Math 5031 - Homework 2

Due 9/16/05

1. **Semidirect products**. We define G to be a *semidirect product* of subgroups H and N if N is normal, $G = NH$ and $H \cap N = \{e\}$.

   (a) Let G be a group and let H, N be subgroups with N normal. Let γ_x be conjugation by an element $x \in G$. Show that $x \to \gamma_x$ induces a homomorphism $f : H \mapsto \mathrm{Aut}(N)$.

   (b) If $H \cap N = \{e\}$, show that the map $H \times N \to HN$ given by $(x, y) \mapsto xy$ is a bijection, and that this map is an isomorphism if and only if f is trivial, i.e., $f(x) = \mathrm{id}_N$ for all $x \in H$.

   (c) Conversely, let N and H be groups, and let $\psi : H \to \mathrm{Aut}(N)$ be a given homomorphism. Construct a semidirect product as follows. Let G be the set of pairs $(x, h) \in N \times H$. Define the composition law

   $$(x_1, h_1)(x_2, h_2) = (x_1 \psi_1(h_1) x_2, h_1 h_2).$$

   Show that this is a group law, and yields a semidirect product of N and H, identifying N with the set of elements $(x, 1)$, and H with the set of elements $(1, h)$.

   (d) Suppose that N and H are both normal subgroups of G and that the orders of N and H are relatively prime. Prove that HN a subgroup of G isomorphic to the direct product $H \times N$.

   (e) Let G be a finite group and let N be a normal subgroup such that N and G/N have relatively prime orders. Let H be a subgroup of G having the same order as G/H. Show that G is the semidirect product of N and$H$. Also show that if σ is any automorphism of G, then $\sigma(N) = N$.

2. **Group actions**. We say that a group action is *transitive* if it has a single orbit.

   (a) Show that a transitive action of a group G on a set X is equivalent to the action of G on the right-coset space G/H by left-translations.

   (b) Let G be a group acting transitively on a finite set X, where $\#X \geq 2$. Prove that there exists an element g of G which has no fixed point, i.e., $gx \neq x$ for all $x \in X$. (Hint: first prove the next item.)

   (c) Let H be a proper subgroup of a finite group G. Show that G is not the union of all the conjugates of H.