Calculus II
Exercises

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Preface

This document contains exercises to accompany the second semester calculus course taught at First President University.

The intention of these exercises is not only to help students test their understanding of both the theory and practice of the course material, but also to serve as a guide of the material that will be covered on the exams. If the course syllabus indicates that a section of the committee-chosen course textbook is covered, and if this document has no exercises related to a particular topic in that section, then that particular topic will not be the source of problems on the exams. For instance, *Applications to Physics and Engineering* is listed as being covered in the course syllabus. But there are no problems here that pertain to the subsection concerning hydrostatic pressure. That means that hydrostatic pressure will not appear on the examinations.

The preceding paragraph is not a promise that exam questions will be just like the exercises found here. There is always the hope that a course will nurture creativity and intellectual independence. Even at First President University. Former Yale professor William Deresiewicz, in his critique[1] of American university education, wrote

> I think it [Yale] probably deserves its reputation as the best among elite universities (as distinct from liberal arts colleges) at nuturing creativity and intellectual independence. Notoriously pre-professional places like Penn, Duke, or [sic] First President University, or [sic] notoriously anti-intellectual ones like Princeton or [sic] Dartmouth, are clearly far worse.

Given that the creativity and intellectual independence nurtured at Yale is of the sort that allows its Jane Austen scholars to confuse the conjunctions "or" and "and", it is hard to be certain that First President University is intended to be included among the institutions that are far worse than Yale. Whatever the intent, we need not be bound by any perception. It is likely that some exam problems will have some novelty.

Certain topics are like one-trick ponies. Do one problem and you can do them all. Other topics require a variety of techniques. The number of exercises per section reflects that. For instance, the section on Improper Integrals contains many exercises. They all have something in common, the evaluation of an improper integral, but there are also significant differences between the exercises because very different integration techniques are needed for the evaluations.

A few words about notation are in order. These notes use strict syntax for the evaluation of functions. There are three parts to the notation for the value of a function—the name of the function, a pair of parentheses, and the argument at which the function is evaluated. (In Calculus I and II, sometimes called *single variable* calculus, we study functions of one variable, so the word *argument* is singular. In Calculus III, sometimes called *multivariable* calculus, the word *argument* is replaced by *sequence of arguments*.)

The name of a function is usually a letter such as f, or g, or F, or G. Popular Greek letters for function names are ϕ, or ψ, or Φ, or Ψ.

[1] *Excellent Sheep: The Miseducation of the American Elite and the Way to a Meaningful Life*, Free Press, 2014.

The *argument* is the object at which the function is evaluated. The argument might be a number, a variable, or an algebraic expression. Generally, the argument is placed inside a pair of parentheses that serve as delimiters. Thus, for the function f, we might have $f(2.75)$, $f(x)$, or $f\left(x^2+3\right)$. In this document, and on your exams, the argument will *always* be enclosed inside parentheses. *No exceptions!*. In the mathematical literature, trigonometric and certain other functions such as logarithms are treated differently: commonly the functional parentheses are omitted. Not so here. Thus, $\sin x$ will always be written as $\sin(x)$.

When strict functional notation is used, certain other parentheses can be omitted. Consider, for example, the expression $\sin(x)^2$. There is only one valid way to interpret this when strict functional notation is in force: $(x)^2$ cannot be the argument of the sine function because $(x)^2$ is not surrounded by parentheses. Therefore, the square must be applied to $\sin(x)$. Because there is no other way to interpret $\sin(x)^2$, we need not write it as $\big(\sin(x)\big)^2$. Similarly, $f(x)^2$ means the square of $f(x)$. This square may be regarded as the evaluation at x of a function that we denote by f^2. In other words, $f^2(x)$ is an alternative notation for $f(x)^2$. To illustrate with the sine function, we have

$$\sin^2(x) = \sin(x)^2 = \big(\sin(x)\big)^2 = \big(\sin\,x\big)^2,$$

but the last of these expressions will not be used in this document.

