FANO 3-FOLDS AND CLASSIFICATION OF CONSTANTLY CURVED HOLOMORPHIC 2-SPHERES OF DEGREE 6 IN THE COMPLEX GRASSMANNIAN $G(2,5)$

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Abstract. Up to now the only known constantly curved sextic curve, i.e., holomorphic 2-sphere of degree 6, in the complex $G(2,5)$ has been the first associated curve of the Veronese curve of degree 4, which indicates that such curves are rare to find. Exploring the rich interplay between the ramification of harmonic sequences in differential geometry and algebro-geometric properties of projectively equivalent Fano 3-folds of index 2 and degree 5, we invoke the moduli space structure of sextic curves in the Fano 3-fold often referred to as V_5 to confirm the rarity of constancy of curvature, by establishing that the harmonic sequence of a generic sextic curve in $G(2,5)$ is totally unramified. This paper proposes to investigate from the Galois viewpoint the way ramification can appear in relation to the constancy of curvature among nongeneric sextic curves in $G(2,5)$.

To this end, we break it into two cases. The first is when the sextic curve is $GL(5,\mathbb{C})$-equivalent to a curve $\gamma \subset V_5$ not living in the PSL_2-invariant tangent developable surface $\mathbf{S}$ of V_5, where we may lift γ to a Galois cover in the $\mathbb{C}P^3$ containing PSL_2. By studying the branch points of the Galois covering in connection with the intersection of γ and $\mathbf{S}$ in V_5, we categorize such γ further into two sub-families, namely, the family consisting of those γ ramified at the singular locus of $\mathbf{S}$ somewhere, to be labeled as the *generally ramified family*, and the family complementary to it. In the second case when the 2-sphere is $GL(5,\mathbb{C})$-equivalent to a γ living in $\mathbf{S}$, we show by the PSL_2-invariant theory that it necessarily belongs to the generally ramified family.

We prove through elaborate PSL_2-transvectant and engaged unitary analyses that, up to the ambient unitary equivalence, the moduli space of constantly curved sextic curves in $G(2,5)$ that are $GL(5,\mathbb{C})$-equivalent to those in the generally ramified family, is semi-algebraic of dimension 2, all members of which barring the above Veronese curve are nonhomogeneous. Many explicit examples can be constructed.

We also outline in general the structure of the Galois covers of the sextic curves in the family complementary to the generally ramified family. It appears to suggest, through all classes of rational Galois covers we have completely classified, each dependent on a single parameter, that the constantly curved sextic curves in $G(2,5)$ that are $GL(5,\mathbb{C})$-equivalent to the ones in this family, be nongeneric among all constantly curved ones in $G(2,5)$.

MSC(2020): Primary 53C42, 53C55; Secondary 14M15, 14J45, 14H45

Keywords. holomorphic curves, Gaussian curvature, linear section, quintic del Pezzo 3-fold

1. Introduction

Minimal surfaces constitute one of the most enduring topics in Differential Geometry that not only enjoys its deep links with partial differential equations, complex analysis, and algebraic curves, but also finds intriguing connections to the physical world. In 1980, Din and Zakrzewski [16] classified complex projective σ-models, or, mathematically, harmonic maps

This work was partially supported by NSFC No. 12171473 for the last two authors. The second author was also partially supported by the Fundamental Research Funds for Central Universities, and the third author was also partially supported by by NSFC No. 12301065, the Fundamental Research Funds for the Central Universities, Nankai University (63241426) and Natural Science Foundation of Tianjin (22JCQNJC00880).

from the 2-sphere to the ambient projective space, to be the (projectivized) basis elements of a Frenet frame of a holomorphic $\mathbb{C}P^1$ into the ambient space. Subsequently, Burstall and Wood [6], Chern and Wolfson [10], and Uhlenbeck [40] independently generalized it to other ambient spaces by different methods.

Of all minimal surfaces, those of constant curvature in different ambient spaces form a model class that have continually drawn attention, such as Calabi [8], Wallach [41], Do Carmo-Wallach [18], Chen [9], Barbosa [2], Kenmotsu [26], and Bryant [5] in the real space forms, Kenmotsu [27], Bando-Ohnita [1], Bolton-Jensen-Rigoli-Wood [3], Chi-Jensen-Liao [11], and Kenmotsu [28] in the complex projective spaces, and Yau [43] in Kähler manifolds of nonnegative constant holomorphic sectional curvature. In particular, constantly curved minimal 2-spheres in the real space forms are Borůvka spheres [4], up to rigid motion. Similarly, constantly curved minimal 2-spheres in the complex projective spaces are, up to rigid motion, the (projectivized) basis elements of the Frenet frame of the Veronese curve of constant curvature, where the proof followed from Calabi's rigidity principle [7] that states that if the isometric embedding from one complex manifold into the complex projective space exists, then it is unique up to rigid motion.

The rigidity principle of Calabi no longer holds for general ambient spaces. Motivated by the Grassmannian σ-models introduced by Din and Zakrzewski [17] and the rigidity principle, the first named author and Zheng [13] classified the noncongruent, constantly curved holomorphic 2-spheres of degree 2 in $G(2,4)$ into two 1-parameter families, by exploring the method of moving frames and Cartan's theory of higher order invariants [22]. Later on, Li and Yu [31] classified all constantly curved minimal 2-spheres in $G(2,4)$, using the Plücker embedding and the theory of harmonic sequences.

The next simplest ambient space is the complex Grassmannian $G(2,5)$. By analyzing a 2×5 matrix representation of a holomorphic $\mathbb{C}P^1$, constantly curved holomorphic 2-spheres in $G(2,5)$ are divided into two classes by Jiao and Peng, the *singular* and the *nonsingular* ones (a technical condition different from the usual geometric meaning, see Section 2.2 for definition). They classified nonsingular constantly curved holomorphic 2-spheres of degree less than or equal to 5 in $G(2,5)$, and proved the nonexistence of such spheres with degree $6 \leq d \leq 9$ [23, 24]. For the singular category, however, as the degree increases the computational complexity involved in their method rises dramatically. It is thus technically difficult to apply the method to construct singular 2-spheres in general. Subsequently, there have emerged several partial classifications (e.g. under the condition of total unramification or homogeneity) of constantly curved holomorphic (minimal) 2-spheres in $G(2,5)$ or $G(2,n)$ in general; see [**?**, 34] and the references therein.

Constantly curved holomorphic 2-spheres in $G(2,4)$ and $G(2,5)$ have also been studied by Delisle, Hussin and Zakrzewski in [15] from the viewpoint of Grassmannian σ-models, where the classification results they obtained coincide with those mentioned above. Moreover, they posed a conjecture about the upper bound of the degrees of constantly curved holomorphic 2-spheres in the Grassmannians. This conjecture was affirmed by them in the case of $G(2,5)$, for which the upper bound equals 6 (see also a recent paper [21] with more detailed proof by He).

At the critical degree $d = 6$, however, there does exist a singular (in the above sense) constantly curved holomorphic 2-sphere of degree 6 in $G(2,5)$, namely,

$$\begin{pmatrix} 1 & 2z & \sqrt{6}z^2 & 2z^3 & z^4 \\ 0 & 1 & \sqrt{6}z & 3z^2 & 2z^3 \end{pmatrix}, \tag{1.1}$$

referred to in this paper as the *standard* Veronese curve in $G(2,5)$. To the authors' knowledge, it has been the only known example in the literature. Surprisingly, we will show in this paper that the moduli space of constantly curved holomorphic 2-spheres in $G(2,5)$ has a 2-dimensional semialgebraic component, modulo rigid motion, out of which many explicit examples can be constructed.

Different from all existing methods, to see whether there are constantly curved holomorphic examples of degree 6 other than the standard Veronese curve in $G(2,5)$, let us return to our paper [12] for motivation, where we investigated constantly curved holomorphic (and minimal) 2-spheres of degree d in the complex hyperquadric. Such a holomorphic 2-sphere is a rational normal curve of degree d sitting in a projective d-plane, so that the 2-sphere lies in the intersection of the d-plane and the hyperquadric called a *linear section* of the hyperquadric, which is itself a quadric (may be singular). Thus, the moduli space of such 2-spheres is essentially a fibered space over the base space that is a semialgebraic subset of the variety of linear sections of the hyperquadric.

In the same vein, albeit more sophisticated, via the Plücker embedding, a holomorphic 2-sphere of degree 6 contained in $G(2,5) \subset \mathbb{C}P^9$ is a rational normal curve (a sextic curve) sitting in a projective 6-plane $\mathbf{L}$ in $\mathbb{C}P^9$; thus, the curve lies in the linear section $\mathbf{L} \cap G(2,5)$. Castelnuovo [14] showed that generic (see Section 3.1 for definition) such linear sections constitute the intriguing class of Fano 3-folds of index 2 and degree 5 all of which are projectively equivalent (see also [36] for a detailed modern account and Section 3.1 for a quick overview).

Employing $PSL(2,\mathbb{C})$-representations, Mukai and Umemura [33] constructed a beautiful Fano 3-fold of index 2 and degree 5, to be denoted by $\mathcal{H}_0^3$ henceforth (often denoted by V_5 in the literature), which can be identified naturally with the linear section of $G(2,5)$ cut out by the 6-plane $\mathbf{L}_0$ containing the above standard Veronese curve, whose tangent developable surface $\mathbf{S} \subset \mathcal{H}_0^3$ plays a crucial role in the sequel, where $\mathbf{L}_0$ turns out to be precisely the projectivization $\mathbb{P}(V_6)$ of the irreducible $PSL(2,\mathbb{C})$-module V_6 of dimension 7. This fits ideally in our differential-geometric framework for computation when the condition of constant curvature is engaged.

Recall a holomorphic curve $F : M \to G(2,5) \subset \mathbb{C}P^9$ is *unramified* at p if the tangent line to F at p does not lie entirely in $G(2,5)$, in which case F is *totally unramified* at p if the curve $[dF \wedge dF] \subset \mathbb{C}P^4$ is unramified as a projective curve at p. Now, transforming by $GL(5,\mathbb{C})$, we can use the sextic curves in $\mathcal{H}_0^3$ to parameterize holomorphic 2-spheres of degree 6 in $G(2,5)$. Takagi and Zucconi's work on the Moduli space (Hilbert scheme) [38, 39] of sextic curves in $\mathcal{H}_0^3$, in which the intersection properties between a general sextic curve in $\mathcal{H}_0^3$ and lines and conics were investigated, turns out to characterize the total unramification of harmonic sequences (see Theorem 4.1), from which we obtain that generic holomorphic 2-spheres of degree 6 in $G(2,5)$ are not of constant curvature (see Theorem 4.2).

To find constantly curved nongeneric holomorphic 2-spheres of degree 6 in $G(2,5)$, we explore the way ramification occurs from the standpoint of Galois covers. We approach this by separating the analysis into two distinct cases. When a sextic curve $\gamma \subset \mathcal{H}_0^3$ does not live in $\mathbf{S}$, by exploring Mukai and Umemura's orbit decomposition structure of $\mathcal{H}_0^3$, we may lift γ to a Galois cover in the natural $\mathbb{C}P^3$ containing $PSL(2,\mathbb{C})$ (see Section 5.2). Studying the Galois covering at its branch points that cover the points of intersection of $\mathbf{S}$ and the sextic curve, enables us to categorize all sextic curves not contained in $\mathbf{S}$ into two classes, namely, the more flexible one labeled as the *generally ramified family*, consisting of those sextic curves ramified at the singular locus of $\mathbf{S}$ somewhere, and the more rigid complementary one labeled as the *exceptional transversal family.*

In contrast, when a sextic curve γ lives in $\mathbf{S}$ but not in its singular locus, through the PSL_2-invariant theory, we may explicitly lift γ to a line in the same $\mathbb{C}P^3$, so that in particular γ also falls in the generally ramified family.

It turns out that a constantly curved holomorphic 2-sphere of degree 6, $GL(5,\mathbb{C})$-equivalent to a sextic curve lying in the generally ramified family, spans a 6-plane $\mathbf{L}$ differing from $\mathbf{L_0}$ by a diagonal transformation of $GL(5,\mathbb{C})$. This is done through elaborate PSL_2-invariant transvectant and engaged unitary analyses (see Section 6) to yield the following.

Theorem. *The moduli space $\mathcal{M}$ of constantly curved holomorphic 2-spheres of degree 6 in $G(2,5)$, which are $GL(5,\mathbb{C})$-equivalent to sextic curves living in the generally ramified family, is a 2-dimensional semialgebraic set, up to the ambient $U(5)$-equivalence.*

The moduli space structure facilitates the computation to verify that the second fundamental form of all members of $\mathcal{M}$ are not of constant norm, and thus all but the standard Veronese curve are nonhomogeneous.

Of particular interest are three points in the moduli space $\mathcal{M}$, for each of which the corresponding Fano 3-fold contains a unique constantly curved holomorphic 2-sphere of degree 6, whereas the Fano 3-fold corresponding to a point other than the three in $\mathcal{M}$ contains exactly two distinct constantly curved holomorphic 2-spheres conjugated to each other in an appropriate sense (see Sections 7 and 8).

Our approach facilitates the explicit construction of many new examples, through algebro-geometric means, of constantly curved 2-spheres of degree 6.

Based on Felix Klein's work [29], we have completely classified all Galois covers of genus zero and their corresponding sextic curves in $\mathcal{H}_0^3$ for the exceptional transversal family (see Table 2, Section 5.4), which consists of a few 1-parameter examples and hence at most finitely many such of constant curvature in $\mathcal{H}_0^3$ by considering total unramification. (Since the classification is long, we only indicate a couple of examples in the current paper. See Section 5.4.) It is tempting to suggest, up to $U(5)$-equivalence, that there would be at most finitely many 1-parameter examples of constantly curved 2-spheres in the transversal exceptional family.

The paper is organized as follows. Section 2 is devoted to recalling the representation theory of $PSL(2,\mathbb{C})$, as well as Jiao and Peng's classification of nonsingular (in their sense) constantly curved holomorphic 2-spheres in $G(2,5)$. In Section 3, we introduce briefly the theory of generic linear sections of $G(2,5)$ and the Fano 3-fold $\mathcal{H}_0^3$ constructed by Mukai and Umemura. In Section 4, we show that generic holomorphic 2-spheres of degree 6 in $GL(2,5)$ are not constantly curved. In Section 5, when a sextic curve $\gamma \subset \mathcal{H}_0^3$ is not in $\mathbf{S}$, we introduce its Galois lift in $\mathbb{C}P^3$, where Galois analyses lead to the generally ramified family that also includes the case when $\gamma \subset \mathbf{S}$ is not in the singular locus of $\mathbf{S}$, as outlined above. Starting from Section 6, we explore PSL_2-transvectant and engaged unitary analyses in preparation for the existence and uniqueness (Theorem 7.1) of constantly curved holomorphic 2-spheres of degree 6 in the generally ramified family in Section 7, and for the moduli space structure of the aforementioned Theorem (Theorem 8.1) and related results in Section 8, from which interesting individual as well as 1-parameter families of new examples are exhibited.

2. Priliminaries

2.1. Irreducible representations of $SL_2(\mathbb{C})$.

Let V_n be the the space of binary forms of degree n in two variables u and v, on which $SL_2(\mathbb{C})$ (to be denoted by SL_2) acts by

$$SL_2 \times V_n \to V_n, \quad (g, f) \quad \mapsto (g \cdot f)(u, v) \triangleq f(g^{-1} \cdot (u, v)^t). \tag{2.1}$$

It is well-known that $V_n,\ n \in \mathbb{Z}_{\geq 0}$, are the only finite-dimensional irreducible representations of SL_2.

Choose the following basis of V_n,

$$e_l \triangleq \tbinom{n}{l}^{\frac{1}{2}} u^{n-l} v^l,\ l = 0, \ldots, n. \tag{2.2}$$

Under this basis, write

$$(e_0, \ldots, e_n)\, \rho^n(g) \triangleq (g \cdot e_0, g \cdot e_1, \ldots, g \cdot e_n). \tag{2.3}$$

The representation $\rho^n(g) : SL_2 \to GL(n+1; \mathbb{C})$ induces the wedge-product representation

$$SL_2 \times V_n \wedge V_n \to V_n \wedge V_n, \quad (g, e_k \wedge e_l) \mapsto (g \cdot e_k) \wedge (g \cdot e_l),\ 0 \leq k, l \leq n. \tag{2.4}$$

For the sake of clarity, we view $V_n \wedge V_n$ as the space of anti-symmetric matrices $\wedge^2 \mathbb{C}^{n+1}$, by identifying $e_k \wedge e_l$ with the anti-symmetric matrix $E_{kl} - E_{lk} \in M_{n+1}(\mathbb{C})$, where the only nonvanishing entry of E_{kl} is 1 at the (k, l) position, $0 \leq k < l \leq n$. With the basis $\{e_k \wedge e_l \mid 0 \leq k < l \leq n\}$ (see (2.2)), it is not difficult to obtain the wedge-product representation in matrix form,

$$\rho^n \wedge \rho^n : SL_2 \times \wedge^2 \mathbb{C}^{n+1} \to \wedge^2 \mathbb{C}^{n+1}, \quad (g, A) \mapsto\ (\rho^n(g)) \cdot A \cdot (\rho^n(g))^t.$$

The Clebsch-Gordan formula states that (assume $m \geq n$)

$$V_m \otimes V_n \cong V_{m+n} \oplus V_{m+n-2} \oplus \cdots \oplus V_{m-n}.$$

Moreover, for any given number $p \in [0, n]$, the projection $V_m \otimes V_n \to V_{m+n-2p}$ can be formulated by

$$(f, h) \mapsto (f, h)_p \triangleq \frac{(m-p)!(n-p)!}{m!n!} \sum_{i=0}^{p} (-1)^i \binom{p}{i} \frac{\partial^p f}{\partial u^{p-i} \partial v^i} \frac{\partial^p h}{\partial u^i \partial v^{p-i}}, \tag{2.5}$$

which is PSL_2-equivariant and is called the *p-th transvectant* [37, Eq (2.1), p. 16]. Moreover,

$$V_n \wedge V_n \cong V_{2n-2} \oplus V_{2n-6} \oplus \ldots \oplus V_r, \tag{2.6}$$

where r is the remainder of $2n - 2$ divided by 4, and the projections are the same as (2.5).

Gordan proved that the binary sextic V_6 has 5 invariants and 26 covariants given by a finite number of iterated transvectants [37, Table 1.1, p. 12; Theorem 2.1.3, p. 18], among which $(f, f)_2,\ (f, f)_4,\ (f, f)_6$ [37, Sections 4.5, 5.6] appear in geometry in an unexpected way (see Proposition 3.1).

2.2. Holomorphic 2-spheres in $G(2, 5)$.

We briefly review some basic facts of constantly curved holomorphic 2-spheres in the complex Grassmannian $G(2, 5)$, and along the way introduce those *nonsingular* ones that Jiao and Peng [23] defined and classified.

Throughout, we equip $G(2, 5)$ with the standard Kähler metric induced from the Fubini-Study metric of $\mathbb{C}P^9$ when $G(2, 5)$ is realized as a subvariety of $\mathbb{C}P^9$ by the Plücker embedding,

$$i : G(2, 5) \to \mathbb{P}(\wedge^2 \mathbb{C}^5) \cong \mathbb{C}P^9,\ \text{span}\{u, v\} \mapsto [u \wedge v].$$

Explicitly, let $\{\epsilon_0, \epsilon_1, \ldots, \epsilon_4\}$ be a basis of $\mathbb{C}^5$. Then $\{\epsilon_i \wedge \epsilon_j \,|\, 0 \le i < j \le 4\}$ forms a basis of $\wedge^2\mathbb{C}^5$ so that $p = [\sum_{i,j} p_{ij}\, \epsilon_i \wedge \epsilon_j]$ belongs to $G(2,5)$ if and only if $p \wedge p = 0$, which is equivalent to

$$\begin{aligned} &p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12} = 0, \quad p_{01}p_{24} - p_{02}p_{14} + p_{04}p_{12} = 0,\\ &p_{01}p_{34} - p_{03}p_{14} + p_{04}p_{13} = 0, \quad p_{02}p_{34} - p_{03}p_{24} + p_{04}p_{23} = 0,\\ &p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0. \end{aligned} \tag{2.7}$$

Remark 2.1. *It follows from the definition that $G(2,5)$ is PSL_2-invariant under the wedge-product action $\rho^4 \wedge \rho^4$ given in* (2.4).

Let $\varphi : \mathbb{C}P^1 \to G(2,5)$ be a holomorphic 2-sphere. It follows from the Normal Form Lemma [35] that there exist two holomorphic curves $f, g : \mathbb{C}P^1 \to \mathbb{C}P^4$, such that $\varphi = \mathrm{span}\{f, g\}$. Explicitly, choosing an affine coordinate z on $\mathbb{C}P^1$, we can write $f(z) = (f_0(z), \ldots, f_4(z))$ and $g(z) = (g_0(z), \ldots, g_4(z))$ as row vectors with polynomial entries except at some isolated points.

In view of Remark 2.1, we obtain that φ is of constant curvature K if and only if $i \circ \varphi$ is of constant curvature K under the Plücker embedding. This guarantees that the rigidity principle of Calabi can be employed to study constantly curved holomorphic 2-spheres in $G(2,5)$, which we rephrase as follows for reference.

Lemma 2.1. *Let $f : \mathbb{C}P^1 \to \mathbb{C}P^n$ be a holomorphic 2-sphere of degree d. The following are equivalent.*

1. *The Gauss curvature K of f is $\frac{4}{d}$. Furthermore, up to the action of $U(n+1)$ and Möbius reparametrization, f is given by the Veronese sphere*
$$Z_d(z) \triangleq [1 : \sqrt{d}z : \cdots : \sqrt{\tbinom{d}{k}}z^k : \cdots : z^d]^t. \tag{2.8}$$
2. *There is an affine chart $z \in \mathbb{C}$ over which $|f|^2 = (1 + |z|^2)^d$.*
3. *There is an affine chart $z \in \mathbb{C}$ over which $f = \sum_{k=0}^{d} \sqrt{\tbinom{d}{k}} A_k z^k$, and $\{A_0, A_1, \cdots, A_6\}$ forms an orthonormal basis of the d-plane spanned by f.*

For a constantly curved holomorphic 2-sphere $\varphi : \mathbb{C}P^1 \to G(2,5)$, it is known [23, 31] that φ can be parameterized as $\varphi = \big(\varphi_1(z)^t, \varphi_2(z)^t\big)^t$ with

$$\varphi_1(z) = \big(1, 0, \varphi_{12}(z), \varphi_{13}(z), \varphi_{14}(z)\big), \;\; \varphi_2(z) = \big(0, 1, \varphi_{22}(z), \varphi_{23}(z), \varphi_{24}(z)\big), \tag{2.9}$$

where $\varphi_{1i}(z)$ and $\varphi_{2i}(z)$ $(2 \le i \le 4)$ are polynomials vanishing at $z = 0$. In the sequel, (2.9) will be called a *standard parameterization* of φ. We point out that this kind of parameterization is not unique. In fact, if $\{\varphi_1,\ \varphi_2\}$ is a standard parameterization of φ, then $\{\alpha\varphi_1 + \beta\varphi_2, -\bar{\beta}\varphi_1 + \bar{\alpha}\varphi_2\}$ is also a standard parameterization after rotating ϵ_0 and ϵ_1 while maintaining $|\alpha|^2 + |\beta|^2 = 1$.

In [23], a holomorphic 2-sphere $\varphi : \mathbb{C}P^1 \to G(2,5)$ is called *nonsingular* if there exists a standard parameterization $\{\varphi_1,\ \varphi_2\}$ of φ, such that $[\varphi_1(\infty)] \ne [\varphi_2(\infty)]$ in $\mathbb{C}P^4$. Otherwise, φ is called *singular*. It is easy to verify that φ is nonsingular if and only if there exists a standard parameterization $\{\varphi_1, \varphi_2\}$ of φ, such that

$$\deg\varphi = \deg\varphi_1 + \deg\varphi_2. \tag{2.10}$$

Using a standard parameterization, one can construct explicitly nonsingular examples as was done by Jiao and Peng in [23]. Indeed, under the nonsingular assumption, Jiao and Peng in the paper proved the following nonexistence result.

Theorem 2.1. *There does not exist* nonsingular *holomorphic constantly curved* 2*-spheres of degree* 6 *in* $G(2,5)$.

The idea goes as follows. By contradiction, otherwise, it would follow from (2.10) that we had only three possibilities that $(\deg \varphi_1, \deg \varphi_2) = (5,1),\ (4,2),\ (3,3)$. In each case, we obtained vectors $A_k,\ 0 \le k \le 6$, where $i \circ \varphi = \varphi_1 \wedge \varphi_2 \triangleq \sum_{k=0}^{6} \sqrt{\binom{d}{k}} A_k z^k$, in terms of undetermined coefficients of φ_1 and φ_2 to violate item (3) of Lemma 2.1.

As the degree of φ increases, however, the number of undetermined coefficients rises dramatically, so that it is technically difficult to apply the method to construct *singular* 2-spheres.

It is readily verified that the Veronese curve (1.1) given in the introduction is singular in terms of Jiao and Peng's definition, where a standard parameterization in the sense of (2.9) can be chosen to be

$$\begin{pmatrix} 1 & 0 & -\sqrt{6}z^2 & -4z^3 & -3z^4 \\ 0 & 1 & \sqrt{6}z & 3z^2 & 2z^3 \end{pmatrix}. \tag{2.11}$$

We point out that this example is smooth (nonsingular) in the usual algebro-geometric sense, which is indeed what we are after.

