splits the Gauss map into factors, $\gamma_f = (\gamma_f^+, \gamma_f^-)$. Blaschke [Bl] and Hoffman–Osserman [HO1] proved that

$$K = \mathcal{J}(\gamma_f^+) + \mathcal{J}(\gamma_f^-)\ ,\ \ K^\perp = \mathcal{J}(\gamma_f^+) - \mathcal{J}(\gamma_f^-),$$

where $\mathcal{J}(.)$ denotes the Jacobian of the map, and K and $K^\perp$ are the Gaussian and normal curvatures, respectively, of f. Integrating these equations, assuming M compact, and using the Chern–Gauss–Bonnet Theorem, they [HO1] generalized a result of

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Chern–Spanier [CS]

$$\chi(M) = \deg(\gamma_f^+) + \deg(\gamma_f^-)\ ,\ \ \chi(TM^\perp) = \deg(\gamma_f^+) - \deg(\gamma_f^-),$$

where χ is the Euler characteristic and deg denotes the degree.

In this paper we allow N to be an arbitrary oriented Riemannian 4–manifold. The Gauss map is replaced by the Gauss lift into the Grassmann bundle $G_2(TN)$ of oriented tangent 2–planes of N. Although the splitting (1.1) holds in the fibers, this space does not in general split. However, the Penrose twistor spaces $Z_\pm$ provide fibrations $G_2(TN) \to Z_\pm$ and consequent factorizations, called the twistor lifts of f. For many generalizations the most interesting results occur when N is a ±self–dual Einstein space. In particular, a special case of our final result, Theorem 8.1, parametrizes a class of harmonic maps from compact Riemann surfaces into $\mathbb{C}P^3$ by compact oriented surfaces immersed in S^4 with parallel mean curvature.

This paper began as an attempt to understand the results of the paper by Eells and Salamon [ES]. Many of the results here were announced in [JR3] where this paper is referred to by the preliminary title "Surfaces in 4–manifolds". Throughout the paper we assume M and N are both connected. We use the Einstein summation convention (sum all repeated indices in a product), and the index conventions $1 \le a,b,c \le 4$; $1 \le i,j,k \le 2$; $3 \le \alpha,\beta,\gamma,\delta \le 4$; $1 \le p,q \le 6$. The paper is organized into eight sections:

§2 Isotropic surfaces in a Riemannian manifold. Let N be a connected n–dimensional Riemannian manifold. Let O(N) denote its principal O(n)–bundle of orthonormal frames. The $\mathbb{R}^n$–valued canonical form on O(N) is denoted $\theta = (\theta^a)$, and the $\mathfrak{o}(n)$–valued Levi–Civita connection and curvature forms on O(N) are denoted $\omega = (\omega^a_b)$ and $\Omega = (\Omega^a_b)$, respectively. Then

$$\Omega^a_b = \tfrac{1}{2} R_{abcd}\theta^c \wedge \theta^d ,$$

where R_{abcd} are functions on $O(N)$ defining the Riemann curvature tensor of N. The structure equations of N are

$$d\theta^a = -\omega^a_b \wedge \theta^c, \quad d\omega^a_b = -\omega^a_c \wedge \omega^c_b + \Omega^a_b .$$

A local orthonormal frame field in N is a local section $e = (e_a)$ of $O(N)$. Its dual coframe field is $(e^*\theta^a)$, for which we will always omit the e^*. Similarly, the connection and curvature forms and components of the curvature tensor with respect to e will be denoted by ω^a_b, Ω^a_b and R_{abcd}, respectively, without explicit indication of e^*.

If N is non–orientable, then $O(N)$ is connected. Otherwise, $O(N)$ has two connected components, $O_{\pm}(N)$, and each of $O_{\pm}(N) \to N$ is a principal $SO(n)$–bundle.

Let M be an m–dimensional manifold, let $f:M \to N$ be an immersion, and let g denote the induced Riemannian metric on M. A local Darboux frame field along f is a local orthonormal frame field e in N such that $e_i \circ f$ is an oriented orthonormal frame field in M and $e_\alpha \circ f$ are normal to M; or, equivalently, $f^*\theta^i$ is an oriented orthonormal coframe in M and

$$f^*\theta^\alpha = 0 .$$

We will almost always suppress the writing of the f^*'s in this context. Exterior differentiation of this equation implies that on M

$$\omega^\alpha_i = h^\alpha_{ij}\theta^j ,$$

where h^α_{ij} are locally defined functions on M, symmetric in i and j. The second fundamental tensor of f is

$$II = h^\alpha_{ij}\theta^i\theta^j \otimes e_\alpha ,$$

a symmetric bilinear form on M with values in the normal bundle $TM^\perp$.

We let

$$H^\alpha = \tfrac{1}{2}(h^\alpha_{11} + h^\alpha_{22})$$

denote the components of the mean curvature vector, $H = H^\alpha e_\alpha$, of f. The Levi–Civita connection of N induces the Levi–Civita connection of g on M given by

$$\nabla e_i = \omega_i^j \otimes e_j \text{ and } \nabla \theta^i = -\omega_j^i \otimes \theta^j ;$$

and a connection on $TM^\perp$ given by

$$\nabla e_\alpha = \omega_\alpha^\beta \otimes e_\beta ;$$

and thus in the standard way on $T^*M \otimes T^*M \otimes TM^\perp$. Then

$$\nabla II = Dh_{ij}^\alpha \otimes \theta^i \theta^j \otimes e_\alpha ,$$

where

$$Dh_{ij}^\alpha = dh_{ij}^\alpha - h_{kj}^\alpha \omega_i^k - h_{ik}^\alpha \omega_j^k + h_{ij}^\beta \omega_\beta^\alpha = h_{ijk}^\alpha \theta^k .$$

From the symmetry of II we have

$$h_{ijk}^\alpha = h_{jik}^\alpha ,$$

while by the Codazzi equations we have

(2.1) $$h_{ijk}^\alpha - h_{ikj}^\alpha = -R_{\alpha ijk} .$$

It is easily verified that the covariant differential of H ,

$$\nabla H = (dH^\alpha + H^\beta \omega_\beta^\alpha) \otimes e_\alpha = H_j^\alpha \theta^j ,$$

is given by

(2.2) $$H_j^\alpha = \frac{1}{2} h_{kkj}^\alpha .$$

We say that f is minimal if $H = 0$, and that f has parallel mean curvature vector if $\nabla H = 0$.

To construct global invariants from this local analysis, we must determine the transformation rules for changes of Darboux frame. For this purpose it is convenient to use the isomorphism

(2.3)
$$\rho: SO(2) \to U(1)$$
$$\begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \mapsto e^{it}$$

We define the Hopf transform from the space of real 2×2 symmetric matrices $h = (h_{ij})$ onto $\mathbb{C}$ by

(2.4)
$$L(h) = \tfrac{1}{2}(h_{11} - h_{22}) - ih_{12} .$$

The kernel of L consists of all scalar matrices, and L has the equivariance property

(2.5)
$$L({}^tAhA) = \rho(A)^2 L(h) ,$$

for any $A \in SO(2)$.

We restrict our attention now to the case $m = 2$, and we suppose that both M and N are oriented. A ± **oriented Darboux frame** along f will mean a Darboux frame $\{e_a\}$ such that $\{e_i\}$ is an oriented frame on M and $\{e_a\}$ is a ± oriented frame in $f^{-1}TM$. Thus $\{e_\alpha\}$ is a ± oriented frame of $TM^\perp$ which is oriented in the way compatible with the orientations of TM and TN , and the decompostion $f^{-1}TN = TM \oplus TM^\perp$.

An arbitrary change of oriented Darboux frame is given by

(2.6)
$$\tilde{e} = eG ,$$

where G is a locally defined function in M with values in

(2.7)
$$SO(2)\times SO(n-2) = \left\{\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} : A\in SO(2), B\in SO(n-2)\right\} .$$

Under such a change the matrices $h^\alpha = (h^\alpha_{ij})$ of II transform by

(2.8)
$$\tilde{h}^\alpha = ({}^tB)^\alpha_\beta \, {}^tAh^\beta A ,$$

where tilded quantities are with respect to $\tilde{e}$. Writing L^α for $L(h^\alpha)$, we have

(2.9)
$$\tilde{L}^\alpha = \rho(A)^2({}^tB)^\alpha_\beta L^\beta ,$$
and

(2.10)
$$\tilde{H}^\alpha = ({}^tB)^\alpha_\beta H^\beta .$$

It is important to use the complex structure of M induced by g. If e is an oriented Darboux frame field along f, then its dual coframe (θ^a) defines a type (1,0) form

$$\varphi = \theta^1 + i\theta^2 , \tag{2.11}$$

which under a change (2.6) of oriented Darboux frame transforms to

$$\tilde{\varphi} = \rho(A)^{-1}\varphi . \tag{2.12}$$

Using the complex structure of M to decompose the second fundamental tensor by type,we have $\omega_1^\alpha + i\omega_2^\alpha = H^\alpha\varphi + L^\alpha\bar{\varphi}$, and thus

$$II = \tfrac{1}{2}\varphi\varphi\otimes L^\alpha e_\alpha + \varphi\bar{\varphi}\otimes H + \tfrac{1}{2}\overline{\varphi\varphi}\otimes\bar{L}^\alpha e_\alpha ,$$

where bars denote complex conjugation. The coefficients $L^\alpha e_\alpha$ and $\bar{L}^\alpha e_\alpha$ are local sections of the complexified normal bundle $TM^\perp_{\mathbb{C}} = TM^\perp\otimes\mathbb{C}$. The Riemannian metric on N induces a fibre metric on $TM^\perp$, which we extend to be complex linear and symmetric on $TM^\perp_{\mathbb{C}}$, and denote by $(.,.)$.

(2.13)Definition The isometric immersion f is **isotropic** at a point p of M if the complex normal vector $L^\alpha e_\alpha(p)$ is isotropic; that is, if

$$(L^\alpha e_\alpha, L^\beta e_\beta) = L^\alpha L^\alpha = 0$$

at p. We say that f is isotropic if it is isotropic at every point of M.

It is evident that the symmetric quartic form

$$\Lambda = L^\alpha L^\alpha \varphi^4 \tag{2.14}$$

is globally defined on M and vanishes at a point if and only if f is isotropic at that point. The function

$$u = \sum_\alpha |L^\alpha|^2 \tag{2.15}$$

is globally defined and C^∞ on M, and vanishes precisely at the umbilic points of f.

We specialize now to the case where dim N = 4 when, as we shall see, the non–simplicity of SO(4) is reflected in the meaning of isotropicity. With respect to a local oriented Darboux frame e along f we define the complex valued functions

(2.16) $$b = H^3 - iH^4, \ S_+ = L^3 - iL^4 \ , \ S_- = L^3 + iL^4 .$$

This strange convention is adopted to match that of the twistor lifts in §4. Under a change of oriented Darboux frame (2.6), these functions transform by

(2.17) $$\tilde{b} = \rho({}^tB)b \ , \ \tilde{S}_\pm = \rho(A^2B^{\pm 1})S_\pm .$$

The absolute values of these functions,

(2.18) $$|b| = \|H\| \ , \ s_\pm = |S_\pm|/\sqrt{2} \ ,$$

are globally defined on M and their squares are of class C^∞.