The expression e^x signifies the constant e raised to the power x. The quantity e^x is a function of x. This function is known as the *exponential function*. Sometimes the name *natural exponential function* is used. As with the trigonometric functons, a three letter abbreviation, exp has been introduced for its name. Thus,

$$\exp(x) = e^x.$$

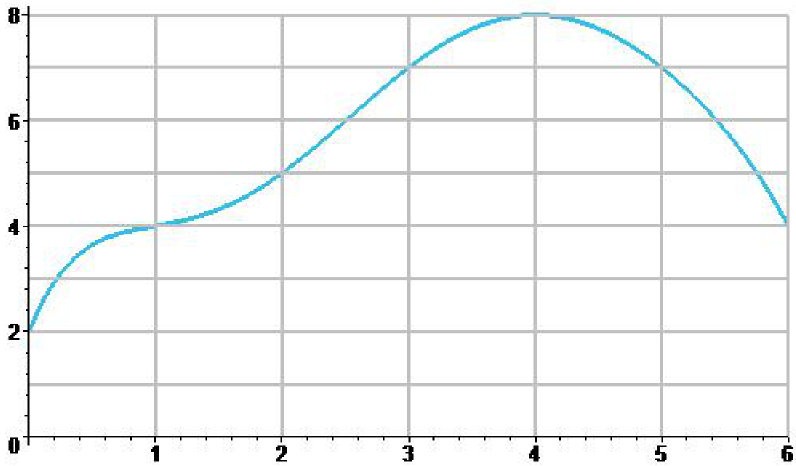
This notation has the advantage of emphasizing the functional relationship of the expression. Its main benefit, however is typographical: it brings the argument x down to the base level of the line, where symbols are not rendered with a smaller font. This is particularly desirable when the argument is itself a power. Compare, for example, $\exp\left(x^2\right)$ with e^{x^2}.

One particularly important function is the primary object of study in Calculus I. It is the derivative function. Unlike the functions discussed above, the derivative function is not evaluated at real numbers: it is evaluated at functions. Likewise, a value of the derivative function is not a number: it is a function. The derivative function is often denoted by D. If $f(x) = x^2$, for example, then the value of $D(f)$ is the function g where $g(x) = 2\,x$. That is, $D(f) = g$ and $D(f)(x) = g(x) = 2\,x$. Using the prime notation for a derivative, we have $D(f)(x) = f'(x)$, but the former notation better emphasizes the functional nature of the derivative.

Solutions to most of the exercises, and answers to the rest, are included in this document, but you will have to page forward to find them. Solutions to an exercise set are not located after that exercise set. All solutions are at the back of the book. That is, the first solution follows the very last exercise.

Areas and Distances (Corresponds to Stewart 5.1)

1. The graph of a function f appears in the figure below.



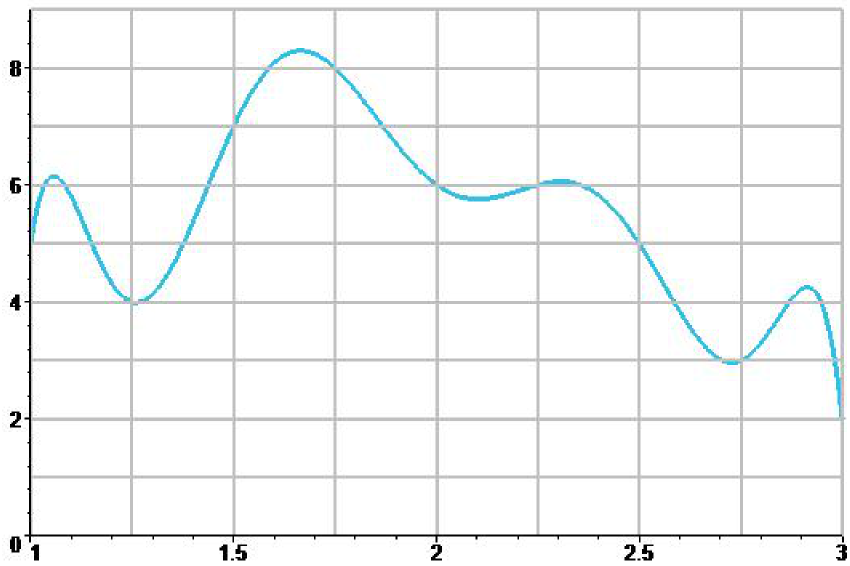
In accordance with the specific instructions given, use rectangles to approximate the area of the region that is under the graph of f and above the interval $[0, 6]$ of the x-axis. (For your interest, the exact area is 34.83571429, to eight decimal places.)

i) Use 6 equal width intervals. The height of each rectangle is the value of f at the left endpoint of the rectangle's base.

ii) Use 6 equal width intervals. The height of each rectangle is the value of f at the right endpoint of the rectangle's base.

iii) Use 3 equal width intervals. The height of each rectangle is the value of f at the midpoint of the rectangle's base.