2.3. **Reducible and Irreducible holomorphic curves in $G(2,5)$.**

For later purposes, we develop the extrinsic geometry of holomorphic curves in $G(2,5)$ from the viewpoint of developable surfaces.

Let $f : M \to G(2,5)$ be a holomorphic map from a Riemann surface M. Composed with the Plücker embedding, $F \triangleq i \circ f$ is a holomorphic curve in $\mathbb{C}P^9 = \mathbb{P}(\wedge^2\mathbb{C}^5)$. Since F lies in $G(2,5)$, we have $F \wedge F \equiv 0$, whose derivative with respect to a local complex coordinate z yields that $F \wedge \partial F/\partial z = 0$. Consider the *tangent developable surface* $\mathcal{D}$ of F in $\mathbb{C}P^9$, spanned by F and its tangent line $\partial F/\partial z$,

$$\mathcal{D} \triangleq \{[u\,F + v\,\partial F/\partial z] \mid [u : v] \in \mathbb{C}P^1\}.$$

Lemma 2.2. *The following are equivalent.*

(1) *The tangent developable surface* $\mathcal{D}$ *of* F *lies in* $G(2,5)$.
(2) $\partial F/\partial z \wedge \partial F/\partial z \equiv 0$.

The lemma follows by differentiating $(u\,F + v\,\partial F/\partial z) \wedge (u\,F + v\,\partial F/\partial z) = 0$ while employing $F \wedge \partial F/\partial z = 0$.

Inspired by the first item in Lemma 2.2, we call a holomorphic curve $f : M \to G(2,5)$ *reducible*, if the tangent developable surface $\mathcal{D}$ of $F = i \circ f$ also lies in $G(2,5)$; otherwise, we call f *irreducible*. If $f : M \to G(2,5)$ is irreducible, then $\partial F/\partial z \wedge \partial F/\partial z$ has isolated zeroes, which we call *ramified points* (with multiplicity) and f is said to be *ramified* at these points.

Remark 2.2. *In the theory of harmonic sequences, a holomorphic curve* $f : M \to G(2,5)$ *is called reducible if the rank of the next term* f_1 *is strictly less than* 2*; see* [25]*. This definition coincides with the above definition. We thank Professor L. He for helpful discussions about it.*

It was proven in [19] that a constantly curved reducible holomorphic 2-sphere of degree 6 is rigid, which is unitarily equivalent to the *standard* Veronese curve (1.1) in $G(2,5)$. As a result, we need only consider irreducible holomorphic 2-spheres in $G(2,5)$ in the sequel.

3. Algebro-geometric preparation

3.1. **Generic linear sections of $G(2,5)$ and Fano 3-folds of index 2 and degree 5.**

To motivate, a holomorphic 2-sphere of degree 6 in $G(2,5)$ lies in a 6-plane $\mathbf{L}$ in $\mathbb{P}(\wedge^2\mathbb{C}^5)\cong\mathbb{C}P^9$, and so it lives in the intersection $\mathbf{L}\cap G(2,5)$ called a *linear section* of $G(2,5)$. The dual 2-plane of $\mathbf{L}$ in $(\wedge^2\mathbb{C}^5)^*$ is given by a linear system

$$[\lambda A+\mu B+\tau C],\qquad [\lambda:\mu:\tau]\in\mathbb{C}P^2, \tag{3.1}$$

where A,B,C are fixed skew-symmetric matrices of size 5×5 identified with elements in $(\wedge^2\mathbb{C}^5)^*$. Following [36], we say that $\mathbf{L}$ is *generic* if all matrices in the linear system are of rank 4, and the associated cut $\mathbf{L}\cap G(2,5)$ is referred to as a *generic* linear section. Let us look at a concrete example next.

By the Clebesch-Gordan formula (2.6), we obtain that $\wedge^2\mathbb{C}^5\cong V_6\oplus V_2$. Here, we identify V_6 with a SL_2-invariant subspace of 5×5 anti-symmetric matrices by

$$\sum_{i=0}^{6}\sqrt{\tbinom{6}{i}}a_iu^{6-i}v^i\mapsto\begin{pmatrix} 0 & a_0 & a_1 & \sqrt{\frac{3}{5}}a_2 & \frac{1}{\sqrt{5}}a_3\\ -a_0 & 0 & \sqrt{\frac{2}{5}}a_2 & \frac{2}{\sqrt{5}}a_3 & \sqrt{\frac{3}{5}}a_4\\ -a_1 & -\sqrt{\frac{2}{5}}a_2 & 0 & \sqrt{\frac{2}{5}}a_4 & a_5\\ -\sqrt{\frac{3}{5}}a_2 & -\frac{2}{\sqrt{5}}a_3 & -\sqrt{\frac{2}{5}}a_4 & 0 & a_6\\ -\frac{1}{\sqrt{5}}a_3 & -\sqrt{\frac{3}{5}}a_4 & -a_5 & -a_6 & 0\end{pmatrix}. \tag{3.2}$$

Let $\{e_i\}$ be an orthonormal basis of $\mathbb{C}^5$. An orthonormal basis of V_6 is given by

$$\begin{aligned}&E_0\triangleq e_0\wedge e_1,\quad E_1\triangleq e_0\wedge e_2,\quad E_2\triangleq\sqrt{3/5}\,e_0\wedge e_3+\sqrt{2/5}\,e_1\wedge e_2,\\ &E_3\triangleq 1/\sqrt{5}\,e_0\wedge e_4+2/\sqrt{5}\,e_1\wedge e_3,\quad E_4\triangleq\sqrt{3/5}\,e_1\wedge e_4+\sqrt{2/5}\,e_2\wedge e_3,\\ &E_5\triangleq e_2\wedge e_4,\quad E_6\triangleq e_3\wedge e_4.\end{aligned} \tag{3.3}$$

It is readily checked that $uv(u^4-v^4)$ (respectively, u^6) in V_6 corresponds to $(E_1-E_5)/\sqrt{6}$ (respectively, E_0). Note that, the dual plane to V_6 is given by a linear system of the form in (3.1), where

$$A\triangleq\sqrt{6}p_{03}-3p_{12}=0,\ B\triangleq 2p_{04}-p_{13}=0,\ C\triangleq\sqrt{6}p_{14}-3p_{23}=0. \tag{3.4}$$

It is also readily checked that the rank of $[\lambda A+\mu B+\tau C]$ is 4 for every $[\lambda:\mu:\tau]\in\mathbb{C}P^2$. Therefore, as a linear section,

$$\mathcal{H}_0^3\triangleq\mathbb{P}(V_6)\cap G(2,5),$$

is generic.

Note also that the space $\mathbb{P}(V_6)$ is the 6-plane spanned by the standard Veronese curve in (1.1), which is precisely the orbit $PSL_2\cdot u^6$ confirmed by a computation with $(E_0,\cdots,E_6)\cdot Z_6(z)$, where Z_6 is given in (2.8), to see that they are agreeable.

We include a short outline of the following well-known fact for the reader's convenience. Our reference is [36].

Theorem 3.1. *All generic linear sections $\mathbf{L}\cap G(2,5)$ are $PGL(5,\mathbb{C})$-equivalent to $\mathcal{H}_0^3$.*

To begin, the *Pfaffian* of a $(2n)\times(2n)$ skew-symmetric matrix M with entries a_{ij} is defined to be

$$pf(M)\triangleq\sum_{\sigma}sgn(\sigma)\,a_{i_1j_1}a_{i_2j_2}\cdots a_{i_nj_n},$$

where $\sigma:\{1,2,\cdots,2n\}\to\{i_1,j_1,i_2,j_2,\cdots,i_n,j_n\}$, in order, runs over permutations of $\{1,2,\cdots,2n\}$ satisfying $i_s<j_s, 1\le s\le n$, and $i_1<i_2<\cdots<i_n$. The Pfaffian enjoys

the property that if N is a $(2n+1)\times(2n+1)$ skew-symmetric matrix of rank $2n$, then the 1-dimensional kernel of N is spanned by the vector $(v_1,\cdots,v_{2n+1})$, where v_i is the diagonal Pfaffian of the $(2n)\times(2n)$ skew-symmetric matrix obtained by deleting the ith row and column.

Now, since the dual 2-plane of a generic 6-plane $\mathbf{L}$ in $\mathbb{P}(\wedge^2(\mathbb{C}^5))$ is a linear system $[\lambda A+\mu B+\tau C]$, $[\lambda:\mu:\tau]\in\mathbb{C}P^2$, all of whose 5×5 skew-symmetric matrices are of rank 4, we can use the associated diagonal Phaffians to define the *center* map

$$\mathbf{c}:[\lambda:\mu:\tau]\in\mathbb{C}P^2\to \text{projectivized center of } [\lambda A+\mu B+\tau C]\in\mathbb{C}P^4.$$

It is then verified that the center map is an embedding of $\mathbb{C}P^2$ into $\mathbb{C}P^4$ of degree 2, and thus the image of $\mathbf{c}$, called the *projected Veronese surface*, is a generic projection from the standard Veronese surface in $\mathbb{C}P^5$ to $\mathbb{C}P^4$. Consequently, any two such 2-plane linear systems are $PGL(5,\mathbb{C})$-equivalent, and so are the corresponding linear sections. In fact, $\mathbf{L}\cap G(2,5)$ is the closure of all lines in $\mathbb{C}P^4$ intersecting the associated projected Veronese surface in three distinct points.

Exploring the center map c, the authors in [36] also obtained the automorphism group of a generic linear section $\mathbf{L}\cap G(2,5)$.

Theorem 3.2. *The automorphism group of a generic linear section* $\mathbf{L}\cap G(2,5)$ *is* PSL_2.

Generic linear sections $\mathbf{L}\cap G(2,5)$ constitute all Fano 3-folds of index 2 and degree 5, first classified by Castelnuovo [14], a typical one of which is to be denoted by $\mathcal{H}^3$ henceforth; here, the degree is that of the Fano 3-fold as a subvariety of $\mathbb{C}P^9$, and the index is the difference between its degree and codimension in $G(2,5)$, so that its anti-canonical bundle is $\simeq\mathcal{O}(2)$. To reference, we call $\mathcal{H}^3_0=\mathbb{P}(V_6)\cap G(2,5)$ introduced earlier the *standard* Fano 3-fold.

We point out that the automorphism group of a Fano 3-fold of index 2 and degree 5 has also been studied by Mukai and Umemura in [33] from the viewpoint of algebraic group actions. By considering the action of PSL_2 on $\mathbb{P}(V_6)$, they proved that the closure of $PSL_2\cdot uv(u^4-v^4)$ is precisely $\mathcal{H}^3_0$. In the same paper, they also obtained the following beautiful orbit decomposition structure on $\mathcal{H}^3_0$.

Theorem 3.3.

$$\mathcal{H}^3_0=\overline{PSL_2\cdot uv(u^4-v^4)}=PSL_2\cdot uv(u^4-v^4)\sqcup PSL_2\cdot u^5v\sqcup PSL_2\cdot u^6.$$

Remark 3.1. *In the above orbit decomposition,* $PSL_2\cdot uv(u^4-v^4)$ *is of dimension* 3*, which is parameterized as*

$$\begin{aligned}
&f_1:PSL_2\mapsto\mathbb{P}(V_6),\quad [\begin{pmatrix}a&b\\c&d\end{pmatrix}]\mapsto[\begin{pmatrix}a&b\\c&d\end{pmatrix}\cdot uv(u^4-v^4)]=[a_0:a_1:\cdots:a_6],\\
&a_0\triangleq-\sqrt{6}d^5c+\sqrt{6}dc^5,\quad a_1\triangleq d^4\,(ad+5\,bc)-c^4(5\,ad+bc),\\
&a_2\triangleq-bd^3\,(ad+2\,bc)\,\sqrt{10}+ac^3\,(2\,ad+bc)\,\sqrt{10},\\
&a_3\triangleq b^2d^2\,(ad+bc)\,\sqrt{30}-a^2c^2\,(ad+bc)\,\sqrt{30},\\
&a_4\triangleq-b^3d\,(2\,ad+bc)\,\sqrt{10}+a^3c\,(ad+2\,bc)\,\sqrt{10},\\
&a_5\triangleq b^4(5\,ad+bc)-a^4\,(ad+5\,bc)\,,\quad a_6\triangleq-\sqrt{6}b^5a+\sqrt{6}ba^5.
\end{aligned}\tag{3.5}$$

Similarly, the orbit $PSL_2\cdot u^6$ *is parameterized as*

$$[\begin{pmatrix}a&b\\c&d\end{pmatrix}]\mapsto[d^6:-\sqrt{6}bd^5:\sqrt{15}b^2d^4:-\sqrt{20}b^3d^3:\sqrt{15}b^4d^2:-\sqrt{6}b^5d:b^6].\tag{3.6}$$

It is precisely the Veronese curve Z_6 in (2.8)*. Its tangent developable surface constitutes the closure of the* 2*-dimensional orbit* (*see* [33])*,*

$$\overline{PSL_2 \cdot u^5 v} = PSL_2 \cdot u^5 v \sqcup PSL_2 \cdot u^6,$$

where $PSL_2 \cdot u^5 v$ has the following parameterization

$$\begin{aligned}
&f_2 : PSL_2 \mapsto \mathbb{P}(V_6), \quad [\begin{pmatrix} a & b \\ c & d \end{pmatrix}] \mapsto [\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot u^5 v] = [b_0 : b_1 : \cdots : b_6], \\
&b_0 \triangleq -\sqrt{6} d^5 c, \quad b_1 \triangleq d^4 \left(ad + 5\,bc\right), \quad b_2 \triangleq -bd^3 \left(ad + 2\,bc\right) \sqrt{10}, \\
&b_3 \triangleq b^2 d^2 \left(ad + bc\right) \sqrt{30}, \quad b_4 \triangleq -b^3 d \left(2\,ad + bc\right) \sqrt{10}, \\
&b_5 \triangleq b^4 \left(5ad + bc\right), \quad b_6 \triangleq -\sqrt{6} b^5 a.
\end{aligned} \tag{3.7}$$

Meanwhile, using the invariants and covariants of the binary sextic (see (2.5)), we remark that the above orbits have another SL_2-invariant characterization.

Proposition 3.1. *Given $f = \sum_{i=0}^{6} \sqrt{\binom{6}{i}} a_i u^{6-i} v^i$ defining $[f] \in \mathbb{P}(V_6)$, we have*

1. *$[f]$ lies in $\mathcal{H}_0^3 = \mathbb{P}(V_6) \cap G(2,5) = \overline{PSL_2 \cdot uv(u^4 - v^4)}$ if and only if the* 4*-th transvectant $(f,f)_4 = 0$,*
2. *$[f]$ lies in the closed* 2*-dim orbit $\overline{PSL_2 \cdot u^5 v}$ if and only if the* 4*-th and* 6*-th transvectants $(f,f)_4$ and $(f,f)_6$ vanish, and*
3. *$[f]$ lies in the* 1*-dimensional orbit $PSL_2 \cdot u^6$ if and only if the* 2*nd transvectant $(f,f)_2 = 0$.*

For later purposes, we quote the following well known calculations: $(f,f)_2 = \mathrm{Hess}(f)/450$.

$$(f,f)_6 = 2a_0 a_6 - 2a_1 a_5 + 2a_2 a_4 - a_3^2. \tag{3.8}$$

$$(f,f)_4 = \sum_{i=0}^{4} \sqrt{\binom{4}{i}} t_i u^{4-i} v^i, \qquad \text{where}$$

$$\begin{aligned}
&t_0 \triangleq \frac{2}{15}(\sqrt{15} a_0 a_4 - \sqrt{30} a_1 a_3 + 3a_2^2), \quad t_1 \triangleq \frac{\sqrt{6}}{15}(5a_0 a_5 - \sqrt{15} a_1 a_4 + \sqrt{2} a_2 a_3), \\
&t_2 \triangleq \frac{\sqrt{6}}{15}(5a_0 a_6 - 3a_2 a_4 + 2a_3^2), \quad t_3 \triangleq \frac{\sqrt{6}}{15}(5a_1 a_6 - \sqrt{15} a_2 a_5 + \sqrt{2} a_3 a_4), \\
&t_4 \triangleq \frac{2}{15}(\sqrt{15} a_2 a_6 - \sqrt{30} a_3 a_5 + 3a_4^2).
\end{aligned} \tag{3.9}$$

We point out that $\frac{1}{\sqrt{6}} f \wedge f = \sum_{i=0}^{4} t_{4-i} e_0 \wedge \cdots \wedge \widehat{e_i} \wedge \cdots \wedge e_4 \in \mathbb{P}(\wedge^4(\mathbb{C}^5))$, where the notation $\widehat{e_i}$ means that we omit the term e_i.

Meanwhile, since $\mathcal{H}_0^3$ and the 5-quadric Q_5 defined by

$$Q_5 \triangleq \{[f] \in \mathbb{P}(V_6) : (f,f)_6 = 0\} \tag{3.10}$$

are both PSL_2-invariant in $\mathbb{P}(V_6)$, the closed 2-dimensional PSL_2-orbit is precisely $Q_5 \cap \mathcal{H}_0^3$.

Note also that $(f,f)_2$ vanishes if and only if f is the 6-th power of a linear form; see [30, Prop 5.3, p. 71] for an algebraic reason.

Remark 3.2. *The isotropy groups of the two orbits of $\mathcal{H}_0^3$ of dimension ≥ 2 are given below.*
(1) *The open orbit $PSL_2 \cdot uv(u^4 - v^4)$: Its isotropy group at $uv(u^4 - v^4)$ is the projective binary octahedral group of order 24, isomorphic to S_4, consisting of the following elements* ($\xi \triangleq e^{2k\pi\sqrt{-1}/8},\ k = 0, 1, \ldots, 3$):

$$\begin{pmatrix} \xi & 0 \\ 0 & 1/\xi \end{pmatrix}, \begin{pmatrix} 0 & \xi \\ -1/\xi & 0 \end{pmatrix}, 1/\sqrt{2} \cdot \begin{pmatrix} 1/\xi & -1/\xi \\ \xi & \xi \end{pmatrix},$$

$$1/\sqrt{2} \cdot \begin{pmatrix} \sqrt{-1}/\xi & -1/\xi \\ \xi & -\sqrt{-1}\xi \end{pmatrix}, 1/\sqrt{2} \cdot \begin{pmatrix} -1/\xi & -1/\xi \\ \xi & -\xi \end{pmatrix}, 1/\sqrt{2} \cdot \begin{pmatrix} -\sqrt{-1}/\xi & -1/\xi \\ \xi & \sqrt{-1}\xi \end{pmatrix}.$$

(2) *The 2-dimensional orbit $PSL_2 \cdot u^5 v$: Its isotropy group at $u^5 v$ is*

$$\{\begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix} \mid a \in \mathbb{C}^*\} \quad \mod \pm I_2.$$

For later computational purposes, we prove the following.

Lemma 3.1. *Let A be a matrix in SL_2. Then*

$$\rho^4(A) \cdot (E_0, E_1, \ldots, E_6) = (E_0, E_1, \ldots, E_6)\, \rho^6(A), \tag{3.11}$$

where the left-hand side with a dot is the $\wedge^2$-action of $\rho^4(A)$ on $V_6 \subset \wedge^2(\mathbb{C}^5)$ and the right-hand side without a dot is a matrix multiplication.

Proof. Since the Clebsch-Gordon transvectant $\pi \triangleq f \wedge g \to (f, g)_1$ in (2.5) is SL_2-equivariant, we obtain from the commutativity of the diagram

$$\begin{array}{ccc} V_4 \wedge V_4 & \xrightarrow{\rho^4(A)} & V_4 \wedge V_4 \\ \downarrow \pi & & \downarrow \pi \\ V_6 & \xrightarrow{\rho^6(A)} & V_6 \end{array} \tag{3.12}$$

that $\rho^6(A) : V_6 \to V_6$ is induced from the $\wedge^2$-action of $\rho^4(A)$ (see (2.4)). □

4. Generic holomorphic 2-spheres of degree 6 in $G(2,5)$

In this section, we prove that generic holomorphic 2-spheres of degree 6 in $G(2,5)$ are not of constant curvature. Here, a holomorphic 2-sphere of degree 6 is called generic if it differs from a *general* (in the sense given in [39, Condition 3.20]) rational normal curve of degree 6 (a sextic curve) in the standard Fano 3-fold $\mathcal{H}_0^3$ by a transformation in $GL(5, \mathbb{C})$.

Firstly, we review some results of general sextic curves in $\mathcal{H}_0^3$ referred to as the quintic del Pezzo 3-fold and denoted by V_5 in [38, 39]. In these two papers, Takagi and Zucconi investigated the moduli space (Hilbert scheme) of sextic curves in $\mathcal{H}_0^3$. (Their results are more general; we only invoke the special case when the curve degree is 6.) Let H^6 be the Hilbert scheme whose general points parameterize sextic curves in $\mathcal{H}_0^3$. The following results (see Corollary 3.10 in [38], Proposition 2.3.1, Proposition 2.3.3 and Proposition 2.3.4 in [39]) were proved.

Proposition 4.1. *The closure of H^6 is an irreducible variety of dimension 12. Moreover, for a general sextic curve C_6 in $\mathcal{H}_0^3$,*
(1) *C_6 intersects the closure of the 2-dimensional orbit $\overline{PSL_2 \cdot u^5 v}$ simply,*
(2) *there exist at most finitely many bi-secant lines of C_6 in $\mathcal{H}_0^3$, and any one of them*

intersects C_6 *simply, and*
(3) $Q|_{C_6}$ *has no point of multiplicity greater than* 2 *for any multi-secant conic* Q.

It turns out the above proposition can be used to prove that general sextic curves in $\mathcal{H}_0^3$ are totally unramified in the sense of harmonic sequences [10], from which we can derive that generic holomorphic 2-spheres of degree 6 in $G(2,5)$ are not of constant curvature. Recall (below Lemma 2.2) that a holomorphic 2-sphere $F : \mathbb{C}P^1 \to G(2,5)$ is unramified if $F' \wedge F'$ is nowhere vanishing, in which case it is called *totally unramified* if, furthermore, the curve $[F' \wedge F'] : \mathbb{C}P^1 \to \mathbb{P}(\Lambda^4\mathbb{C}^5) \cong \mathbb{C}P^4$ is unramified as a projective curve, which is equivalent to saying that $F'' \wedge F'$ is nowhere parallel to $F' \wedge F'$. Our key observation is the following interesting algebro-geometric characterization of total unramification.

Theorem 4.1. *Let* $F : \mathbb{C}P^1 \to G(2,5)$ *be a holomorphic* 2*-sphere of degree* 6*.*
(1) $F' \wedge F'$ *is zero at a point* p *if and only if the tangent line of* F *at* p *lies in* $G(2,5)$*.*
(2) *Assume* $F' \wedge F'$ *is nonzero at* p*. If* $[F' \wedge F']$ *is ramified at* p*, then there exists a conic* Q *tangent to* F *at* p *such that* $Q|_F$ *has multiplicity no less than* 3 *at* p*.*

Proof. The conclusion in (1) follows from $F \wedge F' = 0$ so that

$$(F + tF') \wedge (F + tF') = t^2 F' \wedge F', \ t \in \mathbb{C}.$$

For the conclusion in (2), we assume that $F = f \wedge g$. Since $F' \wedge F'$ does not vanish at p, we can choose a basis $\{e_1, e_2, \cdots, e_5\}$ of $\mathbb{C}^5$ such that

$$F(p) = e_1 \wedge e_2, \ F'(p) = e_1 \wedge e_3 - e_2 \wedge e_4, \tag{4.1}$$

and

$$F''(p) = f''(p) \wedge e_2 - 2e_3 \wedge e_4 + e_1 \wedge g''(p). \tag{4.2}$$

If $[F' \wedge F']$ is ramified at p, then there exist two complex number α and β such that

$$(\alpha F'(p) + \beta F''(p)) \wedge F'(p) = 0.$$

It follows that

$$f''(p), g''(p) \in \{e_1, e_2, e_3, e_4\}. \tag{4.3}$$

Consider the 2-plane P_2 spanned by $\{F(p), F'(p), F''(p)\}$ and its intersection with $G(2,5)$. Using (4.1)$\sim$(4.3), it is easy to verify that $[F(p) + xF'(p) + yF''(p)]$ lies in $G(2,5)$ if and only if

$$-4y + 2x^2 + \lambda y^2 + \mu xy = 0 \tag{4.4}$$

for two constants λ and μ, which means that the intersection $P_2 \cap G(2,5)$ is exactly a conic. We denote this conic by Q.

We choose a local coordinate z near p such that $z(p) = 0$. It follows from the Taylor expansion of F at $z = 0$ that, near p, F can be parameterized as $[1 : z : \frac{z^2}{2} : \frac{z^3}{3!} : \cdots : \frac{z^d}{d!}]$, with respect to the frame $\{F(p), F'(p), F''(p), \cdots, F^{(d)}(p)\}$ on the d-plane containing F. Substituting $x = z$ and $y = \frac{z^2}{2}$ into the left-hand side of (4.4), we have

$$-4y + 2x^2 + \lambda y^2 + \mu xy = z^3(\frac{\mu}{2} + \frac{\lambda}{4}z),$$

which implies $Q|_F$ has multiplicity no less than 3 at p. □

Proposition 4.1, Theorem 4.1, and an easy construction of a totally unramified sextic curve in $G(2,5)$ whose curvature is not constant, imply that a generic holomorphic 2-sphere of degree 6 is totally unramified, and so we obtain the main result of this section.