Proposition 2.1 With respect to any oriented Darboux frame we have

(2.19) $$\Lambda = S_+S_-\varphi^4 \ , \ u = s_+^2 + s_-^2$$

and

(2.20) $$\begin{aligned} s_+^2 + s_-^2 &= \|H\|^2 - K + R_{1212} \\ s_+^2 - s_-^2 &= -K^\perp + R_{1234} \ , \end{aligned}$$

where $K^\perp$ is the curvature of the induced connection in the normal bundle; that is, $K^\perp$ is given by

$$d\theta_4^3 = K^\perp\theta^1\wedge\theta^2 .$$

Proof These equations follow from the Gauss equation

(2.21) $$K = R_{1212} + \sum_\alpha \det h^\alpha ,$$

the Ricci equation

(2.22) $$K^\perp = R_{1234} + h^3_{k1}h^4_{k2} - h^3_{k2}h^4_{k1} ,$$

and the easily derived formulas

(2.23)
$$\text{i)}\quad |L^\alpha|^2 = (H^\alpha)^2 - \det(h^\alpha)$$
$$\text{ii)}\quad L^3\bar{L}^4 - \bar{L}^3 L^4 = i(h^3_{k1}h^4_{k2} - h^3_{k2}h^4_{k1}) . \qquad \square$$

Definition At a point p in M the isometric immersion f is isotropic with **positive** (respectively, **negative**) spin if $s_+(p) = 0$ (respectively, $s_-(p) = 0$). It is isotropic with **positive** (**negative**) spin if it has the respective property at every point of M .

Remarks 1) This definition follows that of Bryant [B] for minimal surfaces in S^4 . In [Ca] Calabi observed that Λ is holomorphic when f is minimal and $N = S^4$. Thus when M is homeomorphic to S^2 , Λ must vanish identically for minimal f . He called isotropic minimal surfaces in S^4 pseudo–holomorphic curves. Our notion of isotropy corresponds to real isotropy of Eells–Wood [EW] and Chern [Ch2]. An isotropic f need not be minimal, even in S^4 . We discuss this further in §5 below.

2) If the orientation of N is reversed, thus reversing the orientation of $TM^\perp$, then s_+ and s_- are interchanged. Thus the notions of positive and negative spin are reversed by a reversal of orientation of N .

Isotropy can be defined geometrically in terms of the ellipse of curvature of the immersion (cf. [EGT] and [JR1]). Fix $p \in M$ and in T_pM consider the parametrized unit circle

$$X = X(t) = \cos t\, e_1 + \sin t\, e_2 \qquad 0 \le t \le 2\pi$$

where e is an oriented Darboux frame field along f . The ellipse of curvature at p is defined to be the curve in $TM_p^\perp$ given parametrically by

$$II(X,X) = H + \tfrac{L}{2}e^{2it} + \tfrac{\bar{L}}{2}e^{-2it},$$

where $L = L^\alpha e_\alpha$. This curve is a circle (with center H and radius $|L|^2/2$) if and only if L is isotropic. It degenerates to a line segment (possibly of zero length) if and only if $L \wedge \bar{L} = 0$ at p, which occurs if and only if $R_{1234} = K^\perp$ at p, by (2.22) and (2.23)ii).

THEOREM 2.1 Let $f:M \to N$ be an isometric immersion of a compact surface. Then

$$\int_M \|H\|^2 dA \geq 2\pi\chi(M) + \left|2\pi\chi(TM^\perp) - \int_M R_{1234} dA\right| - \int_M R_{1212} dA , \tag{2.24}$$

where $\chi(M)$ and $\chi(TM^\perp)$ are the Euler characteristics of M and its normal bundle. Equality holds if and only if f is isotropic with positive, or negative, spin.

PROOF Adding and subtracting the two equations in (2.20), we have

$$\|H\|^2 \geq K + |K^\perp - R_{1234}| - R_{1212} . \tag{2.25}$$

Integrating and using the Gauss–Bonnet theorem we obtain (2.24). Suppose f is isotropic with negative spin. Then from (2.20), $K^\perp - R_{1234} \leq 0$, and equality holds in (2.25), and hence also in (2.24). Similarly, equality holds in (2.24) if f is isotropic with positive spin. Conversely, suppose that

$$\int_M \|H\|^2 dA = 2\pi\chi(M) + 2\pi\chi(TM^\perp) - \int_M R_{1234} dA - \int_M R_{1212} dA .$$

(The same argument works if $2\pi\chi(TM^\perp) - \int_M R_{1234} < 0$.) From (2.20) we have

$$\|H\|^2 = 2s_+^2 + K + K^\perp - R_{1234} - R_{1212} .$$

Integrating and subtracting from the preceding equation we have $\int_M 2s_+^2 dA = 0$, that is, f is isotropic with positive spin. □

Remark Inequality (2.24) generalizes a result of Friedrich [F] (Theorem 1, p.272), and of Wintgen [W] obtained for $N = \mathbb{R}^4$. Indeed, in this case $R_{1212} = R_{1234} = 0$ and $\chi(TM^\perp) = 2q$, where q is the self–intersection number of the compact oriented surface $f(M)$ in $\mathbb{R}^4$. Thus, if g is the genus of M , then (2.24) reduces to Wintgen's inequality

$$\int_M \|H\|^2 dA \geq 4\pi(1 + |q| - g) .$$

Equality in this case was first considered by Weiner [We].

PROPOSITION 2.2 Let $f:M \to N$ be an isometric immersion of a compact surface. If f is isotropic with positive (respectively, negative) spin, with $\chi(TM^\perp) = 0$ and $R_{1234} \geq 0$ (respectively, $R_{1234} \leq 0$), then f is totally umbilical.

PROOF Suppose $s_+ = 0$ (the case $s_- = 0$ is similar). From (2.20), the hypothesis $\chi(TM^\perp) = 0$, and the Chern–Gauss–Bonnet theorem we have

$$\int_M s_-^2 dA = -\int_M R_{1234} dA \leq 0 ,$$

and thus $s_- = 0$ also. Hence $u = 0$, and f is totally umbilical. □

Observe that in case f is totally umbilical and M is compact, then $2\pi\chi(TM^\perp) = \int_M R_{1234} dA$ by (2.20). In particular, if N is the constant curvature 4–sphere S^4, we have

PROPOSITION 2.3 Let $f:M \to S^4$ be a minimal surface where $M \approx S^2$. Then f is totally geodesic if and only if $\chi(TM^\perp) = 0$. If f is not totally geodesic, then $\chi(TM^\perp) = -4 - m$, where m is the total number of umbilical points counted with multiplicities (see Remark 1 below).

PROOF The first part follows from Proposition 2.2. Suppose f is not totally geodesic, or equivalently, that f is full in S^4. Then we apply Theorem 1 of [JR4] to obtain the desired estimates of $\chi(TM^\perp)$ (there f is isotropic with positive spin). □

Remarks 1. If a minimal immersion f is not totally umbilical then the umbilical points are isolated and have well defined multiplicities [JR4].

2. The above estimates of $\chi(TM^\perp)$ improve a result of Salamon [S2].

Now let $f:M \to N$ be a minimal immersion of a compact surface. Then from (2.20) and the Chern–Gauss–Bonnet theorem we have

$$(2.26)\qquad \begin{aligned} \tfrac{1}{2}\{\chi(M) + \chi(TM^\perp)\} &= \frac{-1}{2\pi}\int_M s_+^2 dA + \frac{1}{4\pi}\int_M (R_{1212} + R_{1234}) dA \\ \tfrac{1}{2}\{\chi(M) - \chi(TM^\perp)\} &= \frac{-1}{2\pi}\int_M s_-^2 dA + \frac{1}{4\pi}\int_M (R_{1212} - R_{1234}) dA \end{aligned}$$

The left hand sides of (2.26) are the **twistor degrees** $d_{\pm}$ introduced by Eells–Salamon ([ES], §8), and thus (2.26) gives integral representations of the twistor degrees.

If N is Einstein and anti–self–dual (respectively, self–dual; see §3) with scalar curvature s, then (reading the +, respectively the −) $R_{1212} \pm R_{1234} = s/12$, and therefore

(2.27) $$d_{\pm} = \frac{-1}{2\pi}\int_M s_{\pm}^2 dA + \frac{s}{48\pi}A(M),$$

respectively, where A(M) is the area of M. Furthermore, f is then isotropic with positive (respectively, negative) spin if and only if

(2.28) $$d_{\pm} = \frac{s}{48\pi}A(M).$$

The necessity of this last statement for d_+ was first proved by Friedrich [F] and Poon [P] independently. (See also Salamon [S2].)

§3 Four–dimensional Riemannian geometry. The material of this section is well known (see, for example, [Be] or [S] for excellent expositons). We summarize here the essential points that we need and establish our notation and point of view. In this section we use the index conventions $1 \le i,j,k,l \le 3$, $1 \le a,b,c,d \le 4$.

The standard action of SO(4) on $\mathbb{R}^4$ (as column vectors) induces a representation of SO(4) on $\Lambda_2\mathbb{R}^4$ ($a(u\wedge v) = au\wedge av$), which is reducible into irreducible factors $\Lambda_2\mathbb{R}^4 = \Lambda_+ \oplus \Lambda_-$, where the 3–dimensional subspaces $\Lambda_{\pm}$ are the ±1 eigenspaces of the Hodge *–operator on $\mathbb{R}^4$ with the orientation of the standard basis $\epsilon_1,\dots,\epsilon_4$. Standard bases of $\Lambda_{\pm}$ are given by

(3.1) $$E^{\pm} = (E_1^{\pm},E_2^{\pm},E_3^{\pm})$$

where

$$E_1^{\pm} = (\epsilon_1\wedge\epsilon_2 \pm \epsilon_3\wedge\epsilon_4)/\sqrt{2},\quad E_2^{\pm} = (\epsilon_1\wedge\epsilon_3 \pm \epsilon_4\wedge\epsilon_2)/\sqrt{2},\quad E_3^{\pm} = (\epsilon_1\wedge\epsilon_4 \pm \epsilon_2\wedge\epsilon_3)/\sqrt{2}.$$

The standard metric on $\mathbb{R}^4$ induces an SO(4)–invariant inner product on $\Lambda_2\mathbb{R}^4$, and the restriction of the SO(4)–action to $\Lambda_{\pm}$ thus gives a 2:1 surjective homomorphism

$$\mu: SO(4) \to SO(3)\times SO(3)$$
$$a \mapsto (a_+,a_-)$$

where for $a \in SO(4)$, $a_{\pm} = a|_{\Lambda_{\pm}}$ with respect to the bases (3.1) ($a_+E_i^+ = a_{+ij}E_j^+$ etc.)

There is an isomorphism $\mathfrak{o}(4) \cong \Lambda_2\mathbb{R}^4$ given by: the skew–symmetric matrix $X = (X_{ab}) \leftrightarrow \frac{1}{2}X_{ab}\epsilon_a\wedge\epsilon_b$. If $a \in SO(4)$, then the adjoint action of a on $\mathfrak{o}(4)$ corresponds to the above action of $SO(4)$ on $\Lambda_2\mathbb{R}^4$; namely, $Ad(a)X = aXa^{-1} \leftrightarrow a(\frac{1}{2}X_{ab}\epsilon_a\wedge\epsilon_b)$. The above decomposition of $\Lambda_2\mathbb{R}^4$ thus gives the Lie algebra isomorphism $\mathfrak{o}(4) \cong \mathfrak{o}(3)_+ \oplus \mathfrak{o}(3)_-$, where $\mathfrak{o}(3)_\pm \leftrightarrow \Lambda_\pm$.