2. The graph of a function f appears in the figure below.



In accordance with the specific instructions given, use rectangles to approximate the area of the region that is under the graph of f and above the interval $[1, 3]$ of the x-axis. (For your interest, the exact area is 11.05516755, to eight decimal places.)

i) Use 8 equal width intervals. The height of each rectangle is the value of f at the left endpoint of the rectangle's base.

ii) Use 8 equal width intervals. The height of each rectangle is the value of f at the right endpoint of the rectangle's base.

iii) Use 4 equal width intervals. The height of each rectangle is the value of f at the midpoint of the rectangle's base.

3. The speed $v(t)$ (in ft/s) of a car measured at 10s intervals is:

t	0	10	20	30	40	50	60
$v(t)$	54	57	61	66	65	63	62

In accordance with the specific instructions given, approximate the distance the car travelled during the minute in which the speed measurements were taken.

i) Use 3 equal width intervals and the values of $v(t)$ at the left endpoints.

ii) Use 3 equal width intervals and the values of $v(t)$ at the right endpoints.

iii) Use 3 equal width intervals and the values of $v(t)$ at midpoints.

iv) Use 6 equal width intervals and the values of $v(t)$ at the left endpoints.

v) Use 6 equal width intervals and the values of $v(t)$ at the right endpoints.

vi) Use 6 equal width intervals and the averages of the values of $v(t)$ at left and right endpoints.

Riemann Sums, Riemann Integrals, Definite Integrals, Midpoint Rule (Corresponds to Stewart 5.2)

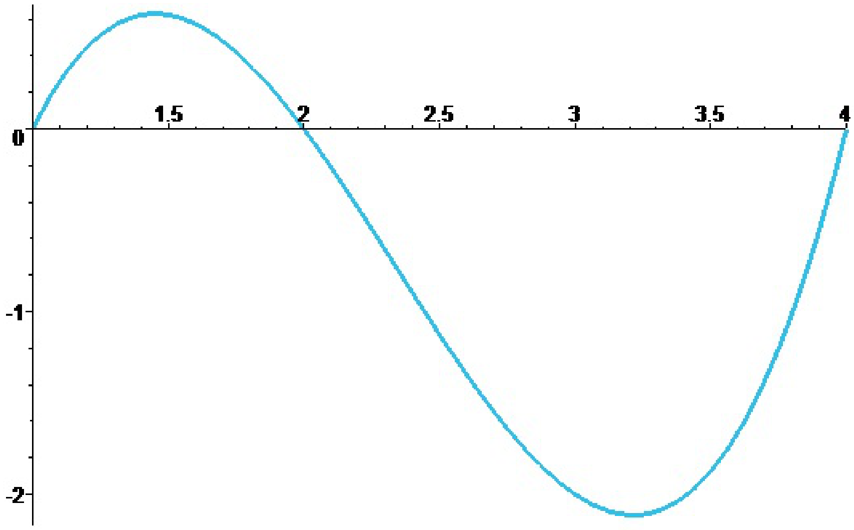
1. For $f(x) = \sqrt{x}$ on the interval $[3, 4]$, calculate the Riemann sums $\sum_j^3 f(s_j)\, \Delta x$ where the sample points s_1, s_2, s_3 are as follows:

   i) left endpoints
   ii) right points
   iii) midpoints
   iv) $s_1 = 47/15$ and the distance between consecutive sample points is Δx.

2. Estimate the Riemann integral $\int_1^4 \frac{1}{x}\, dx$ by the Riemann sum that arises from 4 subintervals and the choice of sample points s_1, s_2, s_3, s_4 in which each sample point s_j is a distance $\frac{1}{3}\Delta x$ from the left endpoint x_{j-1} of the subinterval $[x_{j-1}, x_j]$ that contains it.