Theorem 4.2. *Generic holomorphic 2-spheres of degree 6 in $G(2,5)$ are not of constant curvature.*

5. Galois covering of the holomorphic 2-spheres of degree 6 in $G(2,5)$

We see from the preceding section that holomorphic 2-spheres of degree 6 with constant curvature in $G(2,5)$ are nongeneric. To understand better how and when the ramification in the sense of harmonic sequences can appear, we look at it from the Galois point of view. We divide the discussion according to whether the curve lies in the closed 2-dimensional PSL_2-orbit as follows.

5.1. **The case when the curve lies in the closed 2-dimensional orbit.**

Part of the following theorem is known to algebraists [42]. We give a straightforward proof pertaining to our geometric situation here.

Theorem 5.1. *Let $F : \mathbb{C}P^1 \to \mathcal{H}_0^3$ be a rational normal curve of degree 6. Assume F lies in the closed 2-dimensional orbit $\overline{PSL_2 \cdot u^5 v}$ but does not coincide with the 1-dimensional orbit. Then F can be lifted to a projective line $\phi : \mathbb{C}P^1 \to \mathbb{C}P^3$ in the diagram*

$$\begin{array}{ccc} & & \mathbb{C}P^3 \\ & \overset{\phi}{\nearrow} & \downarrow f_2 \\ \mathbb{C}P^1 & \xrightarrow{F} & \mathbb{C}P^6, \end{array} \tag{5.1}$$

where f_2 is given in (3.7). Moreover, F intersects the 1-dimensional orbit.

Proof. We give a proof based on the PSL_2-invariant theory.

Firstly, we show the existence of the lift ϕ. Assume that $F = \sum_{i=0}^{6} a_i(z)\sqrt{\binom{6}{i}}u^{6-i}v^i$, where $a_i(z)$ are polynomials of z with $a_i(z) \neq 0$ because F is linearly full. Then by (3.9), we obtain

$$\begin{aligned} a_4 &= \frac{\sqrt{10}a_1a_3 - \sqrt{3}a_2^2}{\sqrt{5}a_0}, \qquad a_5 = \frac{\sqrt{30}a_1^2a_3 - 3a_1a_2^2 - \sqrt{2}a_0a_2a_3}{5a_0^2}, \\ a_6 &= \frac{3\sqrt{10}a_1a_2a_3 - 3\sqrt{3}a_2^3 - 2\sqrt{5}a_0a_3^2}{5\sqrt{5}a_0^2}. \end{aligned} \tag{5.2}$$

By (3.8) and (5.2), F lying in the closed 2-dimensional orbit $\overline{PSL_2 \cdot u^5 v}$ is equivalent to

$$0 = Q_5 = -\frac{9}{5}a_3^2 - \frac{2\sqrt{2}(\sqrt{15}a_1^2 - 9a_0a_2)a_1}{5a_0^2}a_3 - \frac{2(8\sqrt{15}a_0a_2 - 15a_1^2)a_2^2}{25a_0^2}. \tag{5.3}$$

We can directly write down the lift $\phi = [\begin{pmatrix} a & b \\ c & d \end{pmatrix}]$ by assigning

$$a = \frac{\sqrt{6}a_1}{a_0} + \frac{-5\sqrt{10}a_1a_2 + 15\sqrt{5}a_0a_3}{10a_1^2 - 4\sqrt{15}a_0a_2}, \ b = \frac{-\sqrt{10}a_1a_2 + 3\sqrt{5}a_0a_3}{10a_1^2 - 4\sqrt{15}a_0a_2}, \tag{5.4}$$

$c = -1$, and $d = 1$. In fact, given $\phi \cdot u^5 v = (u + av)(u - bv)^5$, we derive that

$$\sum_{i=0}^{6} g_i(z)\sqrt{\binom{6}{i}}u^{6-i}v^i \triangleq F - a_0(u + av)(u - bv)^5 = 0 \tag{5.5}$$

in $\mathbb{C}(a_0, a_1, a_2)[a_3]/(Q_5)$, viewing $a_0, \ldots, a_3$ as independent variables. To see this, by direct computations, $g_0 = g_1 = 0$, and $g_i = r_i \cdot Q_5$, $2 \le i \le 6$, for some polynomials $r_i \in \mathbb{C}(a_0, a_1, a_2)[a_3]$ with degree $\deg_{a_3}(r_i) = i-2$, $2 \le i \le 6$, which can be obtained by Euclid's division algorithm; for example, $r_2 = \frac{\text{coeff}(g_2,a_3,2)}{\text{coeff}(Q_5,a_3,2)}$, etc. The only thing to remark is that

$$5a_1^2 - 2\sqrt{15}a_0a_2 \not\equiv 0 \tag{5.6}$$

in (5.4). Otherwise, $a_2 = \frac{\sqrt{15}a_1^2}{6a_0}$, and then by (5.3), $0 = Q_5 = -\frac{(\sqrt{30}a_1^3 - 18a_0^2a_3)^2}{180a_0^4}$ to yield $a_3 = \frac{\sqrt{30}a_1^3}{18a_0^2}$, so that by the graph structure (5.2) we obtain $F = \frac{(\sqrt{6}a_0u + a_1v)^6}{216a_0^5}$, which contradicts the assumption that F does not coincide with the 1-dimensional orbit. (In the following Remark (5.1), we will motivate the choice of a and b given in the lift (5.4).) Moreover, F intersects the 1-dimensional orbit $PSL_2 \cdot u^6$ at the zeros of $a+b$.

In conclusion, we have the above commutative diagram (5.1). Next, we show that $\phi(\mathbb{C}P^1)$ is a projective line. We may assume that a, b, c, d are polynomials of an affine coordinate z, after factoring out the common denominator. Set $\alpha \triangleq \gcd(a, c)$, $\beta \triangleq \gcd(b, d)$, and $A \triangleq \begin{pmatrix} a/\alpha & b/\beta \\ c/\alpha & d/\beta \end{pmatrix}$. Then

$$\phi \cdot u^5v = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot u^5v = (A \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}) \cdot u^5v = (\beta^5\alpha)\, A \cdot u^5v = A \cdot u^5v,$$

after projectivizing. We may thus assume that $\gcd(a, c) = \gcd(b, d) = 1$ and $\phi = A$ in the following arguments.

By (3.7) and that F is nondegenerate in $\mathbb{C}P^6$, none of a, b, c, d are identically zero, from which there induces two non-constant holomorphic maps

$$\phi_1 : \mathbb{C}P^1 \to \mathbb{C}P^1,\ z \mapsto [a : c]; \qquad \phi_2 : \mathbb{C}P^1 \to \mathbb{C}P^1,\ z \mapsto [b : d].$$

Moreover, the coordinates $b_0, b_1, \cdots, b_6$ of F given in (3.7) cannot vanish simultaneously at any point of $\mathbb{C}$. Therefore,

$$\deg(F) = \max_{0\le i\le 6}\{\deg(b_i)\} \le \max\{\deg a, \deg c\} + 5\max\{\deg b, \deg d\} = \deg(\phi_1) + 5\deg(\phi_2), \tag{5.7}$$

where we have used the fact that b_i are homogeneous of bidegree $(1, 5)$ in (a, c) and (b, d), respectively. We assert that the reverse inequality of (5.7) also holds. To this end, multiplying a matrix from the left by interchanging the rows, we may assume that $\deg(a) \ge \deg(c)$. If $\deg(b) \ge \deg(d)$, then

$$\deg(F) \ge \deg(b_6) = \deg(a) + 5\deg(b) = \deg(\phi_1) + 5\deg(\phi_2);$$

otherwise, $\deg(F) \ge \deg(b_1) = \deg(a) + 5\deg(d) = \deg(\phi_1) + 5\deg(\phi_2)$. Hence, we have the equality in (5.7). Lastly, since both ϕ_1 and ϕ_2 are non-constant, it follows from $\deg(F) = 6$ that $\deg(\phi_1) = \deg(\phi_2) = 1$. Therefore ϕ is a projective line in $\mathbb{C}P^3$.

□

Remark 5.1. *It follows from the PSL_2-invariant theory that the curve F lying in the closed 2-dimensional orbit is equivalent to F and $\frac{\partial F}{\partial u}$ having a greatest common divisor G of positive degree in u. Indeed, their resultant with respective to u is $Res_u(F, \frac{\partial F}{\partial u}) = 62208a_0v^{30}Q_5^5 = 0$. Moreover, G can be found by Eculid's algorithm through $F = (\gamma u + \mu v)\frac{\partial F}{\partial u} + G$, where*

$$G = \frac{(2\sqrt{15}a_0a_2 - 5a_1^2)v^2}{6a_0}u^4 - \frac{(a_1a_2\sqrt{2} - 3a_0a_3)\sqrt{5}v^3}{3a_0}u^3 + \cdots \triangleq c_4u^4v^2 + c_3u^3v^3 + \cdots.$$

So, G is of degree 4 in u by (5.6)*. The proof that $\partial F/\partial u$ is divided by G, and G has a root b of multiplicity 4 is similar to* (5.5)*. Thus, by the relations between roots and coefficients for G, we derive $b = \frac{-c_3}{4c_4}$ given in* (5.4)*. Moreover, b is also the root of F with multiplicity 5, and the simple root $-a$ of F can also be found through $-(-a) - 5b = -\frac{\sqrt{6}a_1}{a_0}$.*

5.2. **The case when the curve does not lie in the closed 2-dimensional orbit.**

We identify the projectivization of the space of 2×2 nonzero (complex) matrices with $\mathbb{C}P^3$ by

$$\iota : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto [a : b : c : d].$$

Via ι, the subset of 2×2 matrices of zero determinant is the following PSL_2-invariant hyperquadric Q_2 of dimension 2,

$$Q_2 \triangleq \{[a : b : c : d] \in \mathbb{C}P^3 \mid ad - bc = 0\}. \tag{5.8}$$

Note that we can identify PSL_2 with $\mathbb{C}P^3 \setminus Q_2$.

Theorem 5.2. *Let $F : \mathbb{C}P^1 \to \mathcal{H}_0^3 \subset G(2, 5)$ be a sextic curve. If F does not lie in the closed 2-dimensional orbit $\overline{PSL_2 \cdot u^5 v}$, then there exists a compact Riemann surface $g : M \to \mathbb{C}P^3$ covering F as in the following commutative diagram*

$$\begin{array}{ccc} M & \xrightarrow{g} & \mathbb{C}P^3 \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle f_1} \\ \mathbb{C}P^1 & \xrightarrow{F} & \mathbb{C}P^6 \end{array} \tag{5.9}$$

Moreover, $\varphi : M \to \mathbb{C}P^1$ is a (branched) Galois covering, and the group of covering transformations $G \triangleq \{\sigma \in \mathrm{Aut}(M) \mid \varphi \circ \sigma = \varphi\}$ is a subgroup of S_4 isomorphic to the isotropy group at $uv(u^4 - v^4)$ given in item (1) *of Remark* (3.2)*.*

Proof. Recall the invariant quadric Q_5 defined in (3.8), which cuts the sextic curve F in a divisor of degree 12 with support points $q_1, \cdots, q_l$ by Bezout's theorem.

In the following, we abuse the notation to denote by q either a point of the curve $F(\mathbb{C}P^1)$ or its preimage on $\mathbb{C}P^1$, whenever there is no possibility of confusion.

Consider the complementary set $V \triangleq \mathbb{C}P^1 \setminus \{q_1, \ldots, q_l\}$; $F(V)$ lies in the open 3-dimensional orbit $Y \triangleq PSL_2 \cdot uv(u^4 - v^4)$. Let U be a connected component of the fibered product

$$U \subset V \times_Y PSL_2 \triangleq \{(p, B) \in V \times PSL_2 : F(p) = f_1(B)\}, \tag{5.10}$$

with the two standard projections π_1 and π_2 onto V and $PSL_2 \subset \mathbb{C}P^3$, respectively. Then U is an unramified covering space of V, by item (1) of Remark 3.2. We extend $\pi_1 : U \to V$ to a ramified covering $\varphi : M \to \mathbb{C}P^1$ by the monodromy representation [20, Theorem 8.4, p. 51], where M is a compact Riemann surface. Hence, we obtain the commutative diagram (5.9), where g extends π_2, φ extends π_1, and M is the desingularization of the closure of $\pi_2(U)$ in $\mathbb{C}P^3$.

Furthermore, the group of covering transformations $G = \{\sigma \in \mathrm{Aut}(M) \mid \varphi \circ \sigma = \varphi\}$ is isomorphic to the group

$$\widetilde{G} = \{C \in S_4 \mid \forall\ (q, B) \in U \subset V \times PSL_2,\ \text{s.t.}\ (q, BC) \in U\}.$$

It is easy to see that the elements of the group $\widetilde{G}$ permutes the points on a regular fiber of φ; thus, we obtain the isomorphism

$$\widetilde{G} \to G, \quad C \mapsto \sigma_C \triangleq [(q, B) \in U \mapsto (q, BC^{-1})],$$

whose inverse is given by

$$G \to \widetilde{G} \subset S_4, \quad \sigma \mapsto C_\sigma \triangleq g(\sigma(q))^{-1} g(q),\ \forall q \in U,$$

where C_σ is well-defined due to that U is connected and the isotropy group S_4 is finite.

Furthermore, the order of $\widetilde{G}$ equals $d \triangleq \deg \varphi$, the number of points on a regular fiber. Indeed, given a point $(q_0, B_0) \in U$, the fiber over q_0 is

$$\{(q_0, B_0 C_i),\ \mid C_i \in S_4,\ 1 \le i \le d\}.$$

By definition, we have $\widetilde{G} \subset \{C_1, \ldots, C_d\}$. On the other hand, for a given $1 \le j \le d$, since U and $U \cdot C_j \triangleq \{(q, BC_j) \mid \forall\ (q, B) \in U\}$ are two connected components of the fiber product $V \times_Y PSL_2$ through the same point $(q_0, B_0 C_j)$ and so are identical, we conclude that $C_j \in \widetilde{G}$.

To show the Galoisness of φ, given the data in (3.5) and (5.9), consider the polynomial equation

$$p(z, x) \triangleq \sum_{i=0}^{6} \sqrt{\tbinom{6}{i}} a_i(z) x^i = 0, \tag{5.11}$$

where the entries of $F(z) = [a_0(z) : \cdots : a_6(z)]$ belong to the polynomial ring $\mathbb{C}[z]$ such that the coefficients are relatively prime with the maximum degree 6. Then the splitting field of $p(\varphi, x) \triangleq \varphi^*(p(z, x))$ over $\mathbb{C}(\varphi)$, where φ is given in (5.9), is exactly the function field $\mathbb{C}(M)$ of the covering M.

To see this, the splitting field belongs to $\mathbb{C}(M)$ due to that the six roots of $p(\varphi, x)$ are linear fractions of the coordinate functions a, b, c, d of $g(M)$. In fact, the map g over M in (5.9) splits $f_1 \circ g = u^6 \cdot p(\varphi, \frac{v}{u})$ (see (5.11)) into

$$\begin{aligned} f_1 \circ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot uv(u^4 - v^4) = (du - bv)(av - cu)\big((du - bv) - (av - cu)\big)\times \\ \big((du - bv) + (av - cu)\big)\big((du - bv) + \sqrt{-1}(av - cu)\big)\big((du - bv) - \sqrt{-1}(av - cu)\big), \end{aligned}$$

where the six distinct roots are

$$\sigma_1 \triangleq \frac{d}{b},\ \sigma_2 \triangleq \frac{c}{a},\ \sigma_3 \triangleq \frac{c+d}{a+b},\ \frac{c-d}{a-b},\ \frac{c+\sqrt{-1}\,d}{a+\sqrt{-1}\,b},\ \frac{c-\sqrt{-1}\,d}{a-\sqrt{-1}\,b}. \tag{5.12}$$

Note that the denominators of the six roots can never be identically zero, since either of them being identically zero would imply $a_6 = ab(a^4 - b^4) = 0$ (the last coordinate in (3.5)), contradicting that F is linearly full.

Conversely, let $\mathbf{F} \supset \mathbb{C}(\varphi)$ be any intermediate field of the function field of M. If $\mathbf{F}$ contains the splitting field of (5.11), then we can use the first three roots in (5.12) to solve for $\frac{b}{a} = \frac{\sigma_2 - \sigma_3}{\sigma_3 - \sigma_1}$ so that the curve $g : M \to \mathbb{C}P^3$ is given by

$$[a : b : c : d] = [1 : \frac{b}{a} : \frac{c}{a} : \frac{d}{b} \cdot \frac{b}{a}] = [1 : \frac{\sigma_2 - \sigma_3}{\sigma_3 - \sigma_1} : \sigma_2 : \sigma_1 \frac{\sigma_2 - \sigma_3}{\sigma_3 - \sigma_1}]. \tag{5.13}$$

Thus $\mathbf{F}$ contains $\mathbb{C}(\varphi)(a, b, c, d) = \mathbb{C}(M)$ since $g : M \to \mathbb{C}P^3$ is generically injective.

We conclude that the covering $\varphi : M \to \mathbb{C}P^1$ is Galois of order $d = [\mathbb{C}(M) : \mathbb{C}(\varphi)]$, and the group of covering transformations of φ is the Galois group $\mathrm{Aut}(\mathbb{C}(M)/\mathbb{C}(\varphi))$.

□

For the lift $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : M \to \mathbb{C}P^3$ as in Theorem 5.2, to be referred to as a Galois lift of F, we associate it with two meromorphic functions

$$x \triangleq c/a, \ \ w \triangleq b/a.$$

We employ the geometry of the octahedron to study the Galois covering φ.

The quadric Q_2 defined in (5.8) is a saddle surface in $\mathbb{C}P^3$ isomorphic to $\mathbb{C}P^1 \times \mathbb{C}P^1$ by the parametrization $\begin{pmatrix} 1 & w \\ x & wx \end{pmatrix}$, where each pair (w, x) determines uniquely a point $p \in Q_2$, through which there passes a unique w-ruling $L_w \triangleq \{(w, x) \mid x \in \mathbb{C}P^1\}$. Let S^2 be the unit 2-sphere centered at the origin in $\mathbb{R}^3$, and let $\mathbb{C}$ be the complex plane projected onto by the stereographic projection $\eta : S^2 \setminus \{(0,0,1)\} \longrightarrow \mathbb{C}$, with $(0,0,1)$ mapped to ∞. We identify the regular octahedron in S^2 by sending its top and bottom vertices to $(0,0,1)$ and $(0,0,-1)$, respectively, and identifying the four horizontal vertices with $(\pm 1, 0, 0), (0, \pm 1, 0)$.

It is well-known that the symmetric group S_4 is isomorphic to the projective binary octahedral group, which acts on the regular octahedron as the rotational group of symmetry, consisting of the identity, 6 quarter turns and 3 half turns around the axes passing through two opposite vertices (see the 6 blue points in Figure 1), 6 half turns around the axes passing through two opposite edge centers (see the 12 red points in Figure 1), and 8 one-third turns around the axes passing through the centers of two opposite faces (see the 8 green points in Figure 1). The above 26 points (to be called centers in the following) enumerate all points on the regular octahedron whose stabilizers are nontrivial under the above group action.

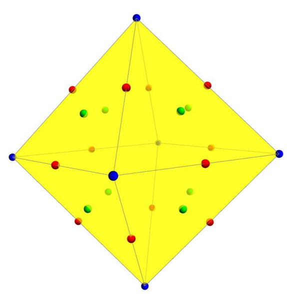

FIGURE 1. Octahedron

Composing the central projection of the regular octahedron on S^2 with the stereographic projection, we can build a one-to-one correspondence between the points on the octahedron and points on the extended plane $\mathbb{C} \cup \{\infty\}$. Under this correspondence, the above 26 centers turn out to be the roots of the polynomial equation

$$(w^5 - w)(w^8 + 14w^4 + 1)(w^{12} - 33w^8 - 33w^4 + 1) = 0, \tag{5.14}$$

where we also count $w = \infty$ as a root. We point out that the above three polynomial factors take exactly the vertices, edge centers, and face centers as their roots, respectively. Furthermore, via this correspondence, the symmetric group S_4 (isomorphic to the projective binary octahedral group) acts on the w-rulings of Q_2, and is exactly the action of the isotropic group in item (1) of Remark (3.2) on $\begin{pmatrix} 1 & w \\ x & wx \end{pmatrix}$ by matrix multiplication on the right, where the 26 centers are related to the 26 distinct eigenvectors of these isotropy matrices.

Now, we introduce two important divisors to study the Galois covering (5.9). In the following, we denote the degree of the covering $\varphi : M \to \mathbb{C}P^1$ by d, and the degree of the curve $g : M \to \mathbb{C}P^3$ by k.

Let $\mathcal{Q}$ be the intersection divisor defined by g and the quadric Q_2. By Bezout's theorem, we have $\deg(\mathcal{Q}) = 2k$.

In the following, we say that a hypersurface $G = 0$ of degree t in $\mathbb{C}P^6$ is *generic* if it does not contain the curve F and it cuts out a divisor on F whose support lives in $V = \mathbb{C}P^1 \setminus F^{-1}(Q_5)$. Projective normality [32, pp.230-231]) of the rational normal curve F warrants the existence of generic hypersurfaces.

A generic hyperplane $H = \sum_{i=0}^{6} c_i\, a_i = 0$ in $\mathbb{C}P^6$ with coordinates $[a_0 : \cdots : a_6]$ cuts $\gamma = F(M)$ in a divisor D_H of degree 6 whose support lies in V, while f_1 pulls the hyperplane $H = 0$ back to a hypersurface of degree 6 in $\mathbb{C}P^3$ that cuts g in a divisor $\mathcal{D}$ of degree $6k$ by Bezout's theorem. Since $\varphi|_U$ is a covering map of degree d over V, the divisor $\mathcal{D}$ contains the pullback divisor $\mathcal{D}_0 \triangleq \varphi^*(D_H)$ of degree $6d$. Define their difference by $\mathcal{F}$,

$$\mathcal{F} \triangleq \mathcal{D} - \mathcal{D}_0. \tag{5.15}$$

In the following, we denote the support of a divisor $\mathbf{D}$ by $\operatorname{Supp} \mathbf{D}$.

Remark 5.2. *$\mathcal{F}$ is the fixed part of the intersection divisors of g with the hypersurfaces of degree 6 obtained by the coordinates of f_1 given in* (3.5)*, namely,*

$$\mathcal{F} = \min_{0 \le i \le 6} \{g^*(a_i \circ f_1)\}. \tag{5.16}$$

Moreover, let $G = 0$ be a generic hypersurface of degree t in $\mathbb{C}P^6$ not containing the curve F. Then

$$g^*(G \circ f_1) = \varphi^*(F^* G) + t\mathcal{F}. \tag{5.17}$$

Proposition 5.1. *Let $F : \mathbb{C}P^1 \to \mathcal{H}_0^3 \subset \mathbb{C}P^6$ be a sextic curve not lying in the closed 2-dim orbit $\overline{PSL_2 \cdot u^5 v}$, and $g : M \to \mathbb{C}P^3$ be the Galois lift of F in the commutative diagram* (5.9)*. Then*

$$\mathcal{F} \le \mathcal{Q}. \tag{5.18}$$

Moreover, for any given $p \in \operatorname{Supp} \mathcal{Q}$,

1. *if $w(p)$ is not associated with any of the 6 vertices, then $\mathrm{ord}_p(\mathcal{F}) = 0$ and $f_1 \circ g(p)$ lies in the 1-dim orbit $PSL_2 \cdot u^6$, and*
2. *if $w(p)$ is associated with one of the 6 vertices, then $\mathrm{ord}_p(\mathcal{F}) > 0$ and $f_1 \circ g(p)$ lies in the 1-dimensional (respectively, 2-dimensional) orbit if and only if $\mathrm{ord}_p(\mathcal{F}) < \mathrm{ord}_p(\mathcal{Q})$ (respectively, $\mathrm{ord}_p(\mathcal{F}) = \mathrm{ord}_p(\mathcal{Q})$).*

Proof. Recall the quadratic Q_5 in (3.8). Via f_1 in (3.5) we have the remarkable SL_2-invariant identity

$$Q_5 = 2a_0 a_6 - 2a_1 a_5 + 2a_2 a_4 - a_3^2 = (ad - bc)^6. \tag{5.19}$$

Therefore, we derive that the support of $\mathcal{F}$ is contained in that of $\mathcal{Q}$, since the former one can be further determined (see Remark 5.2) by setting the coordinate functions zero, i.e.,

$$a_i \circ f_1 \circ g = 0, \quad 0 \le i \le 6. \tag{5.20}$$

Suppose that $p \in \operatorname{Supp} \mathcal{Q}$, i.e., $g(p) \in Q_2$. By the action of PSL_2 on $\mathcal{H}_0^3$, we may assume

$$g(p) = \begin{pmatrix} 1 & w \\ 0 & 0 \end{pmatrix}. \tag{5.21}$$

If $w(p)$ is not associated with any of the 6 vertices, i.e., $w^4 \neq 0, 1, \infty$, then let U_p be a chart around p with local coordinate s and $s(p) = 0$. From (3.5) and $\operatorname{ord}_p(c), \operatorname{ord}_p(d) \geq 1$, we obtain $0 = \operatorname{ord}_p(a_6) < \min_{0 \leq i \leq 5} \{\operatorname{ord}_p(a_i)\}$; thus $f_1 \circ g(p) = [0:0:0:0:0:0:1]$ lies in the 1-dimensional orbit $PSL_2 \cdot u^6$, whence

$$\operatorname{ord}_p(\mathcal{F}) = \min_{0 \leq i \leq 6} \{\operatorname{ord}_p(a_i)\} = \operatorname{ord}_p(a_6) = 0 < \operatorname{ord}_p(ad - bc) = \operatorname{ord}_p(\mathcal{Q}).$$

Next, we assume $w^4 = 0, 1$, or ∞. By the action of PSL_2 and the isotropy group of u^6 (see Remark (3.2)) on $\mathcal{H}_0^3$, we may assume that

$$g(p) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{5.22}$$

Let U_p be a chart around p with local coordinate s and $s(p) = 0$. By (3.5) and the property $\operatorname{ord}_p(b), \operatorname{ord}_p(c), \operatorname{ord}_p(d) \geq 1$, we obtain that if $\operatorname{ord}_p(d) \leq \operatorname{ord}_p(b)$, then $\operatorname{ord}_p(a_5) < \operatorname{ord}_p(a_j)$ for any $j \neq 5$, so that $f_1 \circ g(p) = [0:0:0:0:0:1:0]$ lies in the open 2-dimensional orbit, whence

$$\operatorname{ord}_p(\mathcal{F}) = \min_{0 \leq i \leq 6} \{\operatorname{ord}_p(a_i)\} = \operatorname{ord}_p(a_5) = \operatorname{ord}_p(d) = \operatorname{ord}_p(ad - bc) = \operatorname{ord}_p(\mathcal{Q}) > 0.$$

On the other hand, if $\operatorname{ord}_p(d) > \operatorname{ord}_p(b)$, then $\operatorname{ord}_p(a_6) < \operatorname{ord}_p(a_k)$ for $0 \leq k \leq 5$, so that $f_1 \circ g(p) = [0:0:0:0:0:0:1]$ lies in the 1-dimensional orbit to yield

$$\operatorname{ord}_p(\mathcal{F}) = \operatorname{ord}_p(a_6) = \operatorname{ord}_p(b) < \operatorname{ord}_p(ad - bc) = \operatorname{ord}_p(\mathcal{Q}).$$

In conclusion, $f_1 \circ g(p)$ lies in the open 2-dimensional orbit if and only if $\operatorname{ord}_p(\mathcal{F}) = \operatorname{ord}_p(\mathcal{Q})$. □

Corollary 5.1. *Assume the same setting as in Proposition* 5.1. *We have* $\deg \varphi \leq \deg g \leq \frac{3}{2} \deg \varphi$. *Moreover,*

(1) $\deg g = \deg \varphi$ *if and only if* $\mathcal{F} = 0$.
(2) $\deg g = \frac{3}{2} \deg \varphi$ *if and only if* $\mathcal{F} = \mathcal{Q}$.
(3) *If* $\deg \varphi = 1$, *then* $\deg g = 1$ *so that* $g(M)$ *is a line in* $\mathbb{C}P^3$.