Let N be a connected oriented Riemannian 4–manifold. Let $\theta = (\theta^a)$ and $\Omega = (\Omega^a_b)$ denote the canonical form and the curvature form of the Levi–Civita connection, respectively, on $O(N)$. For any $a \in O(4)$,

$$R_a^*\theta = a^{-1}\theta, \tag{3.2}$$

where R_a denotes right multiplication on $O(N)$ by a. If we define $\mathbb{R}^3$–valued 2–forms on $O(N)$ by

$$\alpha_\pm = \begin{bmatrix}\alpha^1_\pm\\ \alpha^2_\pm\\ \alpha^3_\pm\end{bmatrix} = \begin{bmatrix}\theta^1\wedge\theta^2 \pm \theta^3\wedge\theta^4\\ \theta^1\wedge\theta^3 \pm \theta^4\wedge\theta^2\\ \theta^1\wedge\theta^4 \pm \theta^2\wedge\theta^3\end{bmatrix}/\sqrt{2}, \tag{3.3}$$

then

$$R_a^*\alpha_\pm = a_\pm^{-1}\alpha_\pm . \tag{3.4}$$

The curvature forms Ω^a_b are the components of the $\mathfrak{o}(4)$–valued 2–form Ω with respect to the standard basis of $\mathfrak{gl}(4;\mathbb{R})$, with the linear relations $\Omega^a_b = -\Omega^b_a$, because on $O(N)$, Ω takes values in $\mathfrak{o}(4)$. A fundamental property of the curvature form is that it is given by

$$\Omega^a_b = \tfrac{1}{2}R_{abcd}\theta^c\wedge\theta^d, \tag{3.5}$$

where the R_{abcd} are functions on $O(N)$ satisfying the symmetries of the Riemann curvature tensor: $R_{abcd} = -R_{abdc} = -R_{bacd} = R_{cdab}$.

If we express Ω in terms of the basis (3.1) of $\mathfrak{o}(4)$ and the 2–forms (3.3), then

$$\Omega = A_{ij}E^+_i\otimes\alpha^j_+ + B_{ij}E^-_i\otimes\alpha^j_+ + B_{ji}E^+_i\otimes\alpha^j_- + C_{ij}E^-_i\otimes\alpha^j_-, \tag{3.6}$$

where if $A = (A_{ij})$, $B = (B_{ij})$, and $C = (C_{ij})$, then

$${}^tA = A, \quad {}^tC = C, \quad \text{trace } A = \text{trace } C. \tag{3.7}$$

In matrix notation (3.6) becomes

(3.8) $$\Omega = E^{+}\otimes A\alpha_{+} + E^{-}\otimes B\alpha_{+} + E^{+}\otimes {}^{t}B\alpha_{-} + E^{-}\otimes C\alpha_{-}.$$

For any $a \in O(4)$, we have

(3.9) $$R_a^*\Omega = a^{-1}\Omega a,$$

from which it follows that for any $a \in SO(4)$,

(3.10) $$R_a^*A = a_+^{-1}Aa_+, \quad R_a^*B = a_-^{-1}Ba_+, \quad R_a^*C = a_-^{-1}Ca_-,$$

(explicitly, for any $e \in O(N)$, $A(ea) = a_+^{-1}(e)A(e)a_+(e)$, etc.).

It will be handy to have explicit formulas relating R_{abcd} to A, B, and C. These are found by substituting (3.1) and (3.3) into (3.5). We have

(3.11)
$$A_{11} = \tfrac{1}{2}(R_{1212} + 2R_{1234} + R_{3434}), \quad A_{21} = \tfrac{1}{2}(R_{1312} + R_{1334} - R_{2412} - R_{2434})$$
$$A_{31} = \tfrac{1}{2}(R_{1412} + R_{1434} + R_{2312} + R_{2334}), \quad A_{22} = \tfrac{1}{2}(R_{1313} - 2R_{1324} + R_{2424})$$
$$A_{32} = \tfrac{1}{2}(R_{1413} - R_{1424} + R_{2313} - R_{2324}), \quad A_{33} = \tfrac{1}{2}(R_{1414} + 2R_{1423} + R_{2323})$$

(3.12)
$$B_{11} = \tfrac{1}{2}(R_{1212} - R_{3434}) = \tfrac{1}{4}(R_{11} + R_{22} - R_{33} - R_{44})$$
$$B_{21} = \tfrac{1}{2}(R_{1312} + R_{1334} + R_{2412} + R_{2434}) = \tfrac{1}{2}(R_{32} - R_{14})$$
$$B_{31} = \tfrac{1}{2}(R_{1412} + R_{1434} - R_{2312} - R_{2334}) = \tfrac{1}{2}(R_{42} + R_{13})$$
$$B_{12} = \tfrac{1}{2}(R_{1213} - R_{1224} - R_{3413} + R_{3424}) = \tfrac{1}{2}(R_{23} + R_{14})$$
$$B_{22} = \tfrac{1}{2}(R_{1313} - R_{2424}) = \tfrac{1}{4}(R_{11} - R_{22} + R_{33} - R_{44})$$
$$B_{32} = \tfrac{1}{2}(R_{1413} - R_{1424} - R_{2313} + R_{2324}) = \tfrac{1}{2}(R_{43} - R_{12})$$
$$B_{13} = \tfrac{1}{2}(R_{1214} + R_{1223} - R_{3414} - R_{3423}) = \tfrac{1}{2}(R_{24} - R_{13})$$
$$B_{23} = \tfrac{1}{2}(R_{1314} + R_{1323} + R_{2414} + R_{2423}) = \tfrac{1}{2}(R_{34} + R_{12})$$
$$B_{33} = \tfrac{1}{2}(R_{1414} - R_{2323}) = \tfrac{1}{4}(R_{11} - R_{22} - R_{33} + R_{44})$$

where $R_{ab} = R_{ba}$ are the components of the Ricci tensor: $R_{ab} = \Sigma R_{cacb}$.

(3.13)

$$C_{11} = \frac{1}{2}(R_{1212} - 2R_{1234} + R_{3434}),\ C_{21} = \frac{1}{2}(R_{1312} - R_{1334} + R_{2412} - R_{2434})$$
$$C_{31} = \frac{1}{2}(R_{1412} - R_{1434} - R_{2312} + R_{2334}),\ C_{22} = \frac{1}{2}(R_{1313} + 2R_{1324} + R_{2424})$$
$$C_{32} = \frac{1}{2}(R_{1413} + R_{1424} - R_{2313} - R_{2324}),\ C_{33} = \frac{1}{2}(R_{1414} - 2R_{1423} + R_{2323})$$

From (3.12) we see that

(3.14) N is Einstein if and only if $B = 0$.

The Weyl curvature form is the $\mathfrak{o}(4)$–valued 2–form $\Psi = (\Psi^a_b)$ on $O(N)$ defined by

$$\Psi^a_b = \Omega^a_b - \frac{1}{2}(R_{ac}\theta^c \wedge \theta^b + R_{bc}\theta^a \wedge \theta^c) + \frac{s}{6}\theta^a \wedge \theta^b ,$$

where $s = \Sigma R_{aa}$ is the scalar curvature. In terms of the bases (3.1) and (3.3) we have

$$(3.15) \qquad \Psi = E^+ \otimes (A - \frac{s}{12} I)\alpha_+ + E^- \otimes (C - \frac{s}{12} I)\alpha_- ,$$

where I is the 3×3 identity matrix.

Let $e: U \subset N \to O_+(N)$ be a local oriented orthonormal frame field in N. For each point $p \in U$, $e(p) = (e_1,\dots,e_4)(p)$ is an oriented orthonormal frame of T_pN (which we interpret as an isomorphism $e(p): \mathbb{R}^4 \to T_pN$ given by $e(p)x = x^a e_a(p)$, where $x = (x^a)$) with dual coframe $e^*\theta(p)$. Thus $e(p)E^\pm$ are bases of $\Lambda_\pm T_pN$, the ± 1 eigenspaces of the Hodge $*$ operator on TN ($e(p)E^+_1 = \frac{1}{\sqrt{2}}(e_1 \wedge e_2 + e_3 \wedge e_4)(p)$, etc.) with dual bases $e^*\alpha_\pm(p)$.

The curvature operator at p is $R(p) \in \mathfrak{S}^2(\Lambda_2 T_pN) \subset \Lambda_2 T_pN \otimes \Lambda_2 T^*_pN$ given in terms of the basis $\{e_a \wedge e_b(p) : a < b\}$ by

$$R(p) = \frac{1}{4} R_{abcd}(e(p)) e_a \wedge e_b(p) \otimes e^*(\theta^c \wedge \theta^d)(p) ,$$

and in terms of the basis $e(p)E^\pm$ by (e and e^* evaluated at p throughout)

$$(3.16)\ R(p) = eE^+ \otimes A(e) e^*\alpha_+ + eE^- \otimes B(e) e^*\alpha_+ + eE^+ \otimes {}^tB(e) e^*\alpha_- + eE^- \otimes C(e) e^*\alpha_- .$$

The Weyl curvature operator at p preserves the ± 1 eigenspaces of $*$, and thus $W(p) = W^+(p) + W^-(p)$, where $W^\pm(p): \Lambda_\pm T_pN \to \Lambda_\pm T_pN$ are given by

(3.17)
$$W^+(p) = eE^+ \otimes (A(e) - \tfrac{s(p)}{12} I) e^* \alpha_+$$
$$W^-(p) = eE^- \otimes (C(e) - \tfrac{s(p)}{12} I) e^* \alpha_- .$$

If $e: U \to O_-(N)$ is a local negatively oriented orthonormal frame field in N, then for any point $p \in U$, $e(p)E^+$ is a basis of $\Lambda_- T_p N$ and $e(p)E^-$ is a basis of $\Lambda_+ T_p N$. Thus the expressions for $W^\pm(p)$ in (3.17) are reversed. In summary

(3.18) If e is positively oriented, then the matrix of W^+ (W^-) with respect to eE^+ (eE^-) is $A(e) - \frac{s}{12} I$ ($C(e) - \frac{s}{12} I$); while if e is negatively oriented, then the matrix of W^+ (W^-) with respect to eE^- (eE^+) is $C(e) - \frac{s}{12} I$ ($A(e) - \frac{s}{12} I$).

The oriented Riemannian manifold N is **self–dual** (respectively **anti–self–dual**) if at every point of N we have $W^- = 0$ (respectively $W^+ = 0$) [AHS], [Be]. By (3.18) the following are equivalent:

(3.19)
a) N is self–dual (anti–self–dual)
b) $C - \frac{s}{12} I = 0$ on $O_+(N)$ ($A - \frac{s}{12} I = 0$ on $O_+(N)$)
c) $A - \frac{s}{12} I = 0$ on $O_-(N)$ ($C - \frac{s}{12} I = 0$ on $O_-(N)$)

§4 **Metric structure of the Twistor bundle.** In this section $n = 2m$, and we return to the convention of index ranges given in §1. Let N be a connected $2m$–dimensional Riemannian manifold. The twistor space Z of N is defined to be the set of all pairs (p,J), where $p \in N$ and J is an orthogonal complex structure on T_pN; i.e., J is an orthogonal transformation of T_pN satisfying $J^2 = -$identity. The twistor projection

(4.1)
$$T: Z \to N$$

is defined by $T(p,J) = p$. As the set of orthogonal complex structures on T_pN depends only on the conformal class of the inner product on T_pN, it follows that Z depends only on the conformal structure of N.

The projection (4.1) is a fiber bundle over N with standard fiber $O(2m)/U(m)$. We associate Z to $O(N)$, the principal $O(2m)$–bundle of orthonormal frames on N. To do this we must first consider the representation of $U(m)$ in $O(2m)$.