3. Estimate the definite integral $\int_1^2 3\, x^2\, dx$ by using the Midpoint Rule with $n = 5$. (The exact value is 7.)

4. The plot of $f(x) = (x-1)(x-2)(x-4) = x^3 - 7\, x^2 + 14\, x - 8, \quad 1 \le x \le 4$ is shown in the figure below.



Notice that the verical axis shown is the line $x = 1$, not the y-axis.

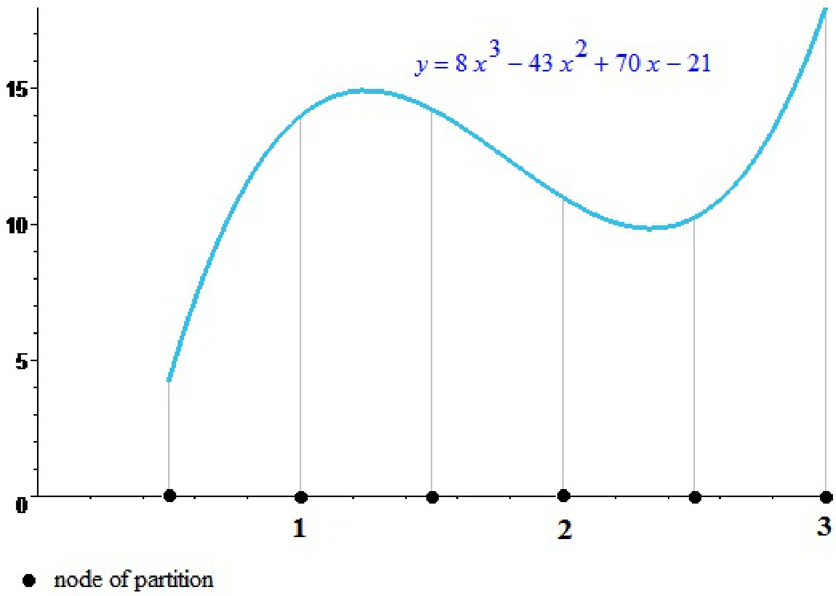
i) The area of the region under the graph of f and above the interval $[1, 2]$ of the x-axis is $5/12$. The area of the region above the graph of f and under the interval $[2, 4]$ of the x-axis is $8/3$. Use the given information to evaluate $\int_1^4 f(x)\,dx$.

ii) Estimate the definite integral $\int_1^2 f(x)\,dx$ using the Midpoint Rule with $n = 3$.

iii) Estimate the definite integral $\int_2^4 f(x)\,dx$ using the Midpoint Rule with $n = 6$.

iv) Estimate the definite integral $\int_1^4 f(x)\,dx$ using the Midpoint Rule with $n = 9$.

5. Let $f(x) = 8\,x^3 - 43\,x^2 + 70\,x - 21$. In this exercise, the definite integral $\displaystyle\int_{1/2}^{3} f(x)\,dx$ is to be estimated by two Riemann sums $\displaystyle\sum_{j=1}^{5} f(s_j)\,\Delta x$. The sample points used will be specified momentarily. As an aid to visualizing the locations of the sample points, the following graph of f over the interval of integration may be helpful. For each node x_j, a vertical line segment $x = x_j$ has been drawn from the x-axis to the graph of f.



i) For each j, the sample point s_j in the j^{th} subinterval satisfies $f(x) \leq f(s_j)$ for all x in the j^{th} subinterval. The resulting Riemann sum R^* is called an ***upper Riemann sum***. It is an overestimate of the definite integral it approximates.

ii) For each j, the sample point s_j in the j^{th} subinterval satisfies $f(s_j) \leq f(x)$ for all x in the j^{th} subinterval. The resulting Riemann sum R_* is called a ***lower Riemann sum***. It is an underestimate of the definite integral it approximates.