Proof. Counting the degree of both sides of (5.15), by (5.18) we obtain

$$6 \deg g - 6 \deg \varphi = \deg(\mathcal{F}) \leq \deg(\mathcal{Q}) = 2 \deg g,$$

which implies $\deg \varphi \leq \deg g \leq \frac{3}{2} \deg \varphi$ with the equality conditions as asserted in (1) and (2). In particular, if $\deg \varphi = 1$, then $\deg g = 1$ and $g(M)$ is a line. □

5.3. **The Generally Ramified Family.**

By Theorem 4.1, a sextic curve γ in $\mathcal{H}_0^3$ is ramified in the sense of harmornic sequences at a point q if and only if the tangent line of γ at q lies in $\mathcal{H}_0^3$. An important class of lines in $\mathcal{H}_0^3$ is given by the rulings of the tangent developable surface $\mathbf{S}$ (i.e., the closed 2-dimensional PSL_2-orbit), which are exactly the tangent lines of the 1-dimensional orbit $PSL_2 \cdot u^6$; in particular, that there is a unique line though q in the 1-dimensional orbit implies that γ is ramified at q if and only if γ is tangent to the 1-dimensional orbit at q. Our investigation of various examples and Galois analysis have prompted the following definition.

Definition 5.1. *We say that a sextic curve γ in $\mathcal{H}_0^3$ is in the* **generally ramified family** *if γ is ramified (as always, in the sense of harmonic sequences) at the 1-dimensional orbit $PSL_2 \cdot u^6$ somewhere.*

Now, we give a characterization of tangency of γ at the 1-dimensional orbit in terms of intersection divisors. In the following, we denote the intersection multiplicity at a point $q \in \gamma \cap Q_5$ by $\mathrm{ord}_q(Q_5)$, and we stipulate that $\mathrm{ord}_q(Q_5) = +\infty$ if γ lies in Q_5.

Proposition 5.2. *Let $F : \mathbb{C}P^1 \to \mathcal{H}_0^3$ be a sextic curve. F is ramified at the 1-dimensional orbit at q if and only if $\mathrm{ord}_q(Q_5) \geq 4$.*

We defer the proof of this proposition to that of Proposition 6.1 for the sake of not interrupting the smoothness of exposition. Immediately we obtain the following.

Corollary 5.2. *Let $F : \mathbb{C}P^1 \to \mathcal{H}_0^3$ be a sextic curve. If either the degree of the covering $\varphi : M \to \mathbb{C}P^1$ in Theorem 5.2 equals 1, or F lives in the closed 2-dimensional PSL_2-orbit, then F belongs to the generally ramified family.*

Proof. When F lives in the closed 2-dimensional orbit, the conclusion holds because the curve intersects the 1-dimensional orbit at a point q by Theorem 5.1, while the fact that $Q_5 \equiv 0$ on this curve implies $\mathrm{ord}_q(Q_5) = +\infty$.

For the other case, it follows from Corollary 5.1 that the line $g(M)$ cuts Q_2 in two points p_1 and p_2. Since $\mathcal{F} = 0$, p_1 and p_2 are mapped to points on the 1-dimensional orbit by $f_1 \circ g$, at which there must hold $\mathrm{ord}_{p_i}(Q_5) \geq 6$ for $i = 1$ or 2. Then the conclusion follows from Proposition 5.2. □

Henceforth, we assume that F does not lie in the closed 2-dimensional orbit.

Lemma 5.1. *Consider the Galois covering $\varphi : M \to \mathbb{C}P^1$ with the Galois group $G \subset S_4$ in the same setting as in Theorem 5.2. Given $p \in \mathrm{Supp}\,\mathcal{Q}$, denote by $\mathrm{mult}_p(\varphi)$ the multiplicity of φ at p, i.e., $\varphi : s \mapsto s^{\mathrm{mult}_p(\varphi)}$ for a local uniformizing parameter s with $s(p) = 0$.*

(1) *If $w(p)$ is not associated with any of the 26 centers of the octahedron, then $\mathrm{mult}_p(\varphi)=1$.*
(2) *If $w(p)$ is associated with one of the 12 edge centers, then $\mathrm{mult}_p(\varphi) = 1$ or 2.*
(3) *If $w(p)$ is associated with one of the 8 face centers, then $\mathrm{mult}_p(\varphi) = 1$ or 3.*
(4) *If $w(p)$ is associated with one of the 6 vertices, then $\mathrm{mult}_p(\varphi) = 1$ or 2 or 4.*

Proof. Let $G_p \triangleq \{\sigma \in G \mid \sigma(p) = p\}$ be the stabilizer of p. Then by [32, Proposition 3.1, p. 76; Theorem 3.4, p. 78], we have $\mathrm{mult}_p(\varphi) = |G_p|$, and G_p is a finite cyclic subgroup of G, and hence of S_4. Since the non-trivial finite cyclic subgroups of S_4 are C_2, C_3, and C_4, we infer $1 \leq \mathrm{mult}_p(\varphi) = |G_p| \leq 4$. As said before, G_p is trivial when $w(p)$ does not correspond to any of the 26 centers of the regular octahedron, from which the conclusion in (1) follows. Moreover, with respect to the action of S^4 on the octahedron, the stabilizer of the vertex (respectively, edge center, face center) is isomorphic to C_4 (respectively, C_2, C_3), from which the conclusions in (2) $\sim$ (4) follow.

□

Now, we provide sufficient conditions for the curve F to belong to the generally ramified family.

Theorem 5.3. *Let $F : \mathbb{C}P^1 \to \mathcal{H}_0^3$ be a sextic curve which is not contained in the closed 2-dimensional PSL_2-orbit. If one of the following holds, then F belongs to the generally ramified family.*

(1) *There exists a point* $p \in \operatorname{Supp} \mathcal{Q}$ *such that* $w(p)$ *is not associated with any of the* 26 *centers of the octahedron, i.e.,* $w(p)$ *does not satisfy* (5.14).
(2) *There exists a point* $p \in \operatorname{Supp} \mathcal{Q} \backslash \operatorname{Supp} \mathcal{F}$ *such that either* $\operatorname{mult}_p(\varphi) = 1$*, or* $\operatorname{ord}_p(\mathcal{Q}) \geq 2$ *and* $w(p)$ *is associated with one of the* 12 *edge centers and* 8 *face centers.*
(3) *There exists a point* $p \in M$ *such that* $0 < \operatorname{ord}_p(\mathcal{F}) < \operatorname{ord}_p(\mathcal{Q})$.

Proof. From (5.17) and (5.19), we obtain $\varphi^*(F^*Q_5) + 2\mathcal{F} = 6\mathcal{Q}$; hence, for any point $p \in M$, we have

$$\operatorname{mult}_p(\varphi) \operatorname{ord}_{\varphi(p)}(Q_5) = 6 \operatorname{ord}_p(\mathcal{Q}) - 2 \operatorname{ord}_p(\mathcal{F}) = 6\big(\operatorname{ord}_p(\mathcal{Q}) - \operatorname{ord}_p(\mathcal{F})\big) + 4 \operatorname{ord}_p(\mathcal{F}). \quad (5.23)$$

We will use Proposition 5.1 and Proposition 5.2 to prove this theorem.

The conclusion for condition (1) follows from $\operatorname{ord}_p(\mathcal{Q}) > 0 = \operatorname{ord}_p(\mathcal{F})$, and item (1) of Lemma 5.1.

Under condition (2), we have $\operatorname{ord}_p(\mathcal{F}) = 0$ while

$$\text{either } \operatorname{ord}_p(\mathcal{Q}) \geq 2 \text{ and } \operatorname{mult}_p(\varphi) \leq 3, \quad \text{or } \operatorname{ord}_p(\mathcal{Q}) \geq 1 \text{ and } \operatorname{mult}_p(\varphi) = 1,$$

where items (2) and (3) of Lemma 5.1 are used. Substituting these into (5.23), we obtain $\operatorname{ord}_{\varphi(p)}(Q_5) \geq 4$. Therefore, F is tangent to the 1-dimensional orbit at $F \circ \varphi(p)$.

Under condition (3), we have

$$\operatorname{ord}_p(\mathcal{F}) \geq 1, \ \operatorname{ord}_p(\mathcal{Q}) - \operatorname{ord}_p(\mathcal{F}) \geq 1,$$

which implies $\operatorname{mult}_p(\varphi) \operatorname{ord}_{\varphi(p)}(Q_5) \geq 10$. Note that in this case, $\operatorname{mult}_p(\varphi) = 1, 2$, or 4. The conclusion follows from that the minimal integer of the form in (5.23) is 16 when it is a multiple of 4 and is ≥ 10. □

In contrast to Definition 5.1, we introduce the following definition.

Definition 5.2. *If a sextic curve in* $\mathcal{H}_0^3$ *is not tangent to the* 1*-dimensional orbit* $PSL_2 \cdot u^6$*, then we say that it lies in the* exceptional transversal family.

By Lemma 5.1 and Theorem 5.3, we obtain the following necessary conditions for the exceptional transversal family.

Proposition 5.3. *Let* $F : \mathbb{C}P^1 \to \mathcal{H}_0^3$ *be a sextic curve that belongs to the exceptional transversal family. Then for any point* $p \in \operatorname{Supp} \mathcal{Q}$*, there holds that* $w(p)$ *corresponds to one of the* 26 *centers of the octahedron. Moreover,*

(1) $\operatorname{Supp} \mathcal{F} = \operatorname{Supp} \mathcal{Q}$ *if and only if* $\mathcal{F} = \mathcal{Q}$*, and*
(2) *for any given* $p \in \operatorname{Supp} \mathcal{Q} \setminus \operatorname{Supp} \mathcal{F}$*, we have* $\operatorname{ord}_p(\mathcal{Q}) = 1$*, and* $w(p)$ *is either one of the* 12 *edge centers, for which*

$$\operatorname{mult}_p(\varphi) = 2, \ \operatorname{ord}_{\varphi(p)}(Q_5) = 3,$$

or one of the 8 *face centers, for which*

$$\operatorname{mult}_p(\varphi) = 3, \ \operatorname{ord}_{\varphi(p)}(Q_5) = 2.$$

Note that for points in item (2) of the preceding proposition, F intersects the 1-dimensional orbit $PSL_2 \cdot u^6$ at $f_1 \circ g(p)$ transversally.

5.4. **The Exceptional Transversal Family.**

We now look at the exceptional transversal family in a unified fashion.

Let G be the group of covering transformations of $\varphi : M \to \mathbb{C}P^1$ in the same setting as in Theorem 5.2. As a subgroup of S_4, the Galois group G can only be one of the following subgroups: the trivial group, the cyclic groups $C_i,\ 2 \leq i \leq 4$, the dihedral groups $D_j,\ 2 \leq j \leq 4$, the alternating group A_4, and S_4 itself.

When $M = \mathbb{C}P^1$, the above Galois coverings were classified by Klein [29] as given in Table 1 below.

G	C_i	D_j	A_4	S_4
$\varphi(s)$	s^i	$s^j - 2 + s^{-j}$	$\frac{(s^3-1)^3}{s^3(s^3+8)^3}$	$\frac{(s^8+14s^4+1)^3}{(s(s^4-1))^4}$

Table 1. Rational Galois Coverings

Given a sextic curve belonging to the exceptional transversal family, we label the points of intersection of this curve and the 1-dimensional orbit as points of type I, and the points of intersection of this curve and the open 2-dimensional orbit as points of type II.

Let q be a point of type I. For each ramified point p over $q = \varphi(p)$, it follows from Proposition 5.3 that $\mathrm{ord}_p(\mathcal{Q}) = 1$, and

$$\text{either}\quad \mathrm{mult}_p(\varphi) = 2 \text{ and } \mathrm{ord}_q(Q_5) = 3, \quad \text{or } \mathrm{mult}_p(\varphi) = 3 \text{ and } \mathrm{ord}_q(Q_5) = 2.$$

Let Σ_1 and Σ_2 be the number of points assuming $\mathrm{ord}_q(Q_5) = 2$ and 3, respectively. Then

$$l \triangleq 2\Sigma_1 + 3\Sigma_2 \tag{5.24}$$

satisfies $0 \leq l \leq 12$. It is easy to verify that the total $\mathcal{Q}$-degree for type I is

$$(\deg\varphi/3)\Sigma_1 + (\deg\varphi/2)\Sigma_2 = l\ \deg\varphi/6, \tag{5.25}$$

where $\deg\varphi/3$ (respectively, $\deg\varphi/2$) is the number of ramified points p over q with $\mathrm{mult}_p(\varphi) = 3$ (respectively, $\mathrm{mult}_p(\varphi) = 2$) [32, Lemma 3.6, p. 80].

Let q be a point of type II. For each ramified point p over q, it follows from Proposition 5.3 and (5.23) that $4\,\mathrm{ord}_p(\mathcal{Q}) = \mathrm{mult}_p(\varphi)\,\mathrm{ord}_q(Q_5)$, which implies that the total $\mathcal{Q}$-degree over q is

$$\sum_{p\in\varphi^{-1}(q)} \mathrm{ord}_p(\mathcal{Q}) = \sum_{p\in\varphi^{-1}(q)} \mathrm{mult}_p(\varphi)\ \mathrm{ord}_q(Q_5)/4 = \deg\varphi\ \mathrm{ord}_q(Q_5)/4. \tag{5.26}$$

Therefore the total $\mathcal{Q}$-degree for type II is

$$\sum_{q\,\text{of type II}} \deg\varphi\ \mathrm{ord}_q(Q_5)/4 = \left(\sum_{q\,\text{of type II}} \mathrm{ord}_q(Q_5)\right) \deg\varphi/4 = (12-l)\deg\varphi/4. \tag{5.27}$$

Hence

$$2\deg(g) = \deg(\mathcal{Q}) = l\deg\varphi/6 + (12-l)\deg\varphi/4 = (36-l)\deg\varphi/12,$$

which implies $\deg g = (36-l)\deg\varphi/24$.

To illustrate, consider $\deg\varphi = 2$, for which $\deg g = (36-l)\deg\varphi/24$ gives $l = 0$ or 12.

If $l = 0$ then $\deg g = 3$; all points q of Supp $\mathcal{Q}$ live in the 2-dimensional orbit $PSL_2 \cdot u^5 v$. We seek to find examples where the genus of M is zero. The Riemann-Hurwitz formula dictates that there be exactly two points q_1 and q_2 of type II over each of which there sits a single ramified point p_1 and p_2, respectively, with ramification index 1 ($\mathrm{mult}_{p_i}(\varphi) = 2$), so that the formula $4\,\mathrm{ord}_p(\mathcal{Q}) = \mathrm{mult}_p(\varphi)\,\mathrm{ord}_q(Q_5)$ gives that $\mathrm{mult}_{q_i}(Q_5)$ are multiples of 2.

There may exist other points $q_3, \cdots, q_m$ of type II over each of which there sit two ramified points p_{j1} and $p_{j2}, 3 \leq j \leq m$, each with ramification index 0 ($\mathrm{mult}_{p_{jk}}(\varphi) = 1$) so that $\mathrm{ord}_{q_j}(Q_5)$ is a multiple of 4 for $3 \leq j \leq m$. In the most generic situation, $\mathrm{ord}_{q_i}(Q_5) = 2$ for $i = 1, 2$ and $\mathrm{ord}_{q_j}(Q_5) = 4$ for $3 \leq j \leq m$, for which we have the constraint

$$12 = 2 + 2 + 4(m-2), \quad \text{so } m = 4.$$

In other words, there are four points $q_1, \cdots, q_4$ of type II, where the Galois covering is unramified over q_3 and q_4.

Indeed, up to a PSL_2-transformation on the left and an isotropy group action on the right of the Galois lift, a detailed Galois analysis, which we will report elsewhere, proves that this is the only possibility with the Galois lift $g(s)$ given by

$$g = [1 : w : x : yw],$$
$$x = -\sqrt{-1}(t^2 - 1)/((-t + \sqrt{-1})s^2 + t^3 - \sqrt{-1}), \quad y = (-\sqrt{-1}t^2 + \sqrt{-1}s^2 - ts^2 + t)/(t^3 - t),$$
$$w = (t^3 - t)/(s((-t + \sqrt{-1})s^2 + t^3 - \sqrt{-1})),$$

whenever t is not a zero of a certain polynomial of a large degree which we do not record here. The Galois covering φ is $z = s^2$ corresponding to the group C_2, q_1 and q_2 are $z = 0$ and $z = \infty$, and q_3 and q_4 are $z = 1$ and $z = t^4$.

If $l = 12$ then $\deg g = \deg \varphi = 2$. Since γ is in the exceptional transversal family, each point q of type I has $\mathrm{ord}_q(Q_5) = 3$ (since $\mathrm{mult}_p(\varphi) = 2$ as $\deg \varphi = 2$). As a result, there are four points $q_1, \cdots, q_4$ of type I over each of which there sits a single ramified point $p_1, \cdots, p_4$, respectively, each with ramification index 1. The Riemann-Hurwitz formula implies that there do not exist any such Galois lifts g with genus zero.

Suffices it to say that a detailed Galois analysis proves that when $\deg g = \deg \varphi = 2$, there are two 1-parameter classes of Galois lifts in the generally ramified family.

As another example, let us find a procedure to determine the structure of M with genus zero, for which the Riemann-Hurwitz formula gives

$$-2 \geq -2 \deg \varphi + 2(\deg \varphi/3)\Sigma_1 + (\deg \varphi/2)\Sigma_2,$$

so that $(2\Sigma_1/3 + \Sigma_2/2 - 2) \deg \varphi \leq -2$ from which we determine, since $2\Sigma_1/3 + \Sigma_2/2 - 2 < 0$, an even l to make sure $\deg g = (36 - l) \deg \varphi/24$ is an integer, which comes down to

$$(l, \Sigma_1, \Sigma_2) = (8, 1, 2), (6, 0, 2), (4, 2, 0), (2, 1, 0).$$

If we set $\deg \varphi \geq 4$, then $(2\Sigma_1/3 + \Sigma_2/2 - 2) \deg \varphi \leq -2$ gives $2\Sigma_1/3 + \Sigma_2/2 \leq 3/2$, from which we narrow it down to

$$(l, \Sigma_1, \Sigma_2) = (8, 1, 2), (6, 0, 2), (2, 1, 0).$$

We choose $(8, 1, 2)$ to find an example. Since $l = 8$ for group 1, whose structural constants $\mathrm{ord}_q(Q_5)$ and $\mathrm{ord}_p(\varphi)$ are known to leave the relatively small number 4 for group 2, we calculate

$$(l, \deg \varphi, \deg g) = (8, 4, 6), \quad (8, 6, 7), \quad (8, 12, 14), \quad (8, 24, 28).$$

If we seek to find an example with an irreducible $p(z, x)$ so that $\deg \varphi \geq 6$, we should start with $(8, 6, 7)$, where the genus of M is zero.

In more details, since $\Sigma_1 = 1$ and $\Sigma_2 = 2$, we have three points q_1, q_2, q_3 of type I such that $\mathrm{ord}_{q_1}(Q_5) = 2$ and $\mathrm{ord}_{q_2}(Q_5) = \mathrm{ord}_{q_3}(Q_5) = 3$, where each ramified point p_{1j} sitting over q_1 has $\mathrm{mult}_{p_{1j}}(\varphi) = 3$ while each ramified point p_{2l}, p_{3s} over q_2 and q_3 has

$\mathrm{mult}_{p_{2l}}(\varphi) = \mathrm{mult}_{p_{3s}}(\varphi) = 2$. Therefore, there are two ramified points over q_1 and three ramified points over each of q_2 and q_3.

Switching to the points of type II, since

$$-2\deg\varphi + (2\deg\varphi/3)\Sigma_1 + (\deg\varphi/2)\Sigma_2 = -2$$

already verifies the Riemann-Hurwitz formula, we see that all points of type II are unramified. We have at most 4 such kind of points. To make (5.26) an integer, for such a point $\tilde{q}$ in the formula, there must hold $\mathrm{ord}_{\tilde{q}}(Q_5) \geq 4$ since $\mathrm{mult}_{\tilde{p}}(\varphi) = 1$ for each ramified point $\tilde{p}$ over $\tilde{q}$. But then this means that $\tilde{q}$ is the only such kind of point since the total Q_5-degree for type II is 4. Therefore, there are six ramified points sitting over $\tilde{q}$ each with ramification index 0.

Indeed, a detailed Galois analysis in the case of genus zero proves that this is the only possibility (up to left PSL_2 and right isotropy actions):

$$\begin{aligned}
&g = [a : b : c : d],\\
&a = ((\sqrt{3}\sqrt{-1} - \sqrt{-1} - (1+\sqrt{-1})b_3 s^4)\sqrt{2})/2 - (((-1+\sqrt{-1})b_3\sqrt{3} - 2s^4 + (1-\sqrt{-1})b_3)s^3)/2,\\
&b = ((((-1+\sqrt{-1})\sqrt{3} + 2\sqrt{-1}b_3 s^4 - 1 + \sqrt{-1})\sqrt{2} - 2(-b_3\sqrt{3} + (1+\sqrt{-1})s^4 - b_3)s^3)/(2+2\sqrt{3}),\\
&c = -(((1-\sqrt{-1}) + (1+\sqrt{-1})(s^2+d_3)s^3\sqrt{2} + (1-\sqrt{-1})(-d_3 s^2+1)\sqrt{3} + (1-\sqrt{-1})d_3 s^2)s)/2,\\
&d = -((((s^2-d_3)\sqrt{3} + s^2 + d_3)s^3\sqrt{2}\sqrt{-1} - 2d_3 s^2 - 2)s)/2, \quad \text{where,}
\end{aligned}$$

$$\begin{aligned}
b_3 &= (-1/6 - \sqrt{-1}/6)(\sqrt{-6}t^6 + \sqrt{3}d_3 t^3 - 3d_3 t^3 + \sqrt{-6})\sqrt{3}/t^3,\\
d_3 &= -(\sqrt{-6}t^5 - \sqrt{-2}t^5 + 2)/[t^2(\sqrt{-6}t - \sqrt{-2}t + 2\sqrt{3} - 4)].
\end{aligned}$$

Moreover, q_1 corresponds to $z = \infty$, q_2 corresponds to $z = 0$, q_3 corresponds to $z = -4$, and $\tilde{q}$ corresponds to $z = t^3 - 2 + 1/t^3$. No Galois lifts with $k = 6, 8, 9$ exist. Here, the Galois covering φ is $z = s^3 - 2 + 1/s^3$ with the Dihedral group D_3.

To end this subsection, due to its length, we only summarize our classification of Galois covering when $M = \mathbb{C}P^1$ in Table 2 below.