Let

$$J_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \text{ and } J_m = \begin{bmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_1 \end{bmatrix}.$$

Observe that $J_m \in SO(2m)$ and that $J_m^{\,2} = -I_{2m}$. Then

$$U(m) \cong \{A \in SO(2m):\ {}^tAJ_mA = J_m\}. \tag{4.2}$$

At the Lie algebra level,

$$\mathfrak{u}(m) \cong \{A \in \mathfrak{o}(2m) : {}^tAJ_m + J_mA = 0\}\ . \tag{4.3}$$

It will be useful for us to see this explicitly when $m = 2$ as (see §3)

$$\mathfrak{u}(2) \cong \{\begin{bmatrix} 0 & a & b & c \\ -a & 0 & -c & b \\ -b & c & 0 & d \\ -c & -b & -d & 0 \end{bmatrix} : a,b,c,d \in \mathbb{R}\} = \operatorname{span}\{\mathfrak{o}(3)_-, E_1^+\}. \tag{4.4}$$

Let V be any oriented 2m–dimensional inner product space. For any orthonormal frame $e = (e_1,...,e_{2m})$ of V, define an orthogonal complex structure J_e on V by

$$J_e e_{2j-1} = e_{2j}\ ,\ \ 1 \le j \le m\ ,\ \ J_e^{\,2} = -\text{identity}. \tag{4.5}$$

Thus, the matrix of J_e with respect to e is J_m. It is easily verified that any orthogonal complex structure on V is equal to J_e for some orthonormal frame e, and that $J_e = J_{\tilde e}$ if and only if $\tilde e = eA$ for some $A \in U(m) \subset SO(2m)$. The set of all orthogonal complex structures on V is $O(2m)/U(m)$ which has two connected components, corresponding to the two connected components of $O(2m)$. A component is selected by choosing an orientation on V, in which case $SO(2m)/U(m) = \{J_e : e \text{ is an oriented orthonormal basis of } V\}$.

From these pointwise considerations we see then that the twistor bundle is

$$Z = O(N)\times_{O(2m)}O(2m)/U(m) = O(N)/U(m)\ . \tag{4.6}$$

It is connected if N is non–orientable, while if N is oriented then Z has two connected components

$$Z_\pm = O_\pm(N)/U(m)\ , \tag{4.7}$$

where $O_\pm(N)$ are defined in §2. Let $\sigma:O(N) \to Z$ be the projection, and if N is oriented, let

$$\sigma_\pm:O_\pm(N) \to Z_\pm \tag{4.9}$$

be the separate projections. By (4.6) these are principal U(m)–bundles. In much of the literature (cf. Salamon [S]) Z_- is called the twistor space of N .

Up to constant positive factor, there is a unique O(2m)–invariant Riemannian metric on O(2m)/U(m) . This metric, combined with the parallelism on O(N) defined by the canonical and Levi–Civita forms θ and ω , defines a natural 1–parameter family of metrics on Z which we now describe. This is a special case of a general construction defined in [JR2].

The unique, up to positive factor, Ad(O(2m))–invariant inner product on $\mathfrak{o}(2m)$ is

$$\langle X,Y\rangle = \operatorname{trace} {}^tXY , \tag{4.10}$$

for $X,Y \in \mathfrak{o}(2m)$. We let $\mathfrak{m}$ denote the orthogonal complement of $\mathfrak{u}(m)$ in $\mathfrak{o}(2m)$. Then $\mathfrak{o}(2m) = \mathfrak{u}(m) \oplus \mathfrak{m}$ decomposes ω into $\omega = \mu + \nu$, where

$$\mu = \tfrac{1}{2}(\omega - J_m \omega J_m) \quad\text{and}\quad \nu = \tfrac{1}{2}(\omega + J_m \omega J_m) , \tag{4.11}$$

which in terms of components is

$$\begin{aligned}
\mu^{2j-1}_{2k-1} &= \mu^{2j}_{2k} = (\omega^{2j-1}_{2k-1} + \omega^{2j}_{2k})/2 \\
\mu^{2j-1}_{2k} &= -\mu^{2j}_{2k-1} = (\omega^{2j-1}_{2k} - \omega^{2j}_{2k-1})/2 \\[1ex]
\nu^{2j-1}_{2k-1} &= -\nu^{2j}_{2k} = (\omega^{2j-1}_{2k-1} - \omega^{2j}_{2k})/2 \\
\nu^{2j-1}_{2k} &= \nu^{2j}_{2k-1} = (\omega^{2j-1}_{2k} + \omega^{2j}_{2k-1})/2 .
\end{aligned} \tag{4.12}$$

When m = 2 these are the skew–symmetric matrices

$$\mu = \frac{1}{2}\begin{bmatrix} 0 & 2\omega^1_2 & \omega^1_3+\omega^2_4 & \omega^1_4-\omega^2_3 \\ & 0 & \omega^2_3-\omega^1_4 & \omega^2_4+\omega^1_3 \\ & & 0 & 2\omega^3_4 \\ & & & 0 \end{bmatrix}, \quad \nu = \frac{1}{2}\begin{bmatrix} 0 & 0 & \omega^1_3-\omega^2_4 & \omega^1_4+\omega^2_3 \\ & 0 & \omega^2_3+\omega^1_4 & \omega^2_4-\omega^1_3 \\ & & 0 & 0 \\ & & & 0 \end{bmatrix} \tag{4.13}$$

The fibers of σ are the integral submanifolds of the completely integrable system $\theta = 0$, $\nu = 0$.

We define a symmetric bilinear form Q_t on O(N) , for any $t > 0$, by

$$Q_t = {}^t\theta\theta + t^2\langle\nu,\nu\rangle. \tag{4.14}$$

By (4.10) and (4.12) this is

$$Q_t = \Sigma (\theta^a)^2 + 4t^2 \sum_{j<k} [(\nu^{2j-1}_{2k-1})^2 + (\nu^{2j-1}_{2k})^2] . \tag{4.15}$$

It is easily checked that $R_a^* Q_t = Q_t$ for any $a \in U(m)$, where R_a denotes right multiplication by a on $O(N)$; and that Q_t is horizontal, meaning that it vanishes on any pair of vectors for which either of them is vertical with respect to σ. Thus there exists a unique Riemannian metric g_t on Z such that $\sigma^* g_t = Q_t$. With the Riemannian metric $Q_t + \langle\mu,\mu\rangle$ on $O(N)$ and g_t on Z, σ is a Riemannian submersion with totally geodesic fibers, as we shall see.

Let $U \subset Z$ be an open subset on which there is a local section $u: U \to O(N)$ of $\sigma: O(N) \to Z$. By (4.15) an orthonormal coframe for g_t on U is given by applying u^* to the 1–forms on $O(N)$

$$\theta^a , \quad 2t\nu^{2j-1}_{2k-1} , \quad 2t\nu^{2j-1}_{2k} , \quad j < k . \tag{4.16}$$

For a uniform notation for this coframe we let

$$\begin{aligned} \theta^{jk-} &= -\theta^{kj-} = \nu^{2j-1}_{2k-1} \\ \theta^{jk+} &= -\theta^{kj+} = \nu^{2j-1}_{2k} , \end{aligned} \tag{4.17}$$

which, when $m = 2$, becomes (letting $12- = 5$ and $12+ = 6$)

$$\theta^5 = \tfrac{1}{2}(\omega^1_3 - \omega^2_4) , \quad \theta^6 = \tfrac{1}{2}(\omega^1_4 + \omega^2_3) . \tag{4.18}$$

The Levi–Civita connection forms for g_t with respect to this orthonormal coframe are given by (where $j < k$ and $l < m$)

$$\begin{aligned}
\theta^a_b &= \omega^a_b - t^2(R_{2j,2k,ba} - R_{2j\text{-}1,2k\text{-}1,ba})\theta^{jk-} + t^2(R_{2j\text{-}1,2k,ba} + R_{2j,2k\text{-}1,ba})\theta^{jk+} \\
\theta^{jk-}_b &= \tfrac{t}{2}(R_{2j,2k,ab} - R_{2j\text{-}1,2k\text{-}1,ab})\theta^a \\
\theta^{jk+}_b &= -\tfrac{t}{2}(R_{2j\text{-}1,2k,ab} + R_{2j,2k\text{-}1,ab})\theta^a \\
\theta^{jk-}_{lm-} &= \delta^j_m \mu^{2l-1}_{2k-1} - \delta^j_l \mu^{2m-1}_{2k-1} + \delta^m_k \mu^{2j-1}_{2l-1} - \delta^l_k \mu^{2j-1}_{2m-1} \\
\theta^{jk-}_{lm+} &= \delta^j_m \mu^{2l}_{2k-1} - \delta^j_l \mu^{2m}_{2k-1} - \delta^k_m \mu^{2j}_{2l-1} + \delta^k_l \mu^{2j}_{2m-1} \\
\theta^{jk+}_{lm+} &= \theta^{jk-}_{lm-}
\end{aligned}$$

and of course $\theta^p_q = -\theta^q_p$ for $1 \le p,q \le m(m+1)$.

In the case $m = 2$ these are (letting $12- = 5$ and $12+ = 6$):

$$\begin{aligned} &\theta^a_b = \omega^a_b + t^2(R_{13ba} - R_{24ba})\theta^5 + t^2(R_{14ba} + R_{23ba})\theta^6 \\ (4.19) \qquad &\theta^5_b = \tfrac{t}{2}(R_{24ab} - R_{13ab})\theta^a = -\theta^b_5, \quad \theta^6_b = -\tfrac{t}{2}(R_{14ab} + R_{23ab})\theta^a = -\theta^b_6 \\ &\theta^5_6 = \omega^1_2 + \omega^3_4 = -\theta^6_5 \end{aligned}$$

These last equations show that the fibers of σ are totally geodesic for the metric $Q_t + <\omega_o,\omega_o>$ on $O(N)$.

With respect to the frame field (4.16) the components of the curvature tensor are

$$\begin{aligned} T_{abcd} = R_{abcd} - \tfrac{t^2}{2}\{&\tfrac{1}{2}[(R_{24ca} - R_{13ca})(R_{24db} - R_{13db}) + \\ &(R_{14ca} + R_{23ca})(R_{14db} + R_{23db}) - (R_{24da} - R_{13da})(R_{24cb} - R_{13cb}) - \\ &(R_{14da} + R_{23da})(R_{14cb} + R_{23cb})] + (R_{24ba} - R_{13ba})(R_{24dc} - R_{13dc}) + \\ &(R_{14ba} + R_{23ba})(R_{14dc} + R_{23dc})\} \end{aligned}$$

$$\begin{aligned} &T_{abc5} = \tfrac{t}{2}(R_{13ba,c} - R_{24ba,c}), \quad T_{abc6} = \tfrac{t}{2}(R_{14ba.c} + R_{23ba,c}) \\ &T_{ab56} = \tfrac{t^2}{4}\{(R_{13ca} - R_{24ca})(R_{14bc} + R_{23bc}) - \\ &\qquad (R_{13bc} - R_{24bc})(R_{14ca} + R_{23ca})\} - R_{34ba} - R_{12ba} \\ (4.20) \qquad &T_{5bd5} = \tfrac{t^2}{2}(R_{13ba} - R_{24ba})(R_{24da} - R_{13da}) \\ &T_{5bd6} = \tfrac{1}{2}(R_{12db} + R_{34db}) + \tfrac{t^2}{4}(R_{14ba} + R_{23ba})(R_{24da} - R_{13da}) \\ &T_{6bc6} = -\tfrac{t^2}{4}(R_{14ba} + R_{23ba})(R_{14ca} + R_{23ca}) \\ &T_{5b56} = 0 = T_{6b56}, \quad T_{5656} = \tfrac{1}{t^2} \end{aligned}$$

The remaining components are determined by the symmetries of the curvature tensor, and $R_{abcd,e}$ are the components of the covariant derivative of the curvature tensor of N . Contracting T_{pqrs} on the first and third index, we obtain the components T_{pq} of the Ricci tensor of g_t on Z :

$$\begin{aligned} T_{bd} = R_{bd} - \tfrac{t^2}{2}\{&(R_{14ba} + R_{23ba})(R_{14da} + R_{23da}) + \\ &(R_{13ba} - R_{24ba})(R_{13da} - R_{24da})\} \\ T_{b5} = \tfrac{t}{2}(&R_{13ba,a} - R_{24ba,a}), \\ T_{b6} = \tfrac{t}{2}(&R_{14ba,a} + R_{23ba,a}) \end{aligned}$$

and

$$T_{55} = \frac{1}{t^2} + \frac{t^2}{4} \sum_{a,b} (R_{13ba} - R_{24ba})^2,$$
$$T_{56} = -\frac{t^2}{4} (R_{14ba} + R_{23ba})(R_{24ba} - R_{13ba})$$
$$T_{66} = \frac{1}{t^2} + \frac{t^2}{4} \sum_{a,b} (R_{14ba} + R_{23ba})^2 .$$

From the Ricci identitiy $R_{abcd,d} = R_{ca,b} - R_{cb,a}$, where $R_{ab,c}$ are the components of the covariant derivative of the Ricci tensor R_{ab} of N , we find that

(4.22)
$$T_{b5} = \frac{t}{2}(R_{b1,3} - R_{b3,1} - R_{b2,4} + R_{b4,2})$$
$$T_{b6} = \frac{t}{2}(R_{b1,4} - R_{b4,1} + R_{b2,3} - R_{b3,2}) .$$

These calculations are used to establish the following theorem which was first proved by Friedrich and Grunewald [FG] (the version Theorem 2.1 in [JR3] is incorrect).