$\deg\varphi$	G	$\deg g$	Dimension of the Moduli Spaces	
			Generally Ramified Family	Exceptional Transversal Family
2	C_2	2	1	∅
		3	∅	1
3	C_3	3	1	∅
		4	∅	1
4	C_4	4, 6	∅	∅
		5	2	∅
	D_2	4, 6	∅	∅
		5	2	1
6	D_3	6, 8, 9	∅	∅
		7	∅	1
8	D_4	8, 10, 12	∅	∅
		9	1	∅
		11	∅	1
12	A_4	$12, 14, 16 \sim 18$	∅	∅
		13	1	∅
		15	0	∅
24	S_4	$24, 26 \sim 28, 30, 32 \sim 36$	∅	∅
		25	2	∅
		29	1	∅
		31	∅	1

TABLE 2. Classification of Rational Galois Coverings.

In particular, there are at most finitely many constantly curved sextic curves $\subset \mathcal{H}_0^3$ on the list that belong to the exceptional transversal family, by checking total unramification encountered in Section 4. In view of the above examples and classification, it is tempting to suggest that a constantly curved holomorphic 2-sphere in $G(2,5)$, which differs from a sextic curve in the exceptional transversal family by a $GL(5,\mathbb{C})$-automorphism, be nongeneric among all constantly curved holomorphic 2-spheres of degree 6.

On the other hand, the situation in the generally ramified family is clear-cut. We will show in the next section that a constantly curved holomorphic 2-sphere of degree 6 in $G(2,5)$, which differs from a sextic curve γ in the generally ramified family by a $GL(5,\mathbb{C})$-transformation, is such that the 6-plane $\mathbf{L}$ it spans in $\mathbb{C}P^9$ differs from that spanned by the standard Veronese curve (1.1) only by a diagonal matrix in $GL(5,\mathbb{C})$.

6. Generally ramified holomorphic 2-spheres of degree 6 in $G(2,5)$

Thanks to the discussion in Section 5, we say that a holomorphic 2-sphere of degree 6 in $G(2,5)$ is *generally ramified* if it is projectively equivalent to a sextic curve in $\mathcal{H}_0^3$ belonging to the generally ramified family. In this section, we will first give a useful parameterization to such kind of 2-spheres, then employ it to investigate such 2-spheres of constant curvature. We will show that a generally ramified constantly curved holomorphic 2-sphere of degree 6 can only live in the Fano 3-folds $\mathcal{H}^3$ that differ from the standard $\mathcal{H}_0^3$ by a diagonal transformation in $GL(5,\mathbb{C}^5)$, up to $U(5)$-equivalence.

Definition 6.1. *By the* **diagonal family** *we mean constantly curved holomorphic* 2*-spheres of degree* 6 *in* $G(2,5)$ *parameterized as follows*:

$$\operatorname{diag}(a_{00},\cdots,a_{44})\cdot(E_0,E_1,\ldots,E_6)\ \operatorname{diag}\{\omega_0,\omega_1,\ldots,\omega_6\}\,Z_6(z), \tag{6.1}$$

where $\{E_0,\ldots,E_6\}$ *is the orthonormal basis of* V_6 *defined in* (3.3)*, and* $Z_6(z)$ *is the Veronese* 2*-sphere in* (2.8)*.*

The following is the main result of this section.

Theorem 6.1. *Let* $\gamma:\mathbb{C}P^1\to G(2,5)$ *be a generally ramified holomorphic* 2*-sphere of degree* 6*. If* γ *is of constant curvature, then* γ *belongs the the diagonal family.*

6.1. Sextic curves in $\mathcal{H}_0^3$ ramified at the 1-dimensional orbit.

Proposition 6.1. *Let* $\gamma:\mathbb{C}P^1\to\mathcal{H}_0^3$ *be a sextic curve, and let* p *be a point of the* 1*-dimensional orbit.*
(1) γ *is ramified at* p*, if and only if, up to a transformation in* $SL(2,\mathbb{C})$*,* γ *can be parameterized as*

$$\gamma(z)=L\begin{pmatrix}1 & z & z^2 & \cdots & z^6,\end{pmatrix}^t,\quad \textit{where} \tag{6.2}$$

$$L=\begin{pmatrix}L_{00}&L_{01}&L_{02}&0&0&0&0\\ L_{10}&L_{11}&L_{12}&0&0&0&0\\ L_{20}&L_{21}&L_{22}&L_{23}&0&0&0\\ L_{30}&L_{31}&L_{32}&L_{33}&L_{34}&0&0\\ L_{40}&L_{41}&L_{42}&L_{43}&L_{44}&0&0\\ L_{50}&L_{51}&L_{52}&L_{53}&L_{54}&L_{55}&0\\ L_{60}&L_{61}&L_{62}&L_{63}&L_{64}&L_{65}&1\end{pmatrix}, \tag{6.3}$$

if and only if, the vanishing order of Q_5 *restricted on* γ *at* p *is no less than* 4*.*
(2) γ *is ramified at* p *with multiplicity no less than* 2*, if and only if, one of* L_{02},L_{23},L_{34} *in*

(6.3) *vanishes, if and only if, L is lower-triangular, if and only if, the vanishing order of Q_5 restricted on γ at p is no less than* 6.

Proof. If γ can be parameterized as (6.2) with L taking the form of (6.3), then it is easily checked that γ is ramified at p, which has multiplicity 2 if L is lower-triangular.

Next, we use the transvectant characterization of $\mathcal{H}_0^3$ to prove the reverse part. Choose a coordinate z on $\mathbb{C}P^1$ such that $\gamma(\infty) = p$. Note that by applying a transformation in $SL(2, C)$, we can assume $p = v^6$. Let $\{l_0, l_1, \cdots, l_6\}$ be the columns of L, and set $L_{ij} = \frac{\partial^6 l_j}{\partial v^i u^{6-i}}$. Then we have

$$\gamma = \sum_{j=0}^{6} z^j l_j, \quad l_6 = v^6.$$

Assume γ is ramified at p. By Theorem 4.1, we know the line spanned by l_6 and l_5 lies in $\mathcal{H}_0^3$. It is well-known that the only line passing through v^6 is given by $v^6 + tuv^5$. Therefore, we have $l_5 = \alpha v^6 + \beta uv^5$, with $\beta \neq 0$.

In terms of the transvectant characterization (see Proposition 3.1), γ lying in $\mathcal{H}_0^3$ is equivalent to saying $(\gamma, \gamma)_4 = 0$, which implies, for any $0 \leq j \leq 12$, that we have

$$\sum_{r+s=j} (l_r, l_s)_4 = 0. \tag{6.4}$$

In the following, we use the symbol "$*$" to denote some unimportant nonzero constants.

Take $j = 10$ in (6.4). Since $(l_5, l_5)_4 = 0$, we have

$$0 = (l_6, l_4)_4 = *\frac{\partial^4 l_6}{\partial v^4}\frac{\partial^4 l_4}{\partial u^4} = *\, v^2 \frac{\partial^4 l_4}{\partial u^4}.$$

It follows that $\frac{\partial^4 l_4}{\partial u^4} = 0$.

Taking $j = 9$ in (6.4), we have

$$0 = (l_5, l_4)_4 + (l_6, l_3)_4 = \beta(uv^5, l_4)_4 + (l_6, l_3)_4 = *\, v^2 \frac{\partial^4 l_4}{\partial u^3 \partial v} + *\, v^2 \frac{\partial^4 l_3}{\partial u^4},$$

where we have used $\frac{\partial^4 l_4}{\partial u^4} = 0$. It follows that $\frac{\partial^5 l_3}{\partial u^5} = *\frac{\partial^5 l_4}{\partial u^4 \partial v} = 0$.

From

$$l_6 = v^6,\ l_5 = \alpha v^6 + \beta uv^5,\ \frac{\partial^4 l_4}{\partial u^4} = 0,\ \frac{\partial^5 l_3}{\partial u^5} = 0, \tag{6.5}$$

we can derive that L takes the form as in (6.3).

To calculate the vanishing order of $Q_5|_\gamma$ at p, we use the the transvectant characterization

$$Q_5|_\gamma = (\gamma, \gamma)_6 = \sum_{j=0}^{12} z^j \sum_{r+s=j} (l_j, l_k)_6. \tag{6.6}$$

It follows from (6.5) that

$$(l_6, l_5)_6 = 0,\ (l_6, l_4)_6 = *\frac{\partial^6 l_4}{\partial u^6} = 0,\ (l_6, l_3)_6 = *\frac{\partial^6 l_3}{\partial u^6} = 0,\ (l_5, l_5)_6 = 0,\ (l_5, l_4)_6 = *\frac{\partial^6 l_4}{\partial u^5 \partial v} = 0.$$

Therefore, we have $\deg(Q_5|_\gamma) \leq 8$, which implies the vanishing order of $Q_5|_\gamma$ at $\gamma(\infty) = p$ is no less than 4.

Conversely, assume the vanishing order of $Q_5|_\gamma$ at $\gamma(\infty) = p$ is no less than 4. Then we have

$$2(l_6, l_4)_6 + (l_5, l_5)_6 = 0, \tag{6.7}$$
$$2(l_6, l_4)_4 + (l_5, l_5)_4 = 0, \tag{6.8}$$
$$(l_6, l_3)_4 + (l_5, l_4)_4 = 0, \tag{6.9}$$
$$(l_6, l_3)_6 + (l_5, l_4)_6 = 0. \tag{6.10}$$

It follows from $(l_6, l_5)_4 = 0$ that $\frac{\partial^4 l_5}{\partial u^4} = 0$. By comparing (6.7) with the second derivative with respect to u on (6.8), it is easy to derive $\frac{\partial^3 l_5}{\partial u^3} = 0$. Then substituting this into the first derivative with respect to u of (6.8), we obtain $\frac{\partial^5 l_4}{\partial u^5} = 0$. By comparing (6.9) with the second derivative with respect to u of (6.10), it is easy to derive $\frac{\partial^6 l_3}{\partial u^6} = 0$. Substituting this into (6.9) and combining (6.8), we have $\frac{\partial^2 l_5}{\partial u^2} = 0$ and $\frac{\partial^4 l_4}{\partial u^4} = 0$. Finally, by taking the second derivative with respect to u on both sides of $0 = 2(l_5, l_3)_4 + (l_4, l_4)_4$, we derive $\frac{\partial^5 l_3}{\partial u^5} = 0$. Therefore L has the form of (6.3), and is ramified at p.

In fact, that the multiplicity of γ at the ramification point p is no less than 2 can be characterized by one more equation that $(l_5, l_4)_4 = 0$. It follows from $\frac{\partial^4 l_4}{\partial u^4} = 0$ that

$$(l_5, l_4)_4 = (uv^5, l_4)_4 = *\,\frac{\partial^4 l_4}{\partial u^3 \partial v}.$$

Therefore, that γ is ramified at p with multiplicity no less than 2 is equivalent to saying that L takes the form as in (6.3) and $\frac{\partial^3 l_4}{\partial u^3} = 0$, i.e., $L_{34} = 0$.

Taking $j = 9$ in (6.4), we have

$$-(l_5, l_4)_4 = (l_6, l_3)_4 = *\,v^2 \frac{\partial^4 l_3}{\partial u^4}.$$

Therefore, $\frac{\partial^3 l_4}{\partial u^3} = 0$ is equivalent to $\frac{\partial^4 l_3}{\partial u^4} = 0$, i.e., $L_{23} = 0$ in (6.3).

Choosing $j = 8$ in (6.4), it follows from (6.5) that

$$\begin{aligned} 0 =& 2(l_6, l_2)_4 + 2(l_5, l_3)_4 + (l_4, l_4)_4 \\ =& *\ v^2 \frac{\partial^4 l_2}{\partial u^4} + *\,v^2 \frac{\partial^4 l_3}{\partial u^3 \partial v} + *\,uv \frac{\partial^4 l_3}{\partial u^4} + *\,\frac{\partial^4 l_4}{\partial u^3 \partial v}\frac{\partial^4 l_4}{\partial u \partial v^3} + *\,\frac{\partial^4 l_4}{\partial u^2 \partial v^2}\frac{\partial^4 l_4}{\partial u^2 \partial v^2}. \end{aligned} \tag{6.11}$$

Taking the second partial derivative with respect to u on both sides, we obtain $\frac{\partial^6 l_2}{\partial u^6} = *\,L_{23}^2$, which implies that $L_{23} = 0$ is equivalent to $L_{02} = 0$.

Next, we prove that $L_{34} = 0$ if and only if L is lower-triangular, i.e., that the following equations hold,

$$l_6 = v^6,\ l_5 = \alpha v^6 + \beta u v^5,\ \frac{\partial^3 l_4}{\partial u^3} = 0,\ \frac{\partial^4 l_3}{\partial u^4} = 0,\ \frac{\partial^5 l_2}{\partial u^5} = 0,\ \frac{\partial^6 l_1}{\partial u^6} = 0. \tag{6.12}$$

Note that only the last two equations need to be verified. The second to last follows from taking the partial derivative with respect to u on both sides of (6.11). Taking $j = 7$ in (6.4), we have

$$\begin{aligned} 0 &= (l_6, l_1)_4 + (l_5, l_2)_4 + (l_4, l_3)_4 \\ &= *\,v^2 \frac{\partial^4 l_1}{\partial u^4} + *\,v^2 \frac{\partial^4 l_2}{\partial u^4} + *\,v^2 \frac{\partial^4 l_2}{\partial u^3 \partial v} + *\,\frac{\partial^4 l_4}{\partial u^2 \partial v^2}\frac{\partial^4 l_3}{\partial u^2 \partial v^2} + *\,\frac{\partial^4 l_3}{\partial u^3 \partial v}\frac{\partial^4 l_4}{\partial u \partial v^3}. \end{aligned}$$

The second partial derivative with respect to u on both sides implies that $\frac{\partial^6 l_1}{\partial u^6} = 0$.

Similar to the discussion in the first part, we can derive that the vanishing order of $Q_5|_\gamma$ at $\gamma(\infty) = p$ is no less than 6, and the reverse part is also true. □

The following technical lemma entailing ramification will be used in the proof of Theorem 6.1. It characterizes when the lower-triangular matrix L is diagonal.

Lemma 6.1. *Let $\gamma(z) = L\, Z_6(z)$ be a rational normal curve of degree 6 in $\mathcal{H}_0^3$, with L being lower-triangular and $L_{21} = 0$. If $\gamma(z)$ is also ramified at $z = 0$ with multiplicity no less than 2, then $\gamma(0)$ lies in the closed 2-dimensional orbit. Moreover, the following are equivalent.*

(1) L *is diagonal.* (2) $\gamma(0)$ *lies in the 1-dimensional orbit.* (3) $L_{10} = 0$. (4) $L_{65} = 0$.

Proof. We continue to use the notation given in the proof of the preceding proposition. Write $a_i = \sum_{j=0}^{6} \sqrt{\binom{6}{j}} L_{ij} z^j,\ 0 \le i \le 6$. Then L is lower-triangular if and only if $\deg a_i = i,\ 0 \le i \le 6$. Note that $[a_0 : a_1 : \cdots : a_6]$ is exactly the coordinates of γ, and a_0 is a constant. This implies that a_4, a_5 and a_6 can be solved as polynomials of a_1, a_2 and a_3 as in (5.2).

By the first equation of (5.2) and $L_{21} = 0$, we obtain

$$L_{41} = \frac{\sqrt{2}(L_{10}L_{31} + L_{11}L_{30})}{L_{00}}, \quad L_{42} = \frac{1}{L_{00}}(\sqrt{2}L_{10}L_{32} + \frac{6\sqrt{2}}{\sqrt{15}}L_{11}L_{31} - \frac{6}{\sqrt{15}}L_{20}L_{22}). \tag{6.13}$$

Assume γ is also ramified at $z = 0$ with multiplicity no less than 2. Then we have

$$0 = (l_1, l_1)_4 = * \frac{\partial^4 l_1}{\partial u^4}\frac{\partial^4 l_1}{\partial v^4} + * \frac{\partial^4 l_1}{\partial u^3 \partial v}\frac{\partial^4 l_1}{\partial u \partial v^3} + * \frac{\partial^4 l_1}{\partial u^2 \partial v^2}\frac{\partial^4 l_1}{\partial u^2 \partial v^2}, \tag{6.14}$$

$$0 = (l_1, l_2)_4 = * \frac{\partial^4 l_1}{\partial u^4}\frac{\partial^4 l_2}{\partial v^4} + * \frac{\partial^4 l_1}{\partial u^3 \partial v}\frac{\partial^4 l_2}{\partial u \partial v^3} + * \frac{\partial^4 l_1}{\partial u^2 \partial v^2}\frac{\partial^4 l_2}{\partial u^2 \partial v^2} + * \frac{\partial^4 l_1}{\partial u \partial v^3}\frac{\partial^4 l_2}{\partial u^3 \partial v} + * \frac{\partial^4 l_1}{\partial v^4}\frac{\partial^4 l_2}{\partial u^4}. \tag{6.15}$$

Taking the fourth partial derivative with respect to u on (6.14), it follows from $\frac{\partial^6 l_1}{\partial u^5 \partial v} = L_{11} \neq 0$ that $L_{31} = \frac{\partial^6 l_1}{\partial u^3 \partial v^3} = 0$. Then considering the fourth partial derivative with respect to v on (6.14), we obtain $L_{41} = \frac{\partial^6 l_1}{\partial u^2 \partial v^4} = 0$. Substituting these into the first equation of (6.13), we have $L_{30} = 0$.

Taking the fourth partial derivative with respect to u on (6.15), it follows from $\frac{\partial^6 l_1}{\partial u^4 \partial v^2} = L_{21} = 0$ and $\frac{\partial^6 l_1}{\partial u^3 \partial v^3} = L_{31} = 0$ that $L_{32} = \frac{\partial^6 l_2}{\partial u^3 \partial v^3} = 0$. Then considering the first partial derivative with respect to v followed by the third partial derivative with respect to u on (6.15), we obtain that $L_{42} = \frac{\partial^6 l_2}{\partial u^2 \partial v^4} = 0$. Substituting these into the second equation of (6.13), we have $L_{20} = 0$. Thus, we have proved that

$$L_{20} = 0,\ L_{21} = 0,\ L_{30} = 0,\ L_{31} = 0,\ L_{32} = 0,$$

which implies that the vanishing order of a_2 and a_3 at $z = 0$ satisfy

$$\operatorname{ord}(a_2) \ge 2,\ \operatorname{ord}(a_3) \ge 3. \tag{6.16}$$

It follows from (5.2) that

$$\operatorname{ord}(a_4) \ge 3,\ \operatorname{ord}(a_5) \ge 3,\ \operatorname{ord}(a_6) \ge 5.$$

It also follows that the ramified point $\gamma(0) = u^5(L_{00}u + \sqrt{6}L_{10}v)$ lies in the closed 2-dimensional orbit.

Note that $\gamma(0)$ lies in the 1-dim orbit if and only if $L_{10} = 0$, i.e.,

$$\mathrm{ord}(a_1) \geq 1, \tag{6.17}$$

which is equivalent to one of the following inequalities

$$\mathrm{ord}(a_4) \geq 4,\ \mathrm{ord}(a_5) \geq 5,\ \mathrm{ord}(a_6) \geq 6, \tag{6.18}$$

where (5.2) and (6.16) are used. Note also that one of the seven inequalities in (6.16) $\sim$ (6.18) becomes an equality if and only if all of them do, if and only if L is diagonal. This finishes the proof. $\square$

6.2. **Necessary conditions for generally ramified holomorphic 2-spheres to be of constant curvature.**

Let $\gamma : \mathbb{C}P^1 \to G(2,5)$ be a generally ramified holomorphic 2-sphere of degree 6. By definition and Proposition 6.1, γ can be parametrized as

$$\gamma = A \cdot (E_0, E_1, E_2, E_3, E_4, E_5, E_6)\ L\, Z_6(z), \tag{6.19}$$

where $A \in GL(5,\mathbb{C})$, L is in the form of (6.3), and $Z_6(z) = \begin{pmatrix} 1 & \sqrt{6}z & \cdots & z^6 \end{pmatrix}^T$, with z being the standard parameter for the condition of constant curvature. Note that up to an isometry of $G(2,5)$, i.e., a $U(5)$-transformation, we may assume that A is lower-triangular. Then, by the definition of $\wedge^2$-action, it follows from (3.3) that $C \triangleq A \cdot (E_0, E_1, E_2, E_3, E_4, E_5, E_6)$ is of the form

$$C = \begin{pmatrix} C_{00} & 0 & 0 & 0 & 0 & 0 & 0 \\ C_{10} & C_{11} & 0 & 0 & 0 & 0 & 0 \\ C_{20} & C_{21} & C_{22} & 0 & 0 & 0 & 0 \\ C_{30} & C_{31} & C_{32} & C_{33} & 0 & 0 & 0 \\ C_{40} & C_{41} & C_{42} & 0 & 0 & 0 & 0 \\ C_{50} & C_{51} & C_{52} & C_{53} & 0 & 0 & 0 \\ C_{60} & C_{61} & C_{62} & C_{63} & C_{64} & 0 & 0 \\ C_{70} & C_{71} & C_{72} & C_{73} & C_{74} & 0 & 0 \\ C_{80} & C_{81} & C_{82} & C_{83} & C_{84} & C_{85} & 0 \\ C_{90} & C_{91} & C_{92} & C_{93} & C_{94} & C_{95} & C_{96} \end{pmatrix}, \tag{6.20}$$

which is a 10×7 matrix obtained by column vectors $A \cdot E_k$ written relative to the standard basis $e_i \wedge e_j$, $0 \leq i < j \leq 4$, in the lexicographic order. We point out that C_{ij} are quadratic in terms of the entries of A. The following lemma is important.

Lemma 6.2. *Let G be a 10×7 matrix of rank 7 in the same form as on the right-hand side of (6.20) with $G_{33}G_{53}G_{64}G_{74} \neq 0$, and let the column vectors of G be mutually orthogonal. If the holomorphic 2-sphere $\gamma(z) \triangleq G\, Z_6(z)$ lies in a generic linear section of $G(2,5)$, then G is in the form*

$$\begin{pmatrix} G_{00} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & G_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & G_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & G_{33} & 0 & 0 & 0 \\ 0 & 0 & G_{42} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & G_{53} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & G_{64} & 0 & 0 \\ 0 & 0 & 0 & 0 & G_{74} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & G_{85} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & G_{96} \end{pmatrix}, \tag{6.21}$$

where γ is ramified at $z = 0$ and $z = \infty$ with multiplicities at least 2.

Proof. If (6.21) holds, then the last statement follows from

$$\gamma'(0) = e_0 \wedge e_2 \in G(2,5),\ \gamma''(0) = G_{22}e_0 \wedge e_3 + G_{42}e_1 \wedge e_2,\ \gamma'(0) \wedge \gamma''(0) = 0,$$
$$\gamma'(\infty) = e_2 \wedge e_4 \in G(2,5),\ \gamma''(\infty) = G_{64}e_1 \wedge e_4 + G_{74}e_2 \wedge e_3,\ \gamma'(\infty) \wedge \gamma''(\infty) = 0.$$

Hence, we need only prove (6.21) in the following. Since the first five columns of G are perpendicular to the last two, we have

$$\gamma(z) = \begin{pmatrix} G_{00} & 0 & 0 & 0 & 0 & 0 & 0 \\ G_{10} & G_{11} & 0 & 0 & 0 & 0 & 0 \\ G_{20} & G_{21} & G_{22} & 0 & 0 & 0 & 0 \\ G_{30} & G_{31} & G_{32} & G_{33} & 0 & 0 & 0 \\ G_{40} & G_{41} & G_{42} & 0 & 0 & 0 & 0 \\ G_{50} & G_{51} & G_{52} & G_{53} & 0 & 0 & 0 \\ G_{60} & G_{61} & G_{62} & G_{63} & G_{64} & 0 & 0 \\ G_{70} & G_{71} & G_{72} & G_{73} & G_{74} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & G_{85} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & G_{96} \end{pmatrix} \begin{pmatrix} 1 \\ \sqrt{6}z \\ \sqrt{15}z^2 \\ 2\sqrt{5}z^3 \\ \sqrt{15}z^4 \\ \sqrt{6}z^5 \\ z^6 \end{pmatrix}.$$

We denote by $\{\gamma_j \mid j = 0, \ldots, 9\}$ the coordinates of γ. Then it is easy to see

$$\deg(\gamma_0) = 0,\ \deg(\gamma_1) \leq 1,\ \deg(\gamma_2) \leq 2,\ \deg(\gamma_3) \leq 3,\ \deg(\gamma_4) \leq 2,$$
$$\deg(\gamma_5) \leq 3,\ \deg(\gamma_6) \leq 4,\ \deg(\gamma_7) \leq 4,\ \deg(\gamma_8) = 5,\ \deg(\gamma_9) = 6.$$

It follows from $\gamma \subset G(2,5)$ that

$$\gamma_2\gamma_4 - \gamma_1\gamma_5 + \gamma_0\gamma_7 = 0, \tag{6.22}$$
$$\gamma_3\gamma_4 - \gamma_1\gamma_6 + \gamma_0\gamma_8 = 0, \tag{6.23}$$
$$\gamma_3\gamma_5 - \gamma_2\gamma_6 + \gamma_0\gamma_9 = 0, \tag{6.24}$$
$$\gamma_3\gamma_7 - \gamma_2\gamma_8 + \gamma_1\gamma_9 = 0, \tag{6.25}$$
$$\gamma_6\gamma_7 - \gamma_5\gamma_8 + \gamma_4\gamma_9 = 0. \tag{6.26}$$

Moreover, $\gamma_i \neq 0,\ i = 0 \ldots, 9$, since γ lies in a generic linear section. Meanwhile, by the orthogonality of $\{G_j \mid j = 0, \ldots, 6\}$, we obtain $|G_j|^2\sqrt{\binom{6}{j}}z^j = \langle \gamma, G_j \rangle = \sum_{k=0}^{9} \overline{G_{kj}}\gamma_k$, so that

$$\overline{G_{64}}\gamma_6 + \overline{G_{74}}\gamma_7 = |G_4|^2\sqrt{15}z^4, \tag{6.27}$$
$$\overline{G_{33}}\gamma_3 + \overline{G_{53}}\gamma_5 + \overline{G_{63}}\gamma_6 + \overline{G_{73}}\gamma_7 = |G_3|^2\sqrt{20}z^3. \tag{6.28}$$

In the following, we will use the assumption $G_{33}G_{53}G_{64}G_{74} \neq 0$. Observe that $\gamma_8 = G_{85}z^5$ and $\gamma_9 = G_{96}z^6$. As a polynomial of z, we denote by $m(\gamma_j)$ the order of γ_j at $z = 0$.

Combining (6.30) and $G_{64}G_{74} \neq 0$, and using $\deg(\gamma_6) = \deg(\gamma_7) = 4$, it yields $0 \leq m(\gamma_6) = m(\gamma_7) \leq 4$. Meanwhile (6.26) gives $z^5|\gamma_6\gamma_7$, which implies $5 \leq m(\gamma_6) + m(\gamma_7)$. It follows that $m(\gamma_6) = m(\gamma_7) \geq 3$. Moreover, we obtain $z^5 \mid \gamma_3\gamma_7$ in accord with (6.25).

Claim 1. $\gamma_6 = G_{64}z^4$ and $\gamma_7 = G_{74}z^4$.

Otherwise, we assume $m(\gamma_7) = 3$. Then $2 \leq m(\gamma_3) \leq 3$ and $m(\gamma_6) = 3$. Using (6.28), we have $m(\gamma_5) \geq 2$, which implies $z^4 \mid (\gamma_3\gamma_5 + \gamma_0\gamma_9)$. It follows from (6.24) that $z^4 \mid \gamma_2\gamma_6$. As a result, $m(\gamma_2) \geq 1$, and $z^6 \mid (\gamma_2\gamma_8 - \gamma_1\gamma_9)$. Next, (6.25) yields $z^6 \mid \gamma_3\gamma_7$, and then $m(\gamma_3) = 3$. Using (6.28) again, we obtain $m(\gamma_5) \geq 3$. Coupled with (6.24), $z^6 \mid \gamma_2\gamma_6$ can be deduced. Consequently, $m(\gamma_2) \geq 3$, which contradicts $\deg(\gamma_2) \leq 2$. Hence the claim follows from the degrees of γ_6 and γ_7.

Now that we have $z^4 \mid (\gamma_1\gamma_6 - \gamma_0\gamma_8)$, it follows from (6.23) that $z^4 \mid \gamma_3\gamma_4$. Since $\deg(\gamma_4) = 2$, there follows $m(\gamma_3) \geq 2$.

Claim 2. $\gamma_3 = G_{33}z^3$.

Otherwise, we assume $m(\gamma_3) = 2$. Then $m(\gamma_4) = 2$. Hence $z^8 \mid (\gamma_4\gamma_9 + \gamma_6\gamma_7)$, and $z^8 \mid \gamma_5\gamma_8$, from which we can derive that $m(\gamma_5) \geq 3$. Using (6.28) again, there yields that $m(\gamma_3) \geq 3$ (by $G_{33} \neq 0$), which contradicts the assumption. Therefore $m(\gamma_3) = 3$ and the Claim 2 follows from $\deg(\gamma_3) = 3$.

Now, $\gamma_5 = G_{53}z^3$ follows from (6.28) and $\deg(\gamma_5) = 3$.

Using (6.26), we obtain $z^8 \mid \gamma_4\gamma_9$. Hence $\gamma_4 = G_{42}z^2$ by $\deg(\gamma_4) = 2$.

From (6.24), we have $z^6 \mid \gamma_2\gamma_6$. Therefore, $\gamma_2 = G_{22}z^2$ due to that $\deg(\gamma_2) = 2$.

Lastly, it follows from (6.25) that $z^7 \mid \gamma_1\gamma_9$. So $\gamma_1 = G_{11}z$, as $\deg(\gamma_1) = 1$. □

The method used in the proof of the above lemma can be generalized to prove the following important proposition.

Proposition 6.2. *Let $\gamma : \mathbb{C}P^1 \to G(2,5)$ be a generally ramified holomorphic 2-sphere of degree 6 parametrized by (6.19). If γ is of constant curvature, then L is lower-triangular.*

Proof. To show that L is lower-triangular, it follows from Proposition 6.1 that we need only prove that one of L_{02}, L_{23}, L_{34} vanishes.

Suppose that in the following $L_{02}L_{23}L_{34} \neq 0$. Similarly as before, we assume that A is a lower-triangular matrix. Then $G \triangleq A \cdot (E_0, \ldots, E_6)\, L$ is a 10×7 matrix with orthonormal columns and takes the following form

$$G = \begin{pmatrix} G_{00} & G_{01} & G_{02} & 0 & 0 & 0 & 0 \\ G_{10} & G_{11} & G_{12} & 0 & 0 & 0 & 0 \\ G_{20} & G_{21} & G_{22} & G_{23} & 0 & 0 & 0 \\ G_{30} & G_{31} & G_{32} & G_{33} & G_{34} & 0 & 0 \\ G_{40} & G_{41} & G_{42} & G_{43} & 0 & 0 & 0 \\ G_{50} & G_{51} & G_{52} & G_{53} & G_{54} & 0 & 0 \\ G_{60} & G_{61} & G_{62} & G_{63} & G_{64} & 0 & 0 \\ G_{70} & G_{71} & G_{72} & G_{73} & G_{74} & 0 & 0 \\ G_{80} & G_{81} & G_{82} & G_{83} & G_{84} & G_{85} & 0 \\ G_{90} & G_{91} & G_{92} & G_{93} & G_{94} & G_{95} & G_{96} \end{pmatrix}, \quad G_{02}G_{23}G_{43}G_{34}G_{54} \neq 0, \tag{6.29}$$

where the inequality comes from the product of diagonal entries of A and $L_{02}L_{23}L_{34}$.

Since the first five columns of G are perpendicular to the last two, we have

$$\gamma(z) = G\, Z_6(z) = \begin{pmatrix} G_{00} & G_{01} & G_{02} & 0 & 0 & 0 & 0 \\ G_{10} & G_{11} & G_{12} & 0 & 0 & 0 & 0 \\ G_{20} & G_{21} & G_{22} & G_{23} & 0 & 0 & 0 \\ G_{30} & G_{31} & G_{32} & G_{33} & G_{34} & 0 & 0 \\ G_{40} & G_{41} & G_{42} & G_{43} & 0 & 0 & 0 \\ G_{50} & G_{51} & G_{52} & G_{53} & G_{54} & 0 & 0 \\ G_{60} & G_{61} & G_{62} & G_{63} & G_{64} & 0 & 0 \\ G_{70} & G_{71} & G_{72} & G_{73} & G_{74} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & G_{85} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & G_{96} \end{pmatrix} \begin{pmatrix} 1 \\ \sqrt{6}z \\ \sqrt{15}z^2 \\ 2\sqrt{5}z^3 \\ \sqrt{15}z^4 \\ \sqrt{6}z^5 \\ z^6 \end{pmatrix}.$$

We denote by $\{\gamma_j \mid j = 0, \ldots, 9\}$ the coordinates of γ. Then it is easy to see

$$\deg(\gamma_0) \leq 2,\ \deg(\gamma_1) \leq 2,\ \deg(\gamma_2) \leq 3,\ \deg(\gamma_3) \leq 4,\ \deg(\gamma_4) \leq 3,$$
$$\deg(\gamma_5) \leq 4,\ \deg(\gamma_6) \leq 4,\ \deg(\gamma_7) \leq 4,\ \deg(\gamma_8) = 5,\ \deg(\gamma_9) = 6,$$

satisfying (6.22) through (6.26). The same constraint between (6.26) and (6.30) gives

$$\overline{G_{00}}\gamma_0 + \overline{G_{10}}\gamma_1 + \overline{G_{20}}\gamma_2 + \overline{G_{30}}\gamma_3 + \overline{G_{40}}\gamma_4 + \overline{G_{50}}\gamma_5 + \overline{G_{60}}\gamma_6 + \overline{G_{70}}\gamma_7 = |G_0|^2, \tag{6.30}$$
$$\overline{G_{23}}\gamma_2 + \overline{G_{33}}\gamma_3 + \overline{G_{43}}\gamma_4 + \overline{G_{53}}\gamma_5 + \overline{G_{63}}\gamma_6 + \overline{G_{73}}\gamma_7 = |G_3|^2\sqrt{20}z^3, \tag{6.31}$$
$$\overline{G_{34}}\gamma_3 + \overline{G_{54}}\gamma_5 + \overline{G_{64}}\gamma_6 + \overline{G_{74}}\gamma_7 = |G_4|^2\sqrt{15}z^4. \tag{6.32}$$

As a polynomial of z, we denote by $m(\gamma_j)$ the order of γ_j at $z = 0$.

Since $\gamma_3 = p_{04}(F),\ \gamma_6 = p_{14}(F)$, we have that γ_3 and γ_6 are linearly independent (since F lies in a generic linear section). Hence combining this with $\deg(\gamma_3), \deg(\gamma_6) \leq 4$, we deduce

$$k \triangleq \min\{m(\gamma_3), m(\gamma_6)\} \leq 3. \tag{6.33}$$

It follows from $m(\gamma_8) = 5$ and $m(\gamma_9) = 6$, (6.25) and (6.26), that $5 \leq m(\gamma_3\gamma_7), m(\gamma_6\gamma_7)$. Since $\deg \gamma_7 \leq 4$, by (6.33), we obtain $1 \leq k \leq 3,\ 2 \leq m(\gamma_7)$, while (6.32) and $G_{54} \neq 0$ yields

$$m(\gamma_5) \geq \min\{k, m(\gamma_7)\} \geq 1. \tag{6.34}$$

Using (6.22) and (6.31), we arrive at

$$m(\overline{G_{23}}\gamma_2 + \overline{G_{43}}\gamma_4) \geq \min\{k, m(\gamma_7)\} \geq 1, \tag{6.35}$$

$$m(\gamma_2\gamma_4) \geq \min\{k, m(\gamma_7)\} \geq 1. \tag{6.36}$$

We claim that

$$m(\gamma_2) \geq [\frac{\min\{k, m(\gamma_7)\} + 1}{2}] \geq 1, \quad m(\gamma_4) \geq [\frac{\min\{k, m(\gamma_7)\} + 1}{2}] \geq 1. \tag{6.37}$$

Indeed, if $m(\gamma_2) = m(\gamma_4)$, then the claim follows from (6.36). If $m(\gamma_2) \neq m(\gamma_4)$, then by $G_{23}G_{43} \neq 0$ and (6.35), we obtain that

$$\min\{m(\gamma_2), m(\gamma_4)\} = m(\overline{G_{23}}\gamma_2 + \overline{G_{43}}\gamma_4) \geq \min\{k, m(\gamma_7)\} \geq [\frac{\min\{k, m(\gamma_7)\} + 1}{2}].$$

This proves our claim. Next, from (6.25), (6.26) and (6.34) we derive (because $\min\{k, m(\gamma_7)\} \geq 1$) that

$$\begin{aligned} m(\gamma_3\gamma_7) &\geq \min\{5 + [\frac{\min\{k, m(\gamma_7)\} + 1}{2}], 6\} \geq 6, \\ m(\gamma_6\gamma_7) &\geq \min\{5 + \min\{k, m(\gamma_7)\}, [\frac{\min\{k, m(\gamma_7)\} + 1}{2}] + 6\} \geq 6. \end{aligned} \tag{6.38}$$

Since $1 \leq k = \min\{\deg\gamma_3, \deg\gamma_6\} \leq 3,\ \deg\gamma_7 \leq 4$, we must have $2 \leq m(\gamma_3), m(\gamma_6)$ and $3 \leq m(\gamma_7)$; hence

$$2 \leq k \leq 3,\ 3 \leq m(\gamma_7) \leq 4. \tag{6.39}$$

Now, we divide the discussion according to $m(\gamma_7)$.

Case 1: Assume that $m(\gamma_7) = 3$. Then $\min\{k, m(\gamma_7)\} = k \geq 2$, so that (6.38) implies

$$m(\gamma_3\gamma_7) \geq 6, \quad 7 \geq \deg\gamma_6 + m(\gamma_7) \geq m(\gamma_6\gamma_7) \geq 7;$$

hence, $k = 2,\ m(\gamma_6) = 4$, and $m(\gamma_3) \geq 3$. But then

$$2 = k = \min\{m(\gamma_3), m(\gamma_6)\} \geq \min\{3, 4\} = 3,$$

a contradiction.

Case 2: Assume that $m(\gamma_7) = 4$. Then by (6.34), (6.37), (6.38) and (6.39), we obtain

$$1 \leq m(\gamma_2), m(\gamma_4),\ 2 \leq m(\gamma_3), m(\gamma_5),\ 3 \leq m(\gamma_6).$$

We conclude that G is in the form

$$G = A \cdot (E_0, \ldots, E_6)\, L = \begin{pmatrix} G_{00} & G_{01} & G_{02} & 0 & 0 & 0 & 0 \\ G_{10} & G_{11} & G_{12} & 0 & 0 & 0 & 0 \\ 0 & G_{21} & G_{22} & G_{23} & 0 & 0 & 0 \\ 0 & 0 & G_{32} & G_{33} & G_{34} & 0 & 0 \\ 0 & G_{41} & G_{42} & G_{43} & 0 & 0 & 0 \\ 0 & 0 & G_{52} & G_{53} & G_{54} & 0 & 0 \\ 0 & 0 & 0 & G_{63} & G_{64} & 0 & 0 \\ 0 & 0 & 0 & 0 & G_{74} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & G_{85} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & G_{96} \end{pmatrix}. \tag{6.40}$$

Consider the QR decomposition of $A \cdot (E_0, \ldots, E_6) = N \cdot L_1$, where N is a 10×7 matrix with orthonormal columns, and $L_1 = (J_{ij})_{0 \leq i,, j \leq 6}$ is a 7×7 lower-triangular matrix. Since

$A \cdot (E_0, \ldots, E_6)$ is in the form (6.20), necessarily N is given by

$$N = \begin{pmatrix} N_{00} & 0 & 0 & 0 & 0 & 0 & 0 \\ N_{10} & N_{11} & 0 & 0 & 0 & 0 & 0 \\ N_{20} & N_{21} & N_{22} & 0 & 0 & 0 & 0 \\ N_{30} & N_{31} & N_{32} & N_{33} & 0 & 0 & 0 \\ N_{40} & N_{41} & N_{42} & 0 & 0 & 0 & 0 \\ N_{50} & N_{51} & N_{52} & N_{53} & 0 & 0 & 0 \\ N_{60} & N_{61} & N_{62} & N_{63} & N_{64} & 0 & 0 \\ N_{70} & N_{71} & N_{72} & N_{73} & N_{74} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} N_0 & 0_{1\times 2} \\ N_1 & 0_{1\times 2} \\ \vdots & \\ N_7 & 0_{1\times 2} \\ 0_{2\times 5} & Id_2 \end{pmatrix},$$

where $N_j,\ 0 \le j \le 7$ are row vectors in $\mathbb{C}^5$. Moreover,

$$N_{64} N_{74} \neq 0, \tag{6.41}$$

since (N_{64}, N_{74}) is parallel to $(\frac{\sqrt{15}}{5} a_{11} a_{44}, \frac{\sqrt{10}}{5} a_{22} a_{33})$ and the diagonal entries of A are not zero. Now, from $G = N \cdot L_2 \cdot L$ and the orthogonality of columns of G and N, respectively, we must have that $L_2 \cdot L = (H_{ij})_{0 \le i,j \le 6} \in U(7)$ is in the same form as (6.3) with

$$H_{23} = J_{22} L_{23} \neq 0,\ H_{34} = J_{33} L_{34} \neq 0. \tag{6.42}$$

Since $L_2 \cdot L \in U(7)$, it is necessary that

$$L_2 \cdot L = \begin{pmatrix} H_0 & H_1 & \cdots & H_4 & 0_{5\times 2} \\ & & 0_{2\times 5} & & \begin{pmatrix} H_{55} & 0 \\ 0 & H_{66} \end{pmatrix} \end{pmatrix}$$

where $H_i,\ 0 \le i \le 4$, are column vectors in $\mathbb{C}^5$ that form an orthonormal basis of $\mathbb{C}^5$, and H_3 and H_4 are in the form

$$H_3 = (0, 0, H_{23}, H_{33}, H_{43})^t, \quad H_4 = (0, 0, 0, H_{34}, H_{44})^t. \tag{6.43}$$

Since $G = N \cdot L_2 \cdot L$, by $G_{6j} = 0,\ 0 \le j \le 2$, and $G_{7i} = 0,\ 0 \le i \le 3$ (see (6.40)), we obtain

$$N_6 \cdot H_j = 0,\ 0 \le j \le 2, \quad N_7 \cdot H_i = 0,\ 0 \le i \le 3;$$

hence, $N_6 \in \operatorname{span}\{\overline{H_3^t}, \overline{H_4^t}\}$ and $N_7 \in \operatorname{span}\{\overline{H_4^t}\}$, so that we conclude by (6.43) that N is in the following form

$$N = \begin{pmatrix} N_{00} & 0 & 0 & 0 & 0 & 0 & 0 \\ N_{10} & N_{11} & 0 & 0 & 0 & 0 & 0 \\ N_{20} & N_{21} & N_{22} & 0 & 0 & 0 & 0 \\ N_{30} & N_{31} & N_{32} & N_{33} & 0 & 0 & 0 \\ N_{40} & N_{41} & N_{42} & 0 & 0 & 0 & 0 \\ N_{50} & N_{51} & N_{52} & N_{53} & 0 & 0 & 0 \\ 0 & 0 & N_{62} & N_{63} & N_{64} & 0 & 0 \\ 0 & 0 & 0 & N_{73} & N_{74} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then the inner product of the third column with fifth column gives $\overline{N_{62}} N_{64} = 0$, and by $N_{64} \neq 0$ (see (6.41)) we obtain $N_{62} = 0$. Meanwhile, from $N_6 \in \operatorname{span}\{\overline{H_3^t}, \overline{H_4^t}\}$ we deduce

$$N_6 = (0, 0, 0, N_{63}, N_{64}) = a \cdot \overline{H_3^t} + b \cdot \overline{H_4^t},$$

for some constant a, b. Then from $H_{23} \neq 0$ (see (6.42) and (6.43)), we infer $a = 0$; hence N_6 is parallel to $\overline{H_4^t}$. Thus, $G_{63} = N_6 \cdot H_3 = 0$, which implies that $m(\gamma_6) = 4$. Then

$$2 \le k = \min\{m(\gamma_3), m(\gamma_6)\} = m(\gamma_3) \le 3.$$

It follows from (6.24) and (6.32) that

$$m(\gamma_3 \gamma_5) \ge 5, \quad m(\overline{G_{34}} \gamma_3 + \overline{G_{54}} \gamma_5) \ge 4. \tag{6.44}$$

From (6.34), we have $m(\gamma_5) \geq k = \min\{m(\gamma_3), m(\gamma_6)\} = m(\gamma_3)$. Combining (6.44), $2 \leq m(\gamma_3) \leq 3$ with $G_{34}G_{54} \neq 0$, we arrive at $m(\gamma_3) = m(\gamma_5)$. Then $k = m(\gamma_3) = m(\gamma_5) \geq \frac{5}{2}$ implies $k = 3$. Next, from (6.37), we see $2 \leq m(\gamma_2), m(\gamma_4)$. Lastly by (6.23), we arrive at

$$m(\gamma_1) + 4 = m(\gamma_1\gamma_6) \geq 5;$$

hence $m(\gamma_1) \geq 1$, so $G_{10} = 0$. Then (6.30) gives $\overline{G_{00}}\gamma_0 = |G_0|^2$, so that $G_{02} = 0$, contradictory to the inequality in (6.29).

In short, one of L_{02}, L_{23}, L_{34} vanishes so that L is lower-triangular. □

Now we can finish the proof of Theorem 6.1.

Proof of Theorem 6.1.

We continue to use the parameterization given in (6.19). Note that by using the automorphism of $\mathcal{H}_0^3$, we can re-choose $A \in GL(5, \mathbb{C})$ such that $L_{21} = 0$. In fact, set $A_1 \triangleq \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ with $b = \frac{L_{21}}{\sqrt{10}L_{11}}$, then by (3.1),

$$A \cdot (E_0, \ldots, E_6)\, L = (A\, \rho^4(A_1^{-1})) \cdot (E_0, \ldots, E_6)\, (\rho^6(A_1)\, L).$$

Since $\rho^6(A_1)$ is lower-triangular, $\rho^6(A_1)L$ is also lower-triangular, and so we derive

$$(\rho^6(A_1)L)_{21} = L_{21} - b\sqrt{10}L_{11} = 0.$$

The constant curvature condition of γ implies that

$$G \triangleq A \cdot (E_0, \ldots, E_6)\, L, \tag{6.45}$$

is a 10×7 matrix with orthonormal columns. Similarly as before, up to a $U(5)$-transformation, we may assume A is lower-triangular. Since L is lower-triangular, we see that G has the form as (6.20). It is easy to verify that $G_{33}G_{53}G_{64}G_{74} \neq 0$.

It follows from Lemma 6.2 that G must be in the form of (6.21). Moreover, $G\, Z_6(z)$ and

$$\mu(z) \triangleq A^{-1} \cdot G\, Z_6(z) = (E_0, \ldots, E_6)\, L\, Z_6(z)$$

are ramified at $z = 0$ and $z = \infty$ with multiplicities at least 2. Thus we can apply Lemma 6.1 to the curve $\mu(z)$. It follows from the proof of Lemma 6.1 that now

$$L_{01} = L_{21} = L_{31} = L_{41} = L_{51} = L_{61} = 0, \quad L_{05} = L_{15} = L_{25} = L_{35} = L_{45}.$$

To prove that L is diagonal, we need only show $L_{65} = 0$.

Since G is in the form of (6.21), comparing the second column of both sides of

$$A^{-1} \cdot G = (E_0, \ldots, E_6)\, L, \tag{6.46}$$

we deduce

$$G_{11}\, A^{-1} \cdot e_0 \wedge A^{-1} \cdot e_2 = L_{11} e_0 \wedge e_2,$$

whence $A^{-1} \cdot e_2 \in \operatorname{span}\{e_0, e_2\}$. Then comparing the penultimate column of both sides of (6.46), we have

$$L_{55} e_2 \wedge e_4 + L_{65} e_3 \wedge e_4 = G_{85}\, A^{-1} \cdot e_2 \wedge A^{-1} \cdot e_4 \in \operatorname{span}\{e_0 \wedge e_4, e_2 \wedge e_4\},$$

which implies $L_{65} = 0$. Hence L is diagonal.

Furthermore, due to that A is lower-triangular, we can also derive that

$$A^{-1} \cdot e_2 \equiv 0 \mod e_2, \qquad A^{-1} \cdot e_4 \equiv 0 \mod e_4,$$

and then $A^{-1} \cdot e_0 \equiv 0 \mod e_0$. By comparing the first and last columns of both sides of (6.46), we have $A^{-1} \cdot e_i \equiv 0 \mod e_i,\ i = 1, 3$. In conclusion, we have arrived at that A is diagonal. Therefore, the curve γ belongs to the diagonal family. $\square$

7. Existence and uniqueness results for the diagonal family.

It follows from Theorem 6.1 that to classify generally ramified constantly curved holomorphic 2-spheres in $G(2,5)$, we need only consider those in the diagonal family, which are determined by diagonal matrices $A \in GL(5,\mathbb{C})$ and complex numbers $\{\omega_0, \omega_1, \ldots, \omega_6\}$ satisfying

$$\begin{aligned} &\omega_0\omega_4 - 4\omega_1\omega_3 + 3\omega_2^2 = 0, \quad \omega_0\omega_5 - 3\omega_1\omega_4 + 2\omega_2\omega_3 = 0, \quad \omega_0\omega_6 - 9\omega_2\omega_4 + \\ &8\omega_3^2 = 0, \quad \omega_2\omega_6 - 4\omega_3\omega_5 + 3\omega_4^2 = 0, \quad \omega_1\omega_6 - 3\omega_2\omega_5 + 2\omega_3\omega_4 = 0, \end{aligned} \tag{7.1}$$

to guarantee that the holomorphic 2-sphere parameterized as in (6.1) lives in $G(2,5)$.

In this section, we will pin down the class of diagonal matrices $A \in GL(5,\mathbb{C})$ that warrants the existence of constantly curved holomorphic 2-spheres of degree 6, and meanwhile find the number of such 2-spheres in each of these Fano 3-folds $A(\mathcal{H}_0^3)$.