Theorem 4.1 Let N be a four dimensional oriented Riemannian manifold. Then the metric g_t on Z_- (respectively Z_+) is Einstein if and only if N is self–dual (respectively anti–self–dual) Einstein with positive scalar curvature $s = 6/t^2$ or $s = 12/t^2$.

§5 Almost complex structures on the twistor bundle We consider now two natural almost complex structures $J_\pm$ on the twistor space Z of a 2m–dimensional oriented Riemannian manifold N . We do this by defining locally the type (1,0) forms on Z . As usual, this we do by defining complex forms on $O(N)$ and pulling them back to Z with local sections. We continue with the index conventions of §1.

It is convenient to begin with a more detailed description of the representation $U(m) \subset SO(2m)$ introduced in (4.2). If $\{\epsilon_i\}$ and $\{\epsilon_1,\epsilon_2\}$ denote the standard bases of $\mathbb{R}^m$ and $\mathbb{R}^2$, respectively, then we take as the standard basis of $\mathbb{R}^{2m} = \mathbb{R}^m \otimes \mathbb{R}^2$ the set of vectors $\{\epsilon_i \otimes \epsilon_1, \epsilon_i \otimes \epsilon_2\}$ ordered lexicographically. Thus $x \in \mathbb{R}^{2m}$ is given by $x = x^{2i-1}\epsilon_i \otimes \epsilon_1 + x^{2i}\epsilon_i \otimes \epsilon_2$. We define a real isomorphism $\alpha : \mathbb{R}^{2m} \to \mathbb{C}^m$ by

(5.1)
$$\alpha(x) = (x^{2j-1} + ix^{2j})\epsilon_j .$$

If we let $\rho : U(m) \to SO(2m)$ denote the faithful representation (4.2), then

(5.2)
$$\rho(A + iB) = A \otimes I_2 + B \otimes J_1 .$$

The induced representation on the Lie algebra we denote by ρ_*. It is easily verified that for any $a \in U(m)$ and any $x \in \mathbb{R}^{2m}$, we have

(5.3) $$a\alpha(x) = \alpha(\rho(a)x) .$$

The same formula holds for $a \in \mathfrak{u}(m)$ and ρ_* in place of ρ.

Using (4.11) and (4.12), one can see that the orthogonal complement $\mathfrak{m}$ of $\mathfrak{u}(m)$ in $\mathfrak{o}(2m)$ has the simple description

(5.4) $$\mathfrak{m} = \{X\otimes\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + Y\otimes\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} : X,Y \in \mathfrak{o}(m)\} .$$

If we define the real isomorphism $\beta:\mathfrak{m} \to \mathfrak{o}(m;\mathbb{C})$ by

(5.5) $$\beta(X\otimes\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + Y\otimes\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}) = X + iY ,$$

then for any $a \in U(m)$ and any $z \in \mathfrak{m}$,

(5.6) $$\beta(\mathrm{Ad}(\rho(a))z) = a\beta(z){}^ta ,$$

and for any $u \in \mathfrak{u}(m)$

(5.7) $$\beta(\mathrm{ad}(\rho_*u)z) = u\beta(z) + \beta(z){}^tu .$$

On the bundle of orthonormal frames $O(N)$ we have the canonical form θ and the Levi–Civita connection form ω, which decomposes by (4.11) into $\omega = \mu + \nu$. We define a $\mathbb{C}^m$–valued 1–form φ and an $\mathfrak{o}(m;\mathbb{C})$–valued 1–form Φ on $O(N)$ by

(5.8) $$\varphi = \alpha(\theta) , \quad (\text{thus } \varphi^j = \theta^{2j-1} + i\theta^{2j})$$
$$\Phi = \beta(\nu) , \quad (\text{thus } \Phi^{jk} = \theta^{jk-} + i\theta^{jk+}) \qquad \text{(cf (4.17).}$$

Then, letting R_b denote right multiplication by $b \in SO(2m)$,we have for any $a \in U(m)$

(5.9) $$R^*_{\rho(a^{-1})}\varphi = a\varphi, \quad R^*_{\rho(a^{-1})}\Phi = a\Phi{}^ta .$$

It follows immediately that an almost complex structure J_+ is defined on Z by defining the type (1,0) vectors to be spanned by the pull back to $Z_\pm$ by local sections $u_\pm$ of $O_\pm(N) \to Z_\pm$, respectively, of the complex 1–forms

(5.10) $$\varphi^i\,,\ \Phi^{jk}\,, j<k\,.$$

In fact, by (5.9) this span over $\mathbb{C}$ does not depend on the choice of $u_\pm$, and it is easily verified that

$$u_\pm^*(\underset{i}{\wedge}\varphi^i \underset{j<k}{\wedge} \Phi^{jk}) \neq 0$$

at every point. Another almost complex structure J_- is defined in the same way, but with Φ^{jk} replaced by their complex conjugates $\bar{\Phi}^{jk}$.

(5.11) **Remarks** 1. $O(2m)/U(m)$ is a Hermitian symmetric space whose $O(2m)$–invariant complex structure is defined by the left–invariant (1,0)–forms $\beta(\nu)$.

2. The almost complex structure J_+ was introduced by Atiyah, Hitchin and Singer [AHS] in their study of self–dual Yang–Mills equations in Euclidean 4–space, while J_- has been studied by Eells and Salamon [ES] for its relation to harmonic maps from Riemann surfaces into N.

When $m = 2$ we can use the notation of (4.18) to write

(5.12) $$\varphi^3 = \Phi^{12} = \theta^5 + i\theta^6.$$

If we let Ω_m denote the m–component of the curvature form Ω (cf. (4.11), and if we let $\hat{\mu}$ denote the u(m)–valued 1–form such that $\rho_*\hat{\mu} = \mu$ (thus $\hat{\mu}_k^j = \mu_{2k-1}^{2j-1} + i\mu_{2k}^{2j-1}$), then from the structure equations of N we obtain

(5.13) $$d\varphi = -\bar{\hat{\mu}}\wedge\varphi - \Phi\wedge\overline{\varphi}, \quad d\Phi = \beta(\Omega_m) - \hat{\mu}\wedge\Phi - \Phi\wedge\hat{\mu}\,.$$

By the Newlander–Nirenberg theorem, an almost complex structure is integrable if and only if the algebraic ideal generated by the (1,0)–forms is closed under exterior differentiation. Thus we conclude from the first equation in (5.13) a result proved in [ES]:

(5.14) J_- is never integrable.

From the second equation in (5.13) and the fact that the curvature form on $O(N)$ is horizontal, we conclude that

(5.15) J_+ is integrable if and only if $\beta(\Omega_m) \equiv 0$ modulo $\{\varphi\}$.

When $m = 2$ the 2×2 skew-symmetric matrices Φ and $\beta(\Omega_{\mathfrak{m}})$ are determined by their entries

(5.16) $$\Phi^{12} = \varphi^3 = \tfrac{1}{2}(\omega_3^1 - \omega_4^2) + \tfrac{i}{2}(\omega_4^1 + \omega_3^2)$$

and

(5.17) $$\beta(\Omega_{\mathfrak{m}})^{12} = \tfrac{1}{2}(\Omega_3^1 - \Omega_4^2) + \tfrac{i}{2}(\Omega_4^1 + \Omega_3^2)\ .$$

By (4.4), and using the notation of §3, we have $\mathfrak{m} = \operatorname{span}\{E_2^+, E_3^+\}$. Thus from (3.6) we have

(5.18) $$\Omega_{\mathfrak{m}} = E_2^+ \otimes \sum_1^3 (A_{2j}\alpha_+^j + B_{j2}\alpha_-^j) + E_3^+ \otimes \sum_1^3 (A_{3j}\alpha_+^j + B_{j3}\alpha_-^j)\ .$$

Hence, as $\beta(E_2^+) = -\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}/\sqrt{2}$ and $\beta(E_3^+) = i\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}/\sqrt{2}$, we have

(5.19) $$\beta(\Omega_{\mathfrak{m}})^{12} = \tfrac{1}{\sqrt{2}}[A_{2j}\alpha_+^j + B_{j2}\alpha_-^j + i(A_{3j}\alpha_+^j + B_{j3}\alpha_-^j)].$$

To express this in terms of φ and $\bar{\varphi}$, we use

(5.20) $$\begin{aligned} \alpha_+^1 &= i(\varphi^1\wedge\bar{\varphi}^1 + \varphi^2\wedge\bar{\varphi}^2)/2\sqrt{2}\ , & \alpha_-^1 &= i(\varphi^1\wedge\bar{\varphi}^1 - \varphi^2\wedge\bar{\varphi}^2)/2\sqrt{2} \\ \alpha_+^2 &= (\varphi^1\wedge\varphi^2 + \bar{\varphi}^1\wedge\bar{\varphi}^2)/2\sqrt{2}\ , & \alpha_-^2 &= (\bar{\varphi}^1\wedge\varphi^2 + \varphi^1\wedge\bar{\varphi}^2)/2\sqrt{2} \\ \alpha_+^3 &= -i(\varphi^1\wedge\varphi^2 - \bar{\varphi}^1\wedge\bar{\varphi}^2)/2\sqrt{2}\ , & \alpha_-^3 &= -i(\bar{\varphi}^1\wedge\varphi^2 - \varphi^1\wedge\bar{\varphi}^2)/2\sqrt{2} \end{aligned}$$

which, when substituted into (5.19), gives

(5.21) $$\beta(\Omega_{\mathfrak{m}})^{12} = P\varphi^1\wedge\varphi^2 + Q\bar{\varphi}^1\wedge\bar{\varphi}^2 + R\varphi^1\wedge\bar{\varphi}^1 + S\varphi^2\wedge\bar{\varphi}^2 + T\bar{\varphi}^1\wedge\varphi^2 + U\varphi^1\wedge\bar{\varphi}^2,$$

where

(5.22) $$\begin{gathered} P = (A_{22} + A_{33})/4\ ,\ \ Q = (A_{22} - A_{33} + 2iA_{23})/4 \\ R = (-A_{31} - B_{13} + i(A_{21} + B_{12}))/4\ ,\ \ S = (-A_{31} + B_{13} + i(A_{21} - B_{12}))/4 \\ T = (B_{22} + B_{33} + i(B_{23} - B_{32}))/4\ ,\ \ U = (B_{22} - B_{33} + i(B_{23} + B_{32}))/4\ . \end{gathered}$$

From these calculations we can conclude the result of [AHS]:

Theorem (5.23) Let N be an oriented Riemannian 4-manifold. Then J_+ on Z_- is integrable if and only if N is self-dual (and J_+ on Z_+ is integrable if and only if N is anti-self-dual).

Proof From (5.15) and (5.21) we have that J_+ on $Z_\pm$ is integrable if and only if

$Q = 0$ on $O_{\pm}(N)$, respectively; which, by (5.22), occurs if and only if

$$A_{23} = 0 \text{ and } A_{22} = A_{33} \tag{5.24}$$

on $O(N)_{\pm}$, respectively.

By (3.10) and the fact that the homomorphism $SO(4) \to SO(3)$ given by $a \mapsto a_{+}$ is surjective, it follows that A has the form (5.24) on $O_{\pm}(N)$ if and only if A is a scalar matrix. As trace $A = s/4$, it follows that (5.24) holds on $O_{\pm}(N)$ if and only if $A = \frac{s}{12} I$ on $O_{\pm}(N)$. Hence (5.23) follows from (3.18). □

We remarked above that the twistor space Z depends only on the conformal class of the metric h on N. To see how the almost complex structures $J_{\pm}$ on Z depend on the choice of metric, consider a conformally related metric $\tilde{h} = \lambda^2 h$, where λ is a nowhere zero C^{∞} function on N. Let $\tilde{\pi}: \tilde{O}(N) \to N$ denote the bundle of $\tilde{h}$–orthonormal frames on N, and let $\tilde{\theta}$, $\tilde{\omega}$ denote the canonical and Levi–Civita forms, respectively on $\tilde{O}(N)$. Then

$$\begin{aligned} F: \tilde{O}(N) &\to O(N) \\ (p, \tilde{e}) &\mapsto (p, \lambda(p)\tilde{e}) \end{aligned} \tag{5.25}$$

is a bundle isomorphism such that

$$\begin{aligned} F^{*}\theta &= \frac{1}{\lambda}\tilde{\theta} \\ F^{*}\omega^a_b &= \tilde{\omega}^a_b + \frac{\lambda_a}{\lambda}\tilde{\theta}^b - \frac{\lambda_b}{\lambda}\tilde{\theta}^a \end{aligned} \tag{5.26}$$

where we have written λ instead of $\lambda\circ\tilde{\pi}$, and where $d(\lambda\circ\tilde{\pi}) = \lambda_a\tilde{\theta}^a$, $\lambda_a \in C^{\infty}(\tilde{O}(N))$. Thus, using the notation of (4.18) with $t = 1/2$, we have

$$\begin{aligned} F^{*}\theta^5 &= \frac{1}{2\lambda}(\lambda\tilde{\omega}^1_3 + \lambda_1\tilde{\theta}^3 - \lambda_3\tilde{\theta} - \lambda\tilde{\omega}^2_4 - \lambda_2\tilde{\theta}^4 + \lambda_4\tilde{\theta}^2) \\ F^{*}\theta^6 &= \frac{1}{2\lambda}(\lambda\tilde{\omega}^1_4 + \lambda_1\tilde{\theta}^4 - \lambda_4\tilde{\theta}^1 + \lambda\tilde{\omega}^2_3 + \lambda_2\tilde{\theta}^3 - \lambda_3\tilde{\theta}^2) \end{aligned} \tag{5.27}$$

and consequently, using the notation of (5.12), we have

$$\begin{aligned} F^{*}\varphi^i &= \frac{1}{\lambda(p)}\tilde{\varphi}^i, \quad i = 1,2 \\ F^{*}\varphi^3 &= \tilde{\varphi}^3 + \frac{1}{2\lambda}(\lambda_1 + i\lambda_2)\tilde{\varphi}^2 - \frac{1}{2\lambda}(\lambda_3 + i\lambda_4)\tilde{\varphi}^1 . \end{aligned} \tag{5.28}$$

We have U(2)–bundles $\sigma:O(N) \to Z$ and $\tilde{\sigma}:\tilde{O}(N) \to Z$ (see (4.8)) for which it is easily verified that $\sigma \circ F = \tilde{\sigma}$. Thus by definition of $J_\pm$ and $\tilde{J}_\pm$ on Z, defined by h and $\tilde{h}$, respectively, it follows from (5.28) that $J_+ = \tilde{J}_+$, for any conformal factor λ; while $J_- = \tilde{J}_-$ if and only if λ is constant. Thus J_+ is conformally invariant, while J_- is invariant only under change of scale.

§6 Hermitian structures on the twistor bundle. Consider the twistor space Z with metrics g_t of §4 and almost complex structures $J_\pm$. By (4.16) and (5.10), $(Z, J_\pm, g_t)$ is Hermitian and

(6.1) $$\varphi^i\,,\ 2t\Phi^{jk}\,,\ j<k\ \ (\text{resp.}, \varphi^i, 2t\bar{\Phi}^{jk})$$

is a unitary coframe field for (Z,J_+,g_t) (resp., (Z,J_-,g_t)) when pulled back to Z by any section u of $\sigma:O(N) \to Z$. The associated (1,1)–form, i.e., Kaehler form, is then

(6.2) $$\kappa_\pm(t) = \tfrac{i}{2}[\sum_i \varphi^i\wedge\bar{\varphi}^i \pm 4t^2\sum_{j<k} \Phi^{jk}\wedge\bar{\Phi}^{jk}]$$

pulled back to Z by u^*. Taking the exterior derivative of $\kappa_\pm(t)$, using the structure equations of N in the form (5.13), we find

(6.3) $$d\kappa_\pm(t) = i\sum_{j<k}\{\Phi^{jk}\wedge\bar{\varphi}^j\wedge\bar{\varphi}^k - \varphi^j\wedge\varphi^k\wedge\bar{\Phi}^{jk} \pm 2t^2[\beta(\Omega_m)^{jk}\wedge\bar{\Phi}^{jk} - \Phi^{jk}\wedge\overline{\beta(\Omega_m)}^{jk}]\}$$

Suppose now that $m = 2$. Substituting (5.21) into (6.3) we find

(6.4) $$d\kappa_\pm(t) =$$
$$-i\varphi^3\wedge[(-1 \pm 2t^2P)\bar{\varphi}^1\wedge\bar{\varphi}^2 \pm 2t^2(\bar{Q}\varphi^1\wedge\varphi^2 + \bar{R}\bar{\varphi}^1\wedge\varphi^1 + \bar{S}\bar{\varphi}^2\wedge\varphi^2 + \bar{T}\varphi^1\wedge\bar{\varphi}^2 + \bar{U}\bar{\varphi}^1\wedge\varphi^2)]$$
$$+ i[(-1 \pm 2t^2P)\varphi^1\wedge\varphi^2 \pm 2t^2(Q\bar{\varphi}^1\wedge\bar{\varphi}^2 + R\varphi^1\wedge\bar{\varphi}^1 + S\varphi^2\wedge\bar{\varphi}^2 + T\bar{\varphi}^1\wedge\varphi^2 + U\varphi^1\wedge\bar{\varphi}^2)]\wedge\bar{\varphi}^3$$

(6.5) **Definition** Recall that an almost Hermitian manifold (Z, g, J) is **symplectic** if its associated (1,1) form κ is closed; it is **(1,2)–symplectic** if the (1,2) part of $d\kappa$ is zero; and it is **Kaehler** if it is symplectic and J is integrable.

THEOREM 6.1 Let N be an oriented Riemannian 4–manifold. The following are equivalent:

a) (Z_-, g_t, J_+) is (1,2)–symplectic;
b) N is self–dual Einstein with positive scalar curvature s and $t^2 = 12/s$;
c) (Z_-, g_t, J_+) is Kaehler Einstein.

PROOF Recall that the type (1,0) forms of J_+ on Z_- are given by the pull back of $\varphi^1, \varphi^2, \varphi^3$ to Z_- by any local frame u in Z_-. Furthermore, the twelve decomposable 3–forms giving $d\kappa_+(t)$ in (6.4) are linearly independent at each point when pulled back to Z_-. Thus, reading off the type (1,2) part from (6.4), we see that a) holds if and only if

$$P = 1/2t^2 \text{ and } R = S = T = U = 0 \tag{6.6}$$

on $O_-(N)$, which, by (5.22), holds if and only if

$$A_{22} + A_{33} = 2/t^2 \tag{6.7}$$

and

$$A = \begin{bmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & A_{23} \\ 0 & A_{32} & A_{33} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & 0 & 0 \\ B_{21} & 0 & 0 \\ B_{31} & 0 & 0 \end{bmatrix} \tag{6.8}$$

on $O_-(N)$. By the transformation laws (3.10), and the fact that each of the homomorphisms $SO(4) \to SO(3)$, $a \mapsto a_\pm$, is surjective, it follows easily that A and B can have the form (6.8) at each point of $O_-(N)$ if and only if A is a scalar matrix and $B = 0$. If A is scalar, then $A = \frac{s}{12} I$, since trace $A = s/4$. Thus $s/6 = 2/t^2$ by (6.7), from which it follows that $s > 0$ and $t^2 = 12/s$. Hence a) is equivalent to b) by (3.14) and (5.23).

Suppose b) holds. Then J_+ on $O_-(N)$ is integrable by (5.23), and g_t is Einstein by (4.23). As remarked in the proof of (5.23), J_+ on $O_-(N)$ is integrable if and only if $Q = 0$ on $O_-(N)$. This, combined with (6.6) and (6.4), shows that (Z_-, g_t, J_+) is symplectic. Hence b) implies c).

A fortiori, c) implies a). □

Remarks (6.9) A similar result, with evident modifications, holds for (Z_+, g_t, J_+).

(6.10) This theorem and the next generalize Theorem 9.1 in [ES] where N was assumed to be S^4 with its canonical metric.

(6.11) By (4.23), if N is self–dual Einstein with positive scalar curvature s, then g_t on Z_- is Einstein when $t^2 = 6/s$. By our theorem, in this case (Z_-, g_t, J_+) is not even (1,2)–symplectic. (Cf. [FG]).

THEOREM 6.2 Let N be an oriented Riemannian 4–manifold. Then (Z_-, g_t, J_-) is (1,2)–symplectic if and only if N is self–dual Einstein (for any value of $t > 0$), while (Z_-, g_t, J_-) is symplectic if and only if N is self–dual Einstein with negative scalar curvature s and $t^2 = -12/s$.

PROOF The proof is similar to that of theorem 6.1 except that now the type (1,0) forms are the pull backs of $\varphi^1, \varphi^2, \bar{\varphi}^3$ to Z_- by local sections u. Thus, by (6.4), (Z_-, g_t, J_-) is (1,2)–symplectic if and only if $R = S = T = U = Q = 0$. As in the above proof, this is equivalent to N being self–dual Einstein. In this case $d\kappa_-(t) =$ $i(1 + \frac{t^2 s}{12})(\varphi^3\wedge\bar{\varphi}^1\wedge\bar{\varphi}^2 - \varphi^1\wedge\varphi^2\wedge\bar{\varphi}^3)$, which can be zero if and only if $s < 0$ and $t^2 = -12/s$. □

Remarks (6.12) By results of Friedrich and Kurke [FK] and Hitchin [H], if N is a compact 4–dimensional self–dual Einstein space with positive scalar curvature, then it is isometric to either S^4 or $\mathbb{C}P^2$ with their canonical metrics. In these cases Z_- is $\mathbb{C}P^3$ or the flag manifold $\mathbb{F}(1,2)$, respectively. There is no known classification of 4–dimensional self–dual Einstein spaces with negative scalar curvature [V]. Examples are hyperbolic space and Hermitian hyperbolic space with their canonical metrics.

(6.13) The relevance to us of (1,2)–symplectic spaces comes from the result of Lichnerowicz [L]: If $f:M \to N$ is a holomorphic map from a Riemann surface to an almost Hermitian (1,2)–symplectic manifold, then f is harmonic.