Assume φ is a constantly curved holomorphic 2-sphere in the diagonal family given by the data $A = \text{diag}\{a_{00}, a_{11}, \cdots, a_{44}\}$ and $\{\omega_0, \omega_1, \cdots, \omega_6\}$ satisfying (7.1). It follows from Definition 6.1 that

$$\begin{aligned} \varphi(z) &= a_{00}a_{11}\omega_0\, e_0 \wedge e_1 + \sqrt{6}a_{00}a_{22}\,\omega_1\, z\, e_0 \wedge e_2 + 3a_{00}a_{33}\,\omega_2\, z^2\, e_0 \wedge e_3 \\ &+ \sqrt{6}a_{11}a_{22}\,\omega_2\, z^2\, e_1 \wedge e_2 + 2a_{00}a_{44}\,\omega_3\, z^3\, e_0 \wedge e_4 + 4a_{11}a_{33}\,\omega_3\, z^3\, e_1 \wedge e_3 \\ &+ 3a_{11}a_{44}\,\omega_4\, z^4\, e_1 \wedge e_4 + \sqrt{6}a_{22}a_{33}\,\omega_4\, z^4\, e_2 \wedge e_3 + \sqrt{6}a_{22}a_{44}\,\omega_5\, z^5\, e_2 \wedge e_4 \\ &+ a_{33}a_{44}\,\omega_6\, z^6\, e_3 \wedge e_4, \end{aligned} \tag{7.2}$$

and

$$\begin{aligned} &\frac{(9a_{00}^2a_{33}^2 + 6a_{11}^2a_{22}^2)|\omega_2|^2}{15} = \frac{(a_{00}^2a_{44}^2 + 4a_{11}^2a_{33}^2)|\omega_3|^2}{5} = a_{00}^2a_{11}^2|\omega_0|^2 = \\ &\frac{(9a_{11}^2a_{44}^2 + 6a_{22}^2a_{33}^2)|\omega_4|^2}{15} = a_{00}^2a_{22}^2|\omega_1|^2 = a_{22}^2a_{44}^2|\omega_5|^2 = a_{33}^2a_{44}^2|\omega_6|^2 = 1. \end{aligned} \tag{7.3}$$

Remark 7.1. *We point out that φ has the following standard parameterization in the sense of section 2.2.*

$$\begin{pmatrix} \varphi_1(z) \\ \varphi_2(z) \end{pmatrix} = \begin{pmatrix} 1 & 0 & -\sqrt{6}\frac{\omega_2 a_{22}}{\omega_0 a_{00}}\, z^2 & -4\frac{\omega_3 a_{33}}{\omega_0 a_{00}}\, z^3 & -3\frac{\omega_4 a_{44}}{\omega_0 a_{00}}\, z^4 \\ 0 & 1 & \sqrt{6}\frac{\omega_1 a_{22}}{\omega_0 a_{11}}\, z & 3\frac{\omega_2 a_{33}}{\omega_0 a_{11}}\, z^2 & 2\frac{\omega_3 a_{44}}{\omega_0 a_{11}}\, z^3 \end{pmatrix}. \tag{7.4}$$

In Jiao and Peng's approach, they considered collectively the undertermined variables

$$\begin{aligned} &\alpha_2 \triangleq -\sqrt{6}(\omega_2 a_{22})/(\omega_0 a_{00}),\ \beta_3 \triangleq -4(\omega_3 a_{33})/(\omega_0 a_{00}),\ \varphi_4 \triangleq -3(\omega_4 a_{44})/(\omega_0 a_{00}), \\ &u_1 \triangleq \sqrt{6}(\omega_1 a_{22})/(\omega_0 a_{11}),\ v_2 \triangleq 3(\omega_2 a_{33})/(\omega_0 a_{11}),\ z_3 \triangleq 2(\omega_3 a_{44})/(\omega_0 a_{11}). \end{aligned}$$

Then the constant curvature condition (7.3) *is equivalent to*

$$\begin{aligned} &|u_1|^2 = 6,\ |v_2|^2 + |\alpha_2|^2 = 15,\ |z_3|^2 + |\beta_3|^2 = 20 \\ &|\varphi_4|^2 + |\alpha_2 v_2 - \beta_3 u_1|^2 = 15,\ |\alpha_2 z_3 - \varphi_4 u_1|^2 = 6,\ |\beta_3 z_3 - \varphi_4 v_2|^2 = 1. \end{aligned} \tag{7.5}$$

The standard Veronese curve in (1.1) *corresponds to the solution*

$$(\alpha_2, \beta_3, \varphi_4, u_1, v_2, z_3) = (-\sqrt{6}, -4, -3, \sqrt{6}, 3, 2).$$

Branching out, observe that after fixing $(\alpha_2, \varphi_4, u_1, v_2) = (-\sqrt{6}, -3, \sqrt{6}, 3)$*, we have that the system of equations* (7.5) *reduces to*

$$|z_3|^2 + |\beta_3|^2 = 20,\ \ |\beta_3 + 3|^2 = 1,\ \ |z_3 - 3|^2 = 1,\ \ |\beta_3 z_3 + 9|^2 = 1.$$

Set

$$\beta_3 \triangleq -3 + e^{\sqrt{-1}\theta},\ \ z_3 \triangleq 3 + e^{\sqrt{-1}\varphi}. \tag{7.6}$$

From the first equation we derive $\cos\theta = \cos\varphi$*; and so* $\varphi = \pm\theta$*. If* $\varphi = -\theta$*, then the last equation above gives* $\theta = 0$ *or* π*. Therefore without losing generality, we may set* $\varphi = \theta$ *in any event. Consequently, we obtain a* 1*-parameter family of solutions*

$$\begin{pmatrix} 1 & 0 & -\sqrt{6}z^2 & (-3 + e^{\sqrt{-1}\theta})z^3 & -3z^4 \\ 0 & 1 & \sqrt{6}z & 3z^2 & (3 + e^{\sqrt{-1}\theta})z^3 \end{pmatrix}, \tag{7.7}$$

hitherto unknown in the literature, to the authors' knowledge.

Though the simple perturbation (7.6) *generates the explicit* 1*-parameter family* (7.7)*, in general, however, without further geometric clue it is a difficult task to completely classify the system* (7.5)*. As our analysis has revealed up to now, the nature of the classification lies in that one must perturb in certain Fano* 3*-folds dictated by* (7.2) *to achieve the classification. In the following, we will present an algebro-geometric approach to describe all solutions to the diagonal system* (7.2)*.*

Set

$$\omega_i \triangleq \sqrt{t_i} e^{\sqrt{-1}\theta_i}, \qquad i = 0, \ldots, 6.$$

It follows from the condition of constant curvature (7.3) that

$$\begin{aligned} &t_0 = 1/a_{11}^2,\ t_1 = 1/a_{22}^2,\ t_2 = 15/(9a_{00}^2 a_{33}^2 + 6a_{11}^2 a_{22}^2),\ t_6 = 1/(a_{33}^2 a_{44}^2), \\ &t_3 = 5/(a_{00}^2 a_{44}^2 + 4a_{11}^2 a_{33}^2),\ t_4 = 15/(9a_{11}^2 a_{44}^2 + 6a_{22}^2 a_{33}^2),\ t_5 = 1/(a_{22}^2 a_{44}^2). \end{aligned} \tag{7.8}$$

Remark 7.2. *For the detailed analysis to follow on the length constraints* (7.3)*, without loss of generality through scaling, we may assume that* $a_{00} = 1$ *and* $a_{jj} \in \mathbb{R}^+,\ 1 \le j \le 4$ (*by a diagonal unitary transformation in* $U(5)$)*. Moreover, it follows from Lemma* 3.11 *that the transformation* $\rho^4(\mathrm{diag}\{\lambda, 1\}) = \mathrm{diag}\{1, \lambda, \lambda^2, \lambda^3, \lambda^4\}$ *preserves* $\mathcal{H}_0^3$ *for any* $\lambda \in \mathbb{C}^*$*. As a consequence, after multiplying by an appropriate real* λ*, we may furthermore assume* $a_{22} = a_{33}$*. This process is equivalent to applying a Möbius reparametrization to the* 2*-sphere* φ *by* $z \mapsto \lambda z$*.*

Similarly, we assume further that $\theta_0 = \theta_6 = 0$*, which follows from dehomogenizing to eliminate* θ_0 *and introducing a rotational reparametrization of the* 2*-sphere* φ *to eliminate* θ_6*.*

Combining (7.8) with the above normalization, we have

$$t_2 = \frac{5t_0 t_1}{(3t_0 + 2)},\ t_3 = \frac{5t_0 t_1 t_6}{(t_0 t_1^2 + 4t_6)},\ t_4 = \frac{5t_0 t_1^2 t_6}{(3t_1^3 + 2t_0 t_6)},\ t_5 = t_6. \tag{7.9}$$

Moreover, it follows from (7.1) that the angles θ_i of ω_i satisfy

$$\begin{aligned} \sqrt{t_0 t_4} &= 4\sqrt{t_1 t_3} e^{\sqrt{-1}(\theta_1 + \theta_3 - \theta_4)} - 3t_2 e^{\sqrt{-1}(2\theta_2 - \theta_4)}, \\ \sqrt{t_0 t_5} &= 3\sqrt{t_1 t_4} e^{\sqrt{-1}(\theta_1 + \theta_4 - \theta_5)} - 2\sqrt{t_2 t_3} e^{\sqrt{-1}(\theta_2 + \theta_3 - \theta_5)}, \\ \sqrt{t_0 t_6} &= 9\sqrt{t_2 t_4} e^{\sqrt{-1}(\theta_2 + \theta_4)} - 8t_3 e^{\sqrt{-1} 2\theta_3}. \end{aligned} \tag{7.10}$$

Remark 7.3. *Conversely, given a solution* $\{t_0, t_1 \cdots, t_6\} \subset \mathbb{R}^+$ *and* $\{\theta_1 \cdots, \theta_5\} \subset \mathbb{R}$ *to* (7.9) *and* (7.10)*, by solving* a_{ii} *from* t_i *and defining* $\omega_i = t_i e^{\sqrt{-1}\theta_i}$*, we can obtain a constantly curved holomorphic* 2*-sphere of degree* 6 *in* $G(2,5)$ *parameterized as in* (7.2)*.*

We point out that the three equations in (7.10) are not independent by the following Lemma 7.1. In fact, set

$$\begin{aligned} &x_1 \triangleq e^{\sqrt{-1}(\theta_1+\theta_3-\theta_4)},\ y_1 \triangleq e^{\sqrt{-1}(2\theta_2-\theta_4)},\ x_2 \triangleq e^{\sqrt{-1}(\theta_1+\theta_4-\theta_5)},\\ &y_2 \triangleq e^{\sqrt{-1}(\theta_2+\theta_3-\theta_5)},\ x_3 \triangleq e^{\sqrt{-1}(\theta_2+\theta_4)},\quad y_3 \triangleq e^{\sqrt{-1}(2\theta_3)}. \end{aligned} \tag{7.11}$$

Taking norm squared on both sides of (7.10), we see from the realness of $t_0, \cdots, t_6$ that

$$\begin{aligned} &h_1 \triangleq v - uw = 0, \quad h_2 \triangleq u^2 - Xu + 1 = 0, \quad h_3 \triangleq v^2 - Yv + 1 = 0,\\ &h_4 \triangleq w^2 - Zw + 1 = 0, \end{aligned} \tag{7.12}$$

where,

$$\begin{aligned} &u = x_1/y_1, \quad v = x_2/y_2, \quad w = x_3/y_3,\\ &X = (9t_2^2 + 16t_1t_3 - t_0t_4)/(12t_2\sqrt{t_1t_3}),\\ &Y = (4t_2t_3 + 9t_1t_4 - t_0t_5)/(6\sqrt{t_2t_3}\sqrt{t_1t_4}),\\ &Z = (64t_3^2 + 81t_2t_4 - t_0t_6)/(72t_3\sqrt{t_2t_4}). \end{aligned} \tag{7.13}$$

We first solve (7.12) by viewing $\{X, Y, Z\}$ as indeterminates. Define

$$H \triangleq -XYZ + X^2 + Y^2 + Z^2 - 4. \tag{7.14}$$

Lemma 7.1. *If* $\{v, u, w, X, Y, Z\}$ *solves the system* (7.12)*, then* $H = 0$*. Conversely, given any complex solution* (X_0, Y_0, Z_0) *to* $H = 0$*, there always exits* $(v_0, u_0, w_0) \in \mathbb{C}^3$*, such that* $(v_0, u_0, w_0, X_0, Y_0, Z_0)$ *solves this system.*

Moreover, when the solution X_0, Y, Z_0 *to* $H = 0$ *are real,* $|v_0| = |u_0| = |w_0| = 1$ *if and only if* $X_0, Y_0, Z_0 \in [-2, 2]$*, in which case there are at most two solutions, namely,* (v_0, u_0, w_0) *and its complex conjugate* $(\overline{v_0}, \overline{u_0}, \overline{w_0})$*, which are distinct unless* $X_0^2 = Y_0^2 = Z_0^2 = 4$ *and* $X_0Y_0Z_0 = 8$*.*

Proof. Assume $\{v, u, w\}$ solves the last three equations in (7.12), respectively. It follows that $\{1/v, 1/u, 1/w\}$ also solves them, respectively, with $X = u+1/u$, $Y = v+1/v$, $Z = w+1/w$. By a straightforward calculation, we have

$$H = (uvw - 1)(u - vw)(v - uw)(w - uv)/(u^2v^2w^2),$$

from which the first statement follows by the first equation of (7.12).

To prove the second statement, the realness of X_0, Y_0, Z_0 dictates that $|v_0| = |u_0| = |w_0| = 1$ if and only if the last three equations in (7.12) all have a pair of conjugate solutions, which implies that their discriminants $X_0^2 - 4,\ Y_0^2 - 4,\ Z_0^2 - 4$ are no more than 0.

Furthermore, given $(X_0, Y_0, Z_0) \in [-2, 2]^3$ that solves (7.14), assume $\{(v_i, u_i, w_i) | i = 0, 1\}$ are two pairs of solutions of the system (7.12). It follows that

$$v_1 = v_0 \text{ or } \overline{v_0}, \qquad u_1 = u_0 \text{ or } \overline{u_0}, \qquad w_1 = w_0 \text{ or } \overline{w_0}.$$

By the pigeonhole principle, we may assume $u_1 = \overline{u_0},\ w_1 = \overline{w_0}$ without loss of generality. Then it follows from the first equation h_1 in (7.12) that $v_1 = u_1w_1 = \overline{v_0}$. Therefore, we deduce that these two solutions either coincide or differ by a complex conjugation, where the former case occurs when u_0, v_0, w_0 are all real to satisfy $X_0 = Y_0 = Z_0 = \pm 2$ with $X_0Y_0Z_0 = 8$ to respect $H = 0$. □

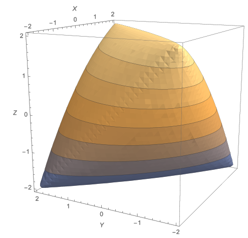

FIGURE 2. Semialgebraic sphere $H = 0$

Remark 7.4. *The cubic surface $H = 0$ with $|x|, |y|, |z| \leq 2$ is a semialgebraic sphere.*

We now analyse the diagonal family in terms of $(t_0, t_1, t_6) \in (\mathbb{R}^+)^3$. By substituting (7.9) and (7.13) into the formula of H in (7.14) and ignoring the nonzero denominator of the fraction and the nonzero factors, we obtain a hypersurface in $(\mathbb{R}^+)^3$ defined by $F(t_0, t_1, t_6) = 0$, where

$$
\begin{aligned}
&F(t_0,t_1,t_6) \triangleq 168750000\ H\, t_0^6 t_1^{11} t_6^4/(t_2 t_3 t_4^2)\\
&= 9\,{t_1}^6{t_6}^3{t_0}^9 + \Big(6912\,{t_1}^9{t_6}^2 - 366\,{t_1}^6{t_6}^3 - 10260\,{t_1}^4{t_6}^4\Big){t_0}^8\\
&+ \Big(435888\,{t_1}^2{t_6}^5 + 299592\,{t_1}^4{t_6}^4 + (-397332\,{t_1}^7 + 2560\,{t_1}^6){t_6}^3 - 58329\,{t_1}^9{t_6}^2 + 63504\,{t_1}^{12}t_6\Big){t_0}^7\\
&+ \Big(65088\,{t_6}^6 + 225504\,{t_1}^2{t_6}^5 + (31968\,{t_1}^5 + 533856\,{t_1}^4){t_6}^4 + (-451260\,{t_1}^7 - 128\,{t_1}^6){t_6}^3 +\\
&\quad (-1296\,{t_1}^{10} - 44868\,{t_1}^9){t_6}^2 + 16416\,{t_1}^{12}t_6\Big){t_0}^6\\
&+ \Big(78720\,{t_6}^6 + (-1366848\,{t_1}^3 + 154368\,{t_1}^2){t_6}^5 + (-2480688\,{t_1}^5 + 203712\,{t_1}^4){t_6}^4 + (2125440\,{t_1}^8 +\\
&\quad 541536\,{t_1}^7){t_6}^3 + (-501336\,{t_1}^{10} + 2560\,{t_1}^9){t_6}^2 + (-190512\,{t_1}^{13} - 58329\,{t_1}^{12})t_6 + 63504\,{t_1}^{15}\Big){t_0}^5\\
&+ \Big(22016\,{t_6}^6 + (15552\,{t_1}^3 + 99840\,{t_1}^2){t_6}^5 + (145152\,{t_1}^6 - 2192448\,{t_1}^5){t_6}^4 + (1076544\,{t_1}^8 +\\
&\quad 533856\,{t_1}^7){t_6}^3 + (31104\,{t_1}^{11} - 451260\,{t_1}^{10}){t_6}^2 + (-1296\,{t_1}^{13} - 366\,{t_1}^{12})t_6 + 6912\,{t_1}^{15}\Big){t_0}^4\\
&+ \Big(-1024\,{t_6}^6 - 645120\,{t_1}^3{t_6}^5 + (5774976\,{t_1}^6 + 154368\,{t_1}^5){t_6}^4 + (-3048192\,{t_1}^9 - 2480688\,{t_1}^8){t_6}^3 +\\
&\quad (2125440\,{t_1}^{11} + 299592\,{t_1}^{10}){t_6}^2 - 397332\,{t_1}^{13}t_6 + 9\,{t_1}^{15}\Big){t_0}^3\\
&+ \Big(22016\,{t_1}^3{t_6}^5 + 15552\,{t_1}^6{t_6}^4 + (145152\,{t_1}^9 + 225504\,{t_1}^8){t_6}^3 + 31968\,{t_1}^{11}{t_6}^2 - 10260\,{t_1}^{13}t_6\Big)\,{t_0}^2\\
&+ \Big(435888\,{t_1}^{11}{t_6}^2 - 1366848\,{t_1}^9{t_6}^3 + 78720\,{t_1}^6{t_6}^4\Big)\,t_0 + 65088\,{t_1}^9{t_6}^3 = 0,
\end{aligned}
\tag{7.15}
$$

with the three necessary discriminant constraints

$$
\begin{aligned}
&(9t_2^2 + 16t_1t_3 - t_0t_4)^2 - 576t_1t_2^2t_3 \leq 0,\ (4t_2t_3 + 9t_1t_4 - t_0t_5)^2 -\\
&144t_1t_2t_3t_4 \leq 0,\ (64t_3^2 + 81t_2t_4 - t_0t_6)^2 - 20736t_2t_3^2t_4 \leq 0,
\end{aligned}
\tag{7.16}
$$

thanks to the assumptions made on $X, Y, Z \in [-2, 2]$ in Lemma 7.1:

Remark 7.5. *The three constraints $|u| = |v| = |w| = 1$ are not independent by the first equation in* (7.12)*. Any two of the three inequalities in* (7.16) *imply the third. Moreover, $Z \in (-2, 2)$ implies $X, Y \in (-2, 2)$ since for a fixed $Z \in (-2, 2)$, $H = 0$ in* (7.14) *defines an ellipse good for the conclusion.*

In conclusion, we obtain the following existence and uniqueness theorem.

Theorem 7.1. *Given a diagonal matrix $A = \mathrm{diag}\{1, a_{11}, a_{22}, a_{22}, a_{44}\}$, normalized as in Remark* 7.2*, there exists a sextic curve γ belonging to the generally ramified family in $\mathcal{H}_0^3$ such that $A(\gamma)$ is of constant curvature, if and only if $\{t_0, t_1, t_6\}$ given by* (7.9) *satisfies the algebraic equation* (7.15) *and inequalities* (7.16).

Moreover, in $A(\mathcal{H}_0^3)$, there exist at most two constantly curved holomorphic 2-spheres of degree 6 belonging to the generally ramified family; they are distinct except when $\{X, Y, Z\}$ defined in (7.13) satisfies $X^2 = Y^2 = Z^2 = 4$ and $XYZ = 8$.

Proof. The necessary part has been verified in the preceding discussion.

Conversely, assume that $\{t_0, t_1, t_6\}$ satisfy the algebraic equation (7.15) and inequalities (7.16). Then we obtain at least a triple (v_0, u_0, w_0) of solution of system (7.12) according to Lemma 7.1. By substituting it into system (7.10), we obtain a unique solution $\{(x_i, y_i)|1 \leq i \leq 3\}$ by the following recipe: The first equation of (7.10) gives that

$$y_1 = \sqrt{t_0 t_4}/(4\sqrt{t_1 t_3}\, u_0 - 3t_2), \quad x_1 = y_1 u_0. \tag{7.17}$$

It follows from $|u_0| = 1$ that both x_1 and y_1 are of unit length. A similar discussion applies to (x_2, y_2) and (x_3, y_3).

Apply the logarithmic function on both sides of (7.11). Since the ranks of the coefficient matrix of of $(\theta_1, \ldots, \theta_5)$ and its enlarged version with the augmented $(\log(x_1), \cdots, \log(y_3))$ are both equal to 5, we can solve θ_j from the arguments of the points $\{(x_i, y_i)|1 \leq i \leq 3\}$ on the plane. Substituting all the data into (7.2) gives a constantly curved holomorphic 2-sphere φ in $A(\mathcal{H}_0^3)$ (see Remark 7.3).

Lastly, we remark that φ is uniquely determined by (v_0, u_0, w_0), owing to that the only difference between any two pairs of solutions $\{\theta_j|1 \leq j \leq 5\}$ and $\{\tilde{\theta_j}|1 \leq j \leq 5\}$ of (7.11) is $\theta_j = \tilde{\theta_j} + 2kj\pi/6$, $1 \leq j \leq 5$, for some $0 \leq k \leq 5$. It is straightforward to show that the corresponding two curves share the same image by introducing a rotational reparametrization $\tilde{z} = z e^{\sqrt{-1}2k\pi/6}$.

In conclusion, any solution (v, u, w) of system (7.12) determines uniquely a constantly curved 2-sphere. Then the second statement follows from Lemma 7.1. □

Corollary 7.1. *The only constantly curved holomorphic 2-sphere of degree 6 in the standard Fano 3-fold $\mathcal{H}_0^3$ tangent to the the standard Veronese curve $PSL_2 \cdot u^6$ is the Veronese curve itself.*

Proof. For the standard Fano 3-fold $\mathcal{H}_0^3$, the associated $\{t_0, t_1, t_6\}$ are all equal to 1. Therefore the corresponding $X = Y = Z = 2$ by (7.13). □

Remark 7.6. *In addition to the standard Fano 3-fold $\mathcal{H}_0^3$, let us take the diagonal $A = \operatorname{diag}\{1, 1, 4, 4, 16\}$, there exists a unique constantly curved holomorphic 2-sphere of degree 6 belonging to the generally ramified family that lies in $A(\mathcal{H}_0^3)$ given by*

$$\begin{pmatrix} 1 & 0 & -\sqrt{6}z^2 & -2z^3 & -3z^4 \\ 0 & 1 & \sqrt{6}z & 3z^2 & 4z^3 \end{pmatrix},$$

since the associated $X = Y = Z = 2$. It turns out that among Fano 3-folds $\mathcal{H}^3$ in $G(2,5)$, only three (up to unitary congruence) contain a unique constantly curved holomorphic 2-sphere of degree 6; the last one will be given in Example 8.5.

8. The moduli space and new examples

Before describing the moduli space of constantly curved holomorphic 2-spheres belonging to the generally ramified family, we first consider the semialgebraic set $S \subseteq (\mathbb{R}^+)^3$ determined by the algebraic equation (7.15) and the three inequalities (7.16).

Proposition 8.1. *The semialgebraic set S is 2-dimensional and equipped with an involution*

$$\sigma : S \to S, \quad t = (t_0, t_1, t_6) \mapsto T = (T_0, T_1, T_6) = (g\, t_0, g\, t_1, g^3\, t_6), \tag{8.1}$$

where $g(t_0, t_1, t_6) \triangleq t_1^3/(t_0^2 t_6)$.

Proof. It is easy to show that σ is an involution of $(\mathbb{R}^+)^3$ restricted to S; consequently, we need only verify that $\sigma(S) \subseteq S$.