§7 The Grassmann bundle. We briefly describe the geometry of the Grassmann bundle of oriented 2–planes tangent to N, $G : G_2(TN) \to N$. An element of $G_2(TN)$ is a pair (p,ξ) where $p \in N$ and ξ is a two–dimensional oriented subspace of T_pN. The Grassmann projection

(7.1) $$G : G_2(TN) \to N$$

is defined by $G(p,\xi) = p$. The projection (7.1) presents $G_2(TN)$ as a fiber bundle over N with standard fiber the Grassmann manifold $\tilde{G}_2(4) = SO(4)/SO(2)\times SO(2)$. We define a map

(7.2) $$\mu : O(N) \to G_2(TN)$$

by $\mu(e) = \{e_1,e_2\}$, where $e = (e_1,\ldots,e_4)$ is an orthonormal frame at a point $p \in N$ and $\{e_1,e_2\}$ is the oriented plane in T_pN spanned by e_1,e_2 with the orientation $e_1\wedge e_2$. Thus

$$G_2(TN) \cong O(N)/SO(2)\times O(2).$$

Notice that μ restricted to $O_\pm(N)$ (which we denote $\mu_\pm$) is a principal $SO(2)\times SO(2)$ bundle. The fibers of μ are the integral submanifolds of the completely integrable system $\theta = 0$, $\zeta = 0$, where $\zeta = \begin{bmatrix} 0 & \omega^i_\alpha \\ \omega^\alpha_i & 0 \end{bmatrix}$.

For $t>0$ the Riemannian metric h_t on $G_2(TN)$ is characterized by

$$\mu^* h_t = P_t \,, \tag{7.3}$$

where P_t is the $O(2)\times O(2)$ invariant symmetric bilinear form on $O(N)$ given by

$$P_t = {}^t\theta\theta + \tfrac{1}{2}\langle t\zeta, t\zeta\rangle \,. \tag{7.4}$$

In terms of components, we have

$$P_t = \Sigma\,(\theta^a)^2 + t^2(\omega^\alpha_i)^2 \,. \tag{7.5}$$

An orthonormal frame for P_t is given by

$$\theta^a \,,\ \ \theta^{\alpha i} = t\omega^\alpha_i \,. \tag{7.6}$$

If $u : U \to O(N)$ is a local section of (7.2), then an orthonormal coframe for h_t on U is given by

$$u^*\theta^a \,,\ \ u^*\theta^{\alpha i} \,. \tag{7.7}$$

From the structure equations of $O(N)$ we find that the pull–back by u^* of the forms

$$\begin{aligned} \theta^a_b &= \omega^a_b + \frac{t^2}{2} R^\alpha_{i\,ab}\,\omega^i_\alpha \\ \theta^{\alpha i}_b &= -\frac{t}{2} R^\alpha_{i\,ab}\,\theta^a = -\theta^b_{\alpha i} \\ \theta^{\alpha i}_{\beta j} &= \delta_{\alpha\beta}\,\omega^i_j + \delta_{ij}\,\omega^\alpha_\beta \end{aligned} \tag{7.8}$$

gives the Levi–Civita connection forms of h_t with respect to the orthonormal frame (7.7).

From (7.8) one can compute the Riemann curvature tensor of h_t and its Ricci tensor. In contrast to the twistor bundle case of §4, there is no value of t for which h_t is Einstein.

We consider now two natural almost complex structures $J^G_\pm$ on the Grassmann bundle $G_2(TN)$. Using the coframe field (7.6) on $O_+(N)$ we let

$$\varphi^1 = \theta^1 + i\theta^2,\ \varphi^2 = \theta^3 + i\theta^4,\ \varphi^\alpha_G = \theta^{\alpha 1} + i\theta^{\alpha 2} = \omega^\alpha_1 + i\omega^\alpha_2 \,, \tag{7.9}$$

complex valued 1–forms on $O_+(N)$. Then J_+^G (respectively, J_-^G) is defined by the condition that its type (1,0) forms are locally spanned by the pull–back of φ^j, φ_G^α (respectively, φ^j, $\overline{\varphi}_G^\alpha$) by any local section u of μ_+ of (7.2).

Using the structure equations of $O(N)$ we find

$$\begin{aligned} d\varphi^1 &= \tfrac{1}{2}(\varphi_G^3 - i\varphi_G^4)\wedge\varphi^2 + \tfrac{1}{2}(\varphi_G^3 + i\varphi_G^4)\wedge\overline{\varphi}^2 \\ d\varphi^2 &= -\tfrac{1}{2}(\overline{\varphi}_G^3 + i\overline{\varphi}_G^4)\wedge\varphi^1 - \tfrac{1}{2}(\varphi_G^3 + i\varphi_G^4)\wedge\overline{\varphi}^1 \end{aligned} \tag{7.10}$$

$$d\varphi_G^\alpha = -i\varphi_G^\alpha\wedge\omega_2^1 - \omega_\beta^\alpha\wedge\varphi_G^\beta + \Omega_1^\alpha + i\Omega_2^\alpha . \tag{7.11}$$

From (7.10) and (7.11) if follows that J_-^G is never integrable, while J_+^G is integrable if and only if

$$\Omega_1^\alpha + i\Omega_2^\alpha \equiv 0 \mod (\varphi^1, \varphi^2) . \tag{7.12}$$

Proposition 7.1 Let N be an oriented Riemannian 4–manifold. Then J_+^G on $G_2(TN)$ is integrable if and only if N is anti–self–dual Einstein.

Proof We need to see that (7.12) holds if and only if N is anti–self–dual Einstein. By (3.6) and (5.20)

$$\begin{aligned} \Omega_3^1 + i\Omega_3^2 &\equiv \tfrac{1}{4}(A_{22} - A_{33} + B_{22} + B_{33} + i(2A_{23} + B_{23} - B_{32}))\overline{\varphi}^1\wedge\overline{\varphi}^2 \\ \Omega_4^1 + i\Omega_4^2 &\equiv \tfrac{1}{4}(2A_{23} + B_{32} - B_{23} + i(A_{33} - A_{22} + B_{22} + B_{33}))\overline{\varphi}^1\wedge\overline{\varphi}^2 \\ &\mod (\varphi^1, \varphi^2). \end{aligned} \tag{7.13}$$

Hence (7.12) holds if and only if

$$\begin{aligned} A_{22} - A_{33} + B_{22} + B_{33} = 0 = 2A_{23} + B_{23} - B_{32} \\ 2A_{23} + B_{32} - B_{23} = 0 = A_{33} - A_{22} + B_{22} + B_{33} \end{aligned}$$

which holds if and only if $A_{23} = 0$, $A_{22} = A_{33}$, $B_{23} = B_{32}$, and $B_{22} + B_{33} = 0$.

By (3.10) this holds if and only if $A = \frac{S}{12} I$, and $B = 0$ on $O_+(N)$, which by (3.19) and (3.14) holds if and only if N is anti–self–dual Einstein. □

Remark Two other almost complex structures, $\tilde{J}^G_\pm$, can be defined on $G_2(TN)$ by pulling back with any positively oriented frame field the forms $\varphi^1, \bar{\varphi}^2, \varphi^\alpha_G$ or $\varphi^1, \bar{\varphi}^2, \bar{\varphi}^\alpha_G$ respectively. It is easily seen that this is equivalent to the structures obtained by pulling back $\varphi^1, \varphi^2, \varphi^\alpha_G$ (respectively, $\varphi^1, \varphi^2, \bar{\varphi}^\alpha_G$) by negatively oriented frame fields $e: U \subset G_2(TN) \to O_-(N)$. In fact, if $K = \mathrm{diag}(1,1,1,-1)$, then $R_K: O_-(N) \to O_+(N)$, $R_K \circ e: U \to O_+(N)$, and $R_K^* \varphi^1 = \varphi^1$, $R_K^* \varphi^2 = \bar{\varphi}^2$, $R_K^* \varphi^3_G = \varphi^3_G$, $R_K^* \varphi^4_G = -\varphi^4_G$. It follows that $\tilde{J}^G_-$ is never integrable, while $\tilde{J}^G_+$ is integrable if and only if N is self–dual Einstein.

§8 Twistor and Gauss lifts. Let $f:M \to N$ be an isometric immersion of an oriented surface into an oriented Riemannian 4–manifold. We define projections

$$\pi_\pm : G_2(TN) \to Z_\pm \tag{8.1}$$

of the Grassmann bundle of oriented tangent 2–planes of N onto the respective twistor spaces as follows. If $\zeta \subset T_pN$ is an oriented 2–dimensional subspace, then $\pi_+(p,\zeta)$ is the almost complex structure on T_pN given by the positive twist (i.e., rotation through $+\pi/2$) in each of ζ and its orthogonal complement $\zeta^\perp$ (with induced orientation from ζ and T_pN); while $\pi_-(p,\zeta)$ is the positive twist in ζ but the negative twist in $\zeta^\perp$. Observe that μ of (7.2) and $\sigma_\pm$ of (4.9) are related to $\pi_\pm$ by $\sigma_\pm = \pi_\pm \circ \mu_\pm$.

The twistor lifts of f,

$$\varphi_\pm : M \to Z_\pm, \tag{8.2}$$

are defined by: $\varphi_+(p)$ is the positive twist in $f_* T_pM$ and in $f_* T_pM^\perp$, while $\varphi_-(p)$ is the positive twist in $f_* T_pM$ but the negative twist in $f_* T_pM^\perp$. Thus $\varphi_\pm = \pi_\pm \circ \gamma_f$, where

$$\gamma_f : M \to G_2(TN)$$

is the Gauss lift: $\gamma_f(p) = f_* T_pM$ (with its orientation from M). These maps are illustrated by the commutative diagram

$$
(8.3)\qquad
\begin{array}{ccccccc}
O_-(N) & \overset{\mu_-}{\nwarrow} & & & \overset{\mu_+}{\nearrow} & O_+(N) \\
\sigma_- \downarrow & \overset{\pi_-}{\swarrow} & G_2(TN) & \overset{\pi_+}{\searrow} & & \downarrow \sigma_+ \\
 & & \uparrow \gamma_f & & & \\
Z_- & \xleftarrow{\varphi_-} & M & \xrightarrow{\varphi_+} & & Z_+ \\
T \nwarrow & & f \downarrow & & \nearrow T & \\
 & & N & & &
\end{array}
$$

Recall the almost complex structures $J_\pm$ defined on Z in §5. The following result was first proved in [ES], Theorem 5.3.

Proposition 8.1 a) $\varphi_\pm$ is J_+ holomorphic if and only if f is isotropic with $\pm$ spin, respectively. b) $\varphi_\pm$ is J_- holomorphic if and only if f is minimal.

Proof Let $e = (e_1,\dots,e_4):U\subset M \to O_+(N)$ be a local oriented Darboux frame along f (see §2), and let $e_- = (e_1,e_2,e_3,-e_4) = R_K \circ e$, where $K = \mathrm{diag}(1,1,1,-1)$. We may assume the existence of local sections $u_\pm : Z_\pm \to O_\pm(N)$ such that $e_\pm = u_\pm \circ \varphi_\pm$, respectively. Thus (see (2.17) and (5.16))

$$
(8.4)\qquad
\begin{aligned}
&\varphi_+^* u_+^* \varphi^1 = e^*\varphi^1 = \varphi, \qquad \varphi_+^* u_+^* \varphi^2 = e^*\varphi^2 = 0,\\
&\varphi_+^* u_+^* \varphi^3 = e^*\varphi^3 = -\tfrac{1}{2}\bar b\varphi - \tfrac{1}{2}\bar S_+\bar\varphi;
\end{aligned}
$$

while

$$
(8.5)\qquad
\begin{aligned}
&\varphi_-^* u_-^* \varphi^1 = e_-^*\varphi^1 = e^* \overset{*}{R_K}\varphi^1 = \varphi\ , \qquad \varphi_-^* u_-^* \varphi^2 = 0\ ,\\
&\varphi_-^* u_-^* \varphi^3 = e^* \overset{*}{R_K}\varphi^3 = -\tfrac{1}{2}b\varphi - \tfrac{1}{2}\bar S_-\bar\varphi
\end{aligned}
$$

since $\overset{*}{R_K}\varphi^1 = \varphi^1$, $\overset{*}{R_K}\varphi^2 = \varphi^2$, and $\overset{*}{R_K}\varphi^3 = \frac{1}{2}(\omega_3^1 + \omega_4^2) + \frac{i}{2}(\omega_3^2 - \omega_4^1)$. Thus a) follows from (8.4) and (8.5), respectively, while b) follows from (8.4) and (8.5) with φ^3 replaced by $\bar\varphi^3$. □

Recall from §6 that a unitary coframe for $(Z, J_\pm, g_t)$ is given (in $O(N)$) by $\varphi^1, \varphi^2, 2t\varphi^3$. By (8.4), using (2.17) and (2.18), we have

$$
(8.6)\qquad
\begin{aligned}
&\varphi_+^* g_t = (1 + t^2\|H\|^2 + 2t^2 s_+^2)\varphi\bar\varphi + t^2\bar b S_+\varphi\varphi + t^2 b\bar S_+\bar\varphi\bar\varphi\\
&\varphi_-^* g_t = (1 + t^2\|H\|^2 + 2t^2 s_-^2)\varphi\bar\varphi + t^2 b S_-\varphi\varphi + t^2\bar b\bar S_-\bar\varphi\bar\varphi.
\end{aligned}
$$

These calculations prove the following.

PROPOSITION 8.2 Let $f:M \to N$ be an isometric immersion of an oriented surface into an oriented Riemannian 4–manifold. Let $\varphi_\pm:M \to Z_\pm$ be its twistor lifts. Let g_t be the Hermitian metric on $Z_\pm$ of §6. Then
(i) $\varphi_\pm$ is conformal if and only if either f is minimal or f is isotropic with $\pm$ spin, respectively;
(ii) $\varphi_\pm$ is an isometry if and only if f is minimal and isotropic with $\pm$ spin, respectively.

Let $\kappa_\pm$ be the Kaehler forms (6.2) of $(Z, J_\pm, g_t)$, respectively. From (8.4) and (8.5) we have

$$(8.7)\qquad \begin{aligned}\varphi_\pm^*\kappa_+ &= (1 + t^2(\|H\|^2 - 2s_\pm^2))dA\\ \varphi_\pm^*\kappa_- &= (1 - t^2(\|H\|^2 - 2s_\pm^2))dA\,.\end{aligned}$$

Consequently

$$(8.8)\qquad \begin{aligned}\varphi_\pm^*(\kappa_+ + \kappa_-) &= 2dA\\ \varphi_\pm^*(\kappa_+ - \kappa_-) &= 2t^2(\|H\|^2 - 2s_\pm^2)dA\,.\end{aligned}$$

Combining this with (2.20), we have

$$(K - R_{1212})dA = \frac{1}{2t^2}(\varphi_+^*\kappa_+ + \varphi_-^*\kappa_+) - t^{-2}dA = \frac{-1}{2t^2}(\varphi_+^*\kappa_- + \varphi_-^*\kappa_-) + t^{-2}dA$$
$$(K^\perp - R_{1234})\,dA = \frac{1}{2t^2}(\varphi_+^*\kappa_+ - \varphi_-^*\kappa_+) = \frac{-1}{2t^2}(\varphi_+^*\kappa_- - \varphi_-^*\kappa_-)\,.$$

The basic contact invariants $\|H\|$ and $s_\pm$ introduced in (2.18) can be interpreted in terms of the (1.0) and (0,1) energy densities of $\varphi_\pm$. If Z has the (J_+,g_t) structure (respectively the (J_-, g_t) structure), then from (8.4) and (8.5), we have

$$\begin{aligned}e_+'(\varphi_\pm) &= 1 + t^2\|H\|^2\\ e_+''(\varphi_\pm) &= 2t^2s_\pm^2\,,\end{aligned}\quad \text{, respectively}\quad \begin{aligned}e_-'(\varphi_\pm) &= 1 + 2t^2s_\pm^2\\ e_-''(\varphi_\pm) &= t^2\|H\|^2\,.\end{aligned}$$

We conclude this section by computing the tension field of φ_-. An analogous result holds for φ_+. As already remarked in §6, $(Z, J_\pm)$ in general is not (1,2)–symplectic (cf. Theorems 6.1 and 6.2). Nevertheless, we know examples (cf. [G] and [ES]) in which a suitable holomorphic map of a Riemann surface into a Hermitian, not (1,2)–symplectic, manifold is still harmonic. It is therefore instructive to face the problem directly and compute the tension of φ_-. One benefit of this is that it does allow us to discover a new feature for φ_-, even in a case for which (Z, J_+, g_t) is (1,2)–symplectic.

Let $u:U\subset Z_- \to O_-(N)$ be a local section of $\sigma_-:O_-(N) \to Z_-$. Recall from §4 that the Riemannian metric g_t on Z_- has a local orthonormal coframe field given by applying u^* to

$$\theta^1,\ldots,\theta^4,\ t(\omega^1_3 - \omega^2_4),\ t(\omega^1_4 + \omega^2_3) \tag{8.12}$$

on $O_-(N)$. For our present calculation it is convenient to modify (4.18) slightly and define

$$\theta^5 = t(\omega^1_3 - \omega^2_4)\ ,\ \ \theta^6 = t(\omega^1_4 + \omega^2_3)\ . \tag{8.13}$$

We may choose u and a negatively oriented Darboux frame e along f such that $e = u\circ\varphi_-$, which means that $\varphi_-^* u^*\theta^p = e^*\theta^p$, $p = 1,\ldots,6$. Let $\theta^j = e^*\theta^j$, an oriented orthonormal coframe in M, and let $e^*\theta^p = B^p_j\theta^j$, where the B^p_j are locally defined functions in M. If $\{E_p\}$ is the orthonormal frame field in Z_- dual to $\{u^*\theta^p\}$, then

$$d\varphi_- = B^p_j\theta^j\otimes E_p.$$

Thus $\nabla d\varphi_- = B^p_{jk}\theta^j\theta^k\otimes E_p$, where

$$dB^p_j - B^p_j\omega^j_k + B^q_j\theta^p_q = B^p_{jk}\theta^k. \tag{8.14}$$

Here θ^p_q are the Levi–Civita connection forms of g_t listed in (4.19), appropriately modified as required by replacing (4.18) with (8.13). The tension field of φ_- is then given by

$$\tau(\varphi_-) = B^p_{jj}E_p.$$

We proceed to make these calculations.

From §2 we have $e^*\theta^j = \theta^j$, $e^*\theta^\alpha = 0$, $e^*\theta^5 = t(h^4_{2j} - h^3_{1j})\theta^j$, and $e^*\theta^6 = -t(h^4_{1j} + h^3_{2j})\theta^j$, from which it follows that

$$B^j_k = \delta^j_k\ ,\ \ B^\alpha_k = 0\ ,\ \ B^5_k = t(h^4_{2k} - h^3_{1k})\ ,\ \ B^6_k = -t(h^4_{1k} + h^3_{2k}). \tag{8.15}$$

Carrying out the calculations of (8.14) and using the notation of §3, we find (summing on k)

$$B^1_{kk} = t^2\{-(A_{21} + B_{12})(h^4_{22} - h^3_{12}) + (A_{31} + B_{13})(h^4_{12} + h^3_{22})\}$$
$$B^2_{kk} = t^2\{(A_{21} + B_{12})(h^4_{21} - h^3_{11}) - (A_{31} + B_{13})(h^4_{11} + h^3_{21})\}$$
$$B^3_{kk} = h^3_{kk} + t^2\{(A_{22} + B_{22})(h^4_{21} - h^3_{11}) + (A_{32} - B_{32})(h^4_{22} - h^3_{12}) -(A_{32} + B_{23})(h^4_{11} + h^3_{21}) - (A_{33} - B_{33})(h^4_{12} + h^3_{22})\}$$
$$B^4_{kk} = h^4_{kk} + t^2\{(A_{32} + B_{32})(h^4_{21} - h^3_{11}) + (B_{22} - A_{22})(h^4_{22} - h^3_{12}) -(A_{33} + B_{33})(h^4_{11} + h^3_{21}) - (B_{23} - A_{32})(h^4_{12} + h^3_{22})\}$$
$$B^5_{kk} = t\{2(H^4_2 - H^3_1) - A_{31} - B_{13}\}$$
$$B^6_{kk} = -t\{2(H^4_1 + H^3_2) - A_{21} - B_{12}\}.$$

Recall (3.14), that N is Einstein if and only if $B = 0$ on $O(N)$, and (3.19), that N is self–dual if and only if $A = \frac{s}{12} I$ on $O\_(N)$, where s is the scalar curvature of N. Thus if N is self–dual Einstein, then

$$B^j_{kk} = 0 \qquad B^\alpha_{kk} = 2H^\alpha(1 - t^2 s/12)$$
$$B^5_{kk} = 2t(H^4_2 - H^3_1) \qquad B^6_{kk} = -2t(H^4_1 + H^3_2).$$

These calculations and their analogs for φ_+ yield the following results.

THEOREM 8.1 Let $f:M \to N$ be an isometric immersion of a Riemann surface into a 4–dimensional self–dual (respectively, anti–self–dual) Einstein manifold with scalar curvature s, twistor space (Z, g_t) and twistor lifts $\varphi_\pm:M \to Z_\pm$.

a) If $st^2 \neq 12$, then f is minimal if and only if φ_- (respectively, φ_+) is harmonic;

b) If $st^2 = 12$, then φ_- (respectively, φ_+) is harmonic if and only if $H^4_1 = -H^3_2$ and $H^4_2 = H^3_1$; i.e., $\nabla H_{(p)}:T_pM \to T_pM^\perp$ is complex.

Remarks

(8.16) Suppose N is self–dual Einstein. Then by Theorem 6.2, $(Z\_, g_t, J\_)$ is (1,2)–symplectic for any $t > 0$. Thus, by (6.13) and Proposition 8.1, if f is minimal, then φ_- is J_–holomorphic, thus harmonic.

(8.17) If N is compact self–dual Einstein with $s > 0$, then it must be S^4 or $\mathbb{C}P^2$ with their canonical metrics (cf. Remark (6.12)). The twistor space Z_ of S^4 is $\mathbb{C}P^3$, and its metric g_t with $t^2 = 12/s$ is the Fubini–Study metric on $\mathbb{C}P^3$. Thus part b) of our theorem parametrizes a class of harmonic maps $\varphi_-:M \to \mathbb{C}P^3$ by immersed surfaces $f:M \to S^4$ whose mean curvature vector H satisfies $H_1^4 = -H_2^3$ and $H_2^4 = H_1^3$. As a consequence, there is a large class of harmonic maps $\varphi_-:M \to \mathbb{C}P^3$ which are not J_--holomorphic.

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DEPARTMENT of MATHEMATICS
CAMPUS BOX 1146
WASHINGTON UNIVERSITY
ST. LOUIS, MO 63130

INTERNATIONAL CENTER for
THEORETICAL PHYSICS
STRADA COSTIERA 11, MIRAMARE
34100 TRIESTE, ITALY

Current Address:
Dipartimento di Matematica
Citta Universitaria
Viale A. Doria 6
95125 Catania, Italy