Assume that $t = (t_0, t_1, t_6) \in S$, i.e., that t satisfies

$$F(t) = 0, \text{ and } X(t), Y(t), Z(t) \in [-2, 2].$$

A direct computation yields that

$$F(T) = g^{21} F(t) = 0, \;\; Z(T) = Z(t) \in [-2, 2].$$

Note that the last equation of (7.1) gives

$$\sqrt{t_1 t_6} = 3\sqrt{t_2 t_5} e^{\sqrt{-1}(\theta_2+\theta_5-\theta_1)} - 2\sqrt{t_3 t_4} e^{\sqrt{-1}(\theta_3+\theta_4-\theta_1)}.$$

Set $q = e^{\sqrt{-1}(\theta_2+\theta_5-\theta_3-\theta_4)}$. Then a similar argument to that deriving (7.12) leads to $q^2 - Qq + 1 = 0$, where

$$Q(t) \triangleq (-t_1 t_6 + 9 t_2 t_5 + 4 t_3 t_4)/(6\sqrt{t_2 t_3 t_4 t_5}).$$

Since $|q| = 1$, it forces $Q(t) \in [-2, 2]$. It is straightforward to show that $Y(T) = Q(t) \in [-2, 2]$. Therefore, combining Remark 7.5, we obtain that the norm of $X(T)$ is also less than or equal to 2. This completes the proof that $T = \sigma(t)$ lies in S.

We are left with showing that the real dimension of the semialgebraic set S is 2. At the generic point $p_0 = (1, \frac{1}{2}, \frac{1}{8}) \in S$ (for the choice of p_0, see Example 8.2 below for details). A calculation gives

$$\nabla F(p_0) = \big(\partial F/\partial t_0, \partial F/\partial t_1, \partial F/\partial t_6\big)(p_0) = (0, -13125/256, 4375/64) \neq 0.$$

Owing to the implicit function theorem, near p_0, S is locally a graph of t_0 and t_1; hence, its real dimension is 2. □

Remark 8.1. *We point out that the involution σ comes from the reciprocal transformation of $\mathbb{C}P^1$ (see the proof of the following Theorem).*

Now, we are in a position to present our main theorem. Denote by $\mathcal{M}$ the moduli space of constantly curved holomorphic 2-spheres belonging to the generally ramified family in $G(2, 5)$, modulo the extrinsic ambient $U(5)$-equivalence and the internal Möbius reparametrization.

Theorem 8.1. *$\mathcal{M} = S/\mathbb{Z}_2$, so that it is a 2-dimensional semialgebraic set.*

Proof. Our first goal is to show that a holomorphic 2-sphere of the diagonal family is also determined by its coefficients of z^k, $k = 2, 3, 4$ in (7.2). Consider the quotients of them respectively to define a map

$$\tau : S \to (\mathbb{R}^+)^3, \quad (t_0, t_1, t_6) \mapsto (A, B, C) \triangleq (\frac{a_{00}a_{33}}{a_{11}a_{22}}, \frac{a_{00}a_{44}}{a_{11}a_{33}}, \frac{a_{11}a_{44}}{a_{22}a_{33}}). \tag{8.2}$$

It follows from (7.8) that $(A, B, C) = (\sqrt{t_0}, \sqrt{\frac{t_0}{t_6}}\, t_1, \sqrt{\frac{t_1}{t_0 t_6}}\, t_1)$. It is straightforward to show that $t_0 = A^2,\; t_1 = A^4 C^2/B^2,\; t_6 = A^{10} C^4/B^6$; therefore τ is injective.

The next step is to describe our moduli space. Let $\varphi_1(z)$ and $\varphi_2(\tilde{z})$ be two holomorphic 2-spheres of the diagonal family corresponding to $t = (t_0, t_1, t_6)$ and $\tilde{t} = (\tilde{t_0}, \tilde{t_1}, \tilde{t_6})$, respectively.

If there exists a $U \in U(5)$ such that the image of $U \cdot \varphi_1$ agrees with that of φ_2, then U induces a Möbius transformation $\tilde{z} = f(z)$ on $\mathbb{C}P^1$. Since the ramified points of φ_1 and φ_2 are both $\{0, \infty\}$ by Lemma 6.2, this set is invariant under φ. Hence $\tilde{z} = \mu z$ or $\frac{\mu}{z}$, where $\mu \in \mathbb{C}^*$. Our aim is to establish that $\tilde{t} = t$ or $\tilde{t} = \sigma(t)$, which suffices to complete the proof. We divide the argument into two cases.

Case (1): Suppose that $\tilde{z} = \mu z$. Comparing the first two and last two terms of φ_1 and φ_2, we obtain that (see (7.2))

$$U \cdot e_0 \wedge U \cdot e_1 \equiv 0 \mod (e_0, e_1),\ U \cdot e_0 \wedge U \cdot e_2 \equiv 0 \mod (e_0, e_2),$$
$$U \cdot e_2 \wedge U \cdot e_4 \equiv 0 \mod (e_2, e_4),\ U \cdot e_3 \wedge U \cdot e_4 \equiv 0 \mod (e_3, e_4).$$

Hence, $U = \mathrm{diag}\{u_{00}, \ldots, u_{44}\}$ is diagonal as U is unitary. As a result, they share the same quotients in (8.2), i.e., $\tau(t) = \tau(\tilde{t})$, so that $t = \tilde{t}$ by the injectivity of τ.

Case (2): Suppose that $\tilde{z} = \frac{\mu}{z}$. Following a similar argument as in Case (1), we see that U is anti-diagonal. Consequently, the quotients in (8.2) satisfy $A(\tilde{t}) = C(t),\ B(\tilde{t}) = B(t),\ C(\tilde{t}) = A(t)$. By the exposition below (8.2), it is easy to show that $\tilde{t} = \sigma(t)$.

Now, the conclusion follows from Theorem 6.1. □

The end of this section is devoted to the construction of several interesting individual as well as 1-parameter families of examples.

Recall the involution $\sigma : S \to S$ and its invariant subset S_1 defined by setting $g = 1$, so that $1 = g = t_1^3/(t_0^2\, t_6)$. It is a piecewise smooth simple closed curve. Indeed, substitute $t_6 = t_1^3/t_0^2$ into (7.15) and ignore the non-zero denominator and the non-zero factors. The level set S_1 is the semialgebraic set defined by the three inequalities in (7.16) and

$$\begin{aligned}&\big(441\, t_0^8 - 42\, t_0^7 + t_0^6 - 72\, t_0^5 t_1 - 5136\, t_0^4 t_1 - 1592\, t_0^3 t_1 + 7056\, t_0^2 t_1^2 - 672\, t_0 t_1^2 + 16\, t_1^2\big)\cdot \\ &(t_0 - 1)\big(2\, t_0^3 - 3\, t_1 t_0 + t_1\big) = 0.\end{aligned}$$

In the t_0t_1-coordinate plane, S_1 is plotted in Figure 3. The branch corresponding to $(t_0 - 1) = 0$ is the blue vertical line segment. The second branch described by $\big(2\, t_0^3 - 3\, t_1 t_0 + t_1\big) = 0$ is the end point $(1, 1)$ of the blue line segment. The third branch corresponds to the union of the (upper) brown and (lower) green curves parametrized by

$$\psi_1 = \{(s, F_1(s)) \mid s \in [1, 11/6]\},\ \psi_2 = \{(s, F_2(s)) \mid s \in [1, 11/6]\}, \tag{8.3}$$

respectively, where $F_1 = (t_0^3(199 + 642t_0 + 9t_0^2 + 30\Delta))/(4(21t_0 - 1)^2)$, $F_2 = (t_0^3(199 + 642t_0 + 9t_0^2 - 30\Delta)/(4(21t_0 - 1)^2)$, and $\Delta \triangleq (3t_0 + 2)\sqrt{(4t_0 + 1)\,(11 - 6t_0)}$.

It follows from Theorem 8.1 that the moduli space is $\mathcal{M} = S/\sigma$ with the simple closed curve S_1 on its boundary. By applying the coordinate transformation $(t_0, t_1, t_6) \mapsto (t_0, t_1, \lambda)$ with $\lambda = 1/g$, we can plot $\mathcal{M}$ as in Figure 4. It looks like a horn, with S_1 marked in red, and the level sets of $g = 2$ and $g = 3$ marked in green and blue, respectively. The figure seems to suggest that the moduli space $\mathcal{M}$ is a topological disk. It would be interesting to see whether this is indeed the case.

Example 8.2. We point out that examples on the blue line segment coincide with the 1-parameter family (7.7) in Remark 7.1. In fact, it follows from (7.9) that

$$t_0 = 1,\ t_2 = t_1,\ t_3 = 5t_1^2/(4t_1 + 1),\ t_4 = t_1^2,\ t_5 = t_1^3,\ t_6 = t_1^3,$$
$$a_{00} = 1,\ a_{11} = 1,\ a_{22} = a_{33} = 1/\sqrt{t_1},\ a_{44} = 1/t_1.$$

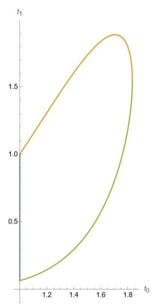

FIGURE 3. The level set S_1

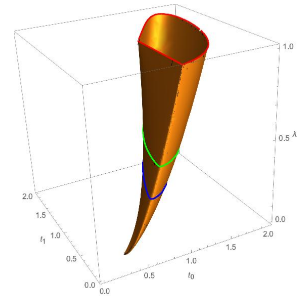

FIGURE 4. The moduli space $\mathcal{M}$

Moreover, substituting all the data into (7.4), we obtain that

$$\begin{pmatrix} 1 & 0 & -\sqrt{6}e^{\sqrt{-1}\theta_2}z^2 & -4\sqrt{\frac{t_3}{t_1}}e^{\sqrt{-1}\theta_3}z^3 & -3e^{\sqrt{-1}\theta_4}z^4 \\ 0 & 1 & \sqrt{6}e^{\sqrt{-1}\theta_1}z & 3e^{\sqrt{-1}\theta_2}z^2 & 2\frac{\sqrt{t_3}}{t_1}e^{\sqrt{-1}\theta_3}z^3 \end{pmatrix}. \tag{8.4}$$

Set $t_1 \triangleq (5+3\cos\theta)/(20-12\cos\theta)$, then $\cos\theta = (20t_1-5)/3(4t_1+1)$. Then $\theta_0 \triangleq 0$, $\theta_6 \triangleq 0$, and

$$\theta_1 \triangleq \theta - \frac{\beta_0-\beta_1}{2},\ \theta_2 \triangleq \theta,\ \theta_3 \triangleq \theta + \frac{\beta_0+\beta_1}{2},\ \theta_4 \triangleq \theta,\ \theta_5 \triangleq \theta - \frac{\beta_0-\beta_1}{2},$$

satisfy (7.10), where

$$\beta_0 = \operatorname{Arg}\left(\frac{3+e^{\sqrt{-1}\theta}}{\sqrt{10+6\cos\theta}}\right),\ \beta_1 = \operatorname{Arg}\left(\frac{3-e^{\sqrt{-1}\theta}}{\sqrt{10-6\cos\theta}}\right).$$

It is straightforward to verify that (7.7) differs from (8.4) by multiplying its third and fourth columns by $e^{\sqrt{-1}(\beta_1-\beta_0+\theta)}$, its last column by $e^{\sqrt{-1}(2(\beta_1-\beta_0)+\theta)}$, and performing a reparameterization $z \mapsto e^{\sqrt{-1}(\beta_0-\beta_1)/2}z$. Note that $\pm\theta$ give the same t_1; they correspond to the two complex-conjugated solutions.

Proposition 8.2. *The second fundamental form A of a constantly curved holomorphic 2-sphere of degree 6 belonging to the generally ramified family is not of constant norm, except for the standard Veronese curve* (1.1).

Proof. It follows from the Gauss equation that

$$||A||^2 = 20/3 - ||\partial F/\partial z \wedge \partial F/\partial z||^2/(9(1+|z|^2)^8), \tag{8.5}$$

where F is the Plücker embedding of the holomorphic 2-sphere in $G(2,5)$ into $\mathbb{C}P^9$ (see [21, p.6, p.9] for details). Note that $||\partial F/\partial z \wedge \partial F/\partial z||^2$ only vanishes at ramified points. Therefore, using Lemma 6.2 we can derive that the second term on the right-hand side of $||A||^2$ is not constant. □

Example 8.3. On the level set S_1, choose $t_0 = 11/6$. Then we can solve for $t_1 = 1331/864$. It gives an exact solution to (7.15),

$$t_0 = \frac{11}{6},\ t_1 = \frac{131}{864},\ t_2 = \frac{14641}{7776},\ t_3 = \frac{73205}{41472},\ t_4 = \frac{1771561}{1119744},\ t_5 = t_6 = \frac{19487171}{17915904},$$

It is checked that $X = Y = 5\sqrt{5}/\sqrt{33}$ and $Z = 2$. from which the angles $\{\theta_1, \cdots, \theta_5\}$ can be solved.

Example 8.4. On the level set S_1, choose $t_0 = \left(2\sqrt{79}+20\right)/21$. Then we can solve for $t_1 = \left(2\sqrt{79}+20\right)/21$. It gives an exact solution to (7.15),

$$\begin{aligned} &t_0 = t_1 = t_5 = t_6 = \left(2\sqrt{79}+20\right)/21,\ t_2 = t_4 = \left(23\sqrt{79}+209\right)/189, \\ &t_3 = (9+\sqrt{79})/8, \end{aligned}$$

from which the angles $\{\theta_1, \cdots, \theta_5\}$ can be solved. Note that for this example, the diagonal matrix A has two distinct eigenvalues $a_{00} = a_{44} \neq a_{11} = a_{22} = a_{33}$.

Example 8.5. Start with the equations $P \triangleq X^2 - 4 = 0$, $Q \triangleq Y^2 - 4 = 0$, $R \triangleq Z^2 - 4 = 0$, with X, Y, Z given in (7.13) to express them in terms of the variables t_0, t_1, g, with $t_6 = t_1^3/(t_0^2 g)$ by (8.1). Continue to compute the derived resultants of the refined numerators P', Q', R' of P, Q, R, in terms of t_0, t_1, g, after removing powers of $g-1$ and those single-variable factors without positive solutions by, e.g., Sturm's algorithm for counting the exact number of distinct positive roots of a real polynomial, while setting aside possible candidate polynomials before proceeding with the next level of resultant computation*;* along the way, we heed the constraint that $(gt_0, gt_1, 1/g)$ is a set of solution if (t_0, t_1, g) is, by Proposition *8.1*, to further narrow down the candidates. We end up with the exact equations for possible t_0, t_1, g :

$$\begin{aligned} &p \triangleq 3004245721g^6 - 139634316726g^5 - 67838574585g^4 - 318786958820g^3 - 67838574585g^2 \\ &\quad - 139634316726g + 3004245721 = 0, \\ &q \triangleq 2537649t_0^6 - 40347234t_0^5 + 36454860t_0^4 - 19711080t_0^3 + 26076060t_0^2 - 17915544t_0 + 3452164 = 0, \\ &r \triangleq 6861904453295341780216896t_1^6 - 5778944084749942749568089 6t_1^5 - 354143212952899964418216 0t_1^4 \\ &\quad + 269578754871582716992368 0t_1^3 - 2425918438750430615250 60t_1^2 - 2610563393624014268141 76t_1 \\ &\quad + 5368957541033807913984 1 = 0. \end{aligned}$$

Compute the Gröbner basis of the ideal $(P', Q,' R', p, q, r)$ to obtain the basis consisting of six elements of which we only record the two essential ones,

$$\begin{aligned} &E \triangleq 3040721913553456992086527928 1g^2t_1 - 568439663135044192248640408 4g^2 \\ &\quad + 482638150820269177521832873 8gt_1 + 878110939074213639282083597 8g \\ &\quad + 220879701772863195482469014 85t_0 - 379527525045034273371934075 59t_1 \\ &\quad - 101296701670107544182707968 64 = 0, \\ &G \triangleq 323983664320381367395969030814241g^3 - 150979192496335081137165367360527 77g^2 \\ &\quad + 240019470529124364905323917771900 00gt_1 - 102972705795702442411637955551124 89g \\ &\quad - 211602161037271546704800657294251 20t_0 + 381555700029075898927185905891242 80t_1 \\ &\quad - 107535291042404279956024533941283 35 = 0. \end{aligned}$$

We obtain $t_0 \triangleq R/S$ and $t_1 = T/U$ in closed form of g, where

$$\begin{aligned} &R \triangleq 323983664320381367395969030814241g^5 - 150464949888530049123291762218259 59g^4 \\ &\quad - 861108557729599525186774059319803 4g^3 + 665801730760386692567772326968836 6g^2 \\ &\quad + 812283095047896987412954048460800 1g + 261329181160908217572369254340993 85, \\ &S \triangleq 211602161037271546704800657294251 20g^2 + 207937978016292208015603247943957 60g \\ &\quad + 130530343528308426646762800276012 0, \\ &T \triangleq -423618308217230277983078980100353g^3 + 268613123953869096710992847894178 65g^2 \\ &\quad + 246468245914607620535805173024672 9g + 267490870599451193235594947969845 59, \\ &U \triangleq 380883889867088784068641183129652 16g^2 + 374288360429325974428085846299123 68g \\ &\quad + 234954618350955167964173040496821 6. \end{aligned}$$

It is then checked that all the remaining equations in the basis are compatible with $p = 0$. Now, $p = 0$ has two positive real roots reciprocal to each other as the coefficients of p are symmetric, which are approximately $g \sim 0.0212731522$ and 47.0076078738 *(Since all the above polynomial equations are exact, the listed numerical values are accurate up to the last digit, checked by the intermediate value theorem, for instance.)* We then derive the corresponding values for t_0 and t_1 through R, S, T, U to yield

$$\begin{aligned}(t_0, t_1, g) &\sim (0.3184944933, 0.1803379951, 47.0076078738), \text{ or} \\ &\sim (14.9716642533, 8.4772577609, 0.0212731522),\end{aligned}$$

accurate up to the last digit, in accord with Proposition *8.1;* both give $X = Y = Z = 2$. The second set gives the pointed end of the horn in Figure 2.

This is the third and the last example, aside from the two given in Remark *7.6* with $g = 1$, for which there is only one constantly curved 2-sphere belonging to the generally ramified family in the corresponding Fano 3-fold $A(\mathcal{H}_0^3)$, where A is computed by (7.8).

Example 8.6. Set $t_1 \triangleq t_0^2/6$ in F given in (7.15) and factor out positive terms to yield

$$\begin{aligned}&f(g, t_0) \triangleq 190512g^4t_0^6 + 20736g^4t_0^5 + 95256g^3t_0^6 + 27g^4t_0^4 - 205416g^3t_0^5 - 401301g^3t_0^4 \\ &- 104328g^2t_0^5 - 6264g^3t_0^3 - 59319g^2t_0^4 + 168282g^2t_0^3 + 32913gt_0^4 + 202140g^2t_0^2 + 35388gt_0^3 \\ &+ 6720gt_0^2 + 2034t_0^3 + 19504gt_0 + 2460t_02 + 688t_0 - 32 = 0.\end{aligned}$$

It defines a plane algebraic curve C. We claim that $C^* \subset C$ falling in the rectangle $\mathcal{R}$ given by $8/15 \le t_0 \le 5$, $1475/10000 \le g \le 3$, is a smooth, connected closed curve contained in S, the double of the moduli space $\mathcal{M}$.

Firstly, observe that $(t_0, g) = (1, 1)$ solves $f = 0$ so that that C^* is not empty. It is also directly checked that $\frac{\partial f}{\partial t_0}/\frac{\partial f}{\partial g} = 2$ at $(t_0, g) = (1, 1)$, so that the implicit function theorem implies that $f = 0$ is locally a curve $(t_0, g(t_0))$ around $(t_0, g) = (1, 1)$ with negative slope.

Setting $t_0 \triangleq 8/15$ or 5, and $g \triangleq 1475/10000$ or 3, respectively, we solve $f(g, t_0) = 0$ to attain (accurate up to the last digit for the exact polynomials)

$$\begin{aligned}&\text{for } t_0 = 8/15, \nexists \text{ real } g, \text{ while for } t_0 = 5,\ g \sim -0.4687373438, \text{ or } -0.0109931977; \\ &\text{for } g = 1475/10000, \ \ t_0 \sim 0.0088038166, \text{ while for } g = 3, \\ &t_0 \sim -0.5591240674, -0.4272041173, -0.0337884110, 0.0005317397.\end{aligned}$$

This means that the set C^* never leaves the rectangle $\mathcal{R}$, so that by analytic continuation of an algebraic curve, C^* consists of closed curves and, a priori, a few isolated points. The latter can be ruled out since these finitely many points must satisfy $f = \partial f/\partial t_0 = \partial f/\partial g = 0$ and the Gröbner basis associated with the ideal $(f, \partial f/\partial t_0, \partial f/\partial g)$ is $\{g - 1, 3t_0 + 2\}$ whose zero locus $(t_0, g) = (-2/3, 1)$ does not fall in the domain $\mathcal{R}$. As a result, it also implies that the finitely many closed curves constituting C^* are smooth and disconnected in $\mathcal{R}$.

By calculating the resultants of $f = \partial f/\partial t_0 = 0$ against g and t_0 and solving for the roots, we verify that none of the possible pairs of (t_0, g) satisfy (7.14) *(*see the remark below for the engaged computational error analysis for rational functions), except possibly for two points (t_0, g) approximately at

$$(0.6547026351, 2.9099350324), \quad \text{or} \quad (4.5794327836, 0.1475263321), \tag{8.6}$$

accurate up to the last digit. Since there exist at least two such points, this proves that C^* is only tangent to the horizontal lines, $g =$ constants, precisely at the two points; likewise, this is also true for the vertical line test. In particular, C^* has only one connected component as, otherwise, we would have more than two points tangent to horizontal or vertical lines.

We calculate the resultants of f and the numerator of $R \triangleq Z^2 - 4$ against g and t_0 and solve for the roots, to confirm that the only point of intersection of the curve C^* and the boundary of $Z^2 \leq 4$ occurs with tangency at

$$(t_0, g) \sim (1.5271772661, 0.4663765333),$$

with the corresponding $X = Y \sim 1.8718004195$ and $Z = 2$. It follows that C^* lies completely in $Z^2 \leq 4$ since $(t_0, g) = (1, 1)$ satisfies $Z^2 < 4$. In particular, the three constraints in (7.16) are satisfied by Remark 7.5.

Figure 5 depicts the curve C^* (in red) in S. Since it extends into the region with $g > 1$, we apply the involution σ to flip it back into $\mathcal{M}$ with $g \leq 1$. Figure 6 shows the resulting self-crossing, flipped C^* (in red), which opens at $g = 1$ for which $t_0 = 1$ or $t_0 \sim 1.4542230103$. The region bounded by the three constraints is colored yellow.

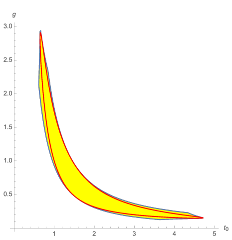

FIGURE 5. The curve C^* in S

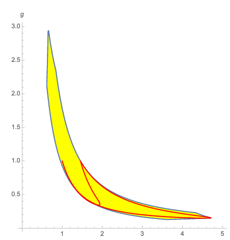

FIGURE 6. Folded C^* in $\mathcal{M}$

Remark 8.2. *Let* $f(x,y) = \sum_{m,n=0}^{M,N} a_{mn}\, x^m y^n$ *and* $l(x,y) \triangleq \sum_{i,j=0}^{I,J} b_{ij}\, x^i y^j$ *over a rectangle* $\mathcal{R} : [a,b] \times [c,d]$ *with* $a, c > 0$. *Assume* $l(x,y) > 0$ *and define the positive function* $||f||(x,y) \triangleq \sum_{m,n=0}^{M,N} |a_{mn}|\, x^m y^n$ *over* $\mathcal{R}$. *Given* $(x_0, y_0), (x,y) \in \mathcal{R}$ *with* $0 < |x_0 - x|, |y_0 - y| < h$, *where* $h > 0$ *is so small that* $nh << 1$ *for* $n = M, N, I$, *or* J, *then* $p(x,y) \triangleq f(x,y)/l(x,y)$ *satisfies the error estimate*

$$|p(x_0, y_0) - p(x,y)| \leq (C(M,N) + C(I,J)) \sup_{(x,y) \in \mathcal{R}} (||f||(x,y)/l(x,y)), \tag{8.7}$$

where, for $n \in \mathbb{N}$ *with* $nh < 1$, *we define* $\gamma_n \triangleq nh/(1 - nh)$, *and*

$$C(p,q) \triangleq (e^{1/a} - 1)\gamma_p + (e^{1/c} - 1)\gamma_q + (e^{1/a} - 1)(e^{1/c} - 1)\gamma_p \gamma_q$$

for $p, q \in \mathbb{N}$. (*We leave it to the reader to verify.*)

In Example 5, $x \triangleq g$ *and* $y \triangleq t_0$, $\mathcal{R}$ *is the rectangle* $[1475/10000, 3] \times [8/15, 5]$, *and* $f(g, t_0)$ *is given in Example* 5. *Write, for* H *in* (7.14),

$$H = f(g, t_0)/l(g, t_0), \quad l(g, t_0) \triangleq 405000\, t_0^3\, g^2\, (3t_0 + 2)\, (3gt_0 + 2) > 0,$$

Since $M = 4, N = 6, A = 3$, *and* $B = 5$, *if we take* $h \triangleq 10^{-20}$, *the error estimate* (8.7) *gives that* $C(M,N) + C(A,B)$ *is in the magnitude of* 10^{-17}, *and an elementary minimax estimate derives* $||f||(x,y)/g(x,y) \leq 1$ *for all* $(x,y) \in \mathcal{R}$, *so that the error is in the magnitude of* 10^{-17}. *Consequently, all the engaged computations for the data satisfying* $H \neq 0$ *to obtain, e.g.,* (8.6) *are accurate up to the tenth decimal place if we set the last significant decimal place to be the twentieth; all the undesired values above, in fact, are such that their third decimal digits are nonzero to satisfy* $H \neq 0$.

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