6. The interval $[1, 5]$ is partitioned into N subintervals of equal width. Let s_1 denote the midpoint of the first subinterval, s_2 the midpoint of the second subinterval, and so on, with s_N denoting the midpoint of the last subinterval. Consider the limit

$$\lim_{N\to\infty} \sum_{j=1}^{N} \frac{4\,s_j^2 - 1}{N}.$$

(i) Identify the sum inside the limit as a Riemann sum, and identify the limit of these Riemann sums as a Riemann integral.
(ii) After the first part of the Fundamental Theorem of Calculus has been covered (in the next section), evaluate the given limit.

7. Identify

$$\sum_{j=1}^{N} \left(1 + \frac{2\,j - 1}{N}\right)^2 \frac{6}{N}$$

as a Riemann sum for a certain function f over the interval $[1, 3]$ partitioned into N subintervals of equal width. After the first part of the Fundamental Theorem of Calculus has been covered (in the next section), evaluate the limit of these Riemann sums,

$$\lim_{N\to\infty} \sum_{j=1}^{N} \left(1 + \frac{2\,j - 1}{N}\right)^2 \frac{6}{N}.$$

The Fundamental Theorem of Calculus, Parts 1 and 2 (Corresponds to Stewart 5.3)

1. Calculate $\int_1^2 \left(6x^2 + 2x - 4\right)\, dx$.
2. Calculate $\int_4^9 \sqrt{x}\, dx$.
3. Calculate $\int_{\pi/3}^{\pi} \sin(\theta)\, d\theta$.
4. Calculate $\int_1^4 \frac{4+x}{\sqrt{x}}\, dx$.
5. Calculate $\int_0^2 (2-w)\sqrt{2w}\, dw$.
6. Calculate $\int_0^1 (\exp(t) + t^e)\, dt$. (Note: $\exp(t) = e^t$. The notation is typographically handy.)
7. Calculate $\int_1^4 \sqrt{\frac{4}{s}}\, ds$.
8. Calculate $\int_{\pi/6}^{\pi/3} \sec^2(\alpha)\, d\alpha$.
9. Calculate $\int_{\pi/4}^{\pi/3} 3\csc(t)\cot(t)\, dt$.
10. Calculate $\int_0^1 \left(1+x^2\right)^{-1}\, dx$.
11. Calculate $\int_3^6 \frac{2}{z}\, dz$.
12. Calculate $\int_{1/2}^{\sqrt{3}/2} \frac{1}{\sqrt{1-x^2}}\, dx$.

13. Calculate $\int_0^1 \frac{3x^2-1}{x^2+1}\,dx.$ (Rewrite the integrand as $A + \frac{B}{x^2+1}$ for certain constants A and B).

14. Calculate $\int_{\pi/6}^{\pi/3} \tan^2(\alpha)\,d\alpha.$ (Use a trigonometric identity and then Exercise 8.)

15. Calculate $\int_0^1 3^{2+t}\,dt.$

16. Calculate $F'(x)$ and $F'(2)$ for $F(x) = \int_0^x \sqrt{1+t^3}\,dt.$

17. Calculate $G'(s)$ and $G'(3)$ for $G(s) = \int_0^s \frac{19+u^4}{1+u^2}\,du.$

18. Calculate $D(F)(x)$ and $D(F)(\pi/6)$ for $F(x) = \int_x^{2\pi} \sqrt{17+16\sin(s)}\,ds.$
Note: $D(F)(x)$ is an alternative notation for $F'(x)$.

19. Calculate $D(F)(x)$ and $D(F)(3)$ for $F(x) = \int_1^{x^2} \sqrt{1+\sqrt{t}}\,dt.$

20. Calculate $G'(x)$ and $G'(2)$ for $G(x) = \int_{3x}^{2\pi} \sin\left(\frac{\pi}{t}\right)\,dt.$

21. Calculate $F'(x)$ and $F'(2)$ for $F(x) = \int_{x^2}^{x^3} \exp\left(\frac{16}{z}\right)\,dz.$

The So-Called Net Change Theorem (Corresponds to Stewart 5.4)

Note: This section of the text is primarily a repetition of material from previous sections. The exercises provided for this section are limited to the Net Change Theorem. In fact, even the "Net Change Theorem" is merely a reformulation of the part of the Fundamental Theorem of Calculus that concerns the evaluation of definite integrals.

1. For $t \geq 0$ measured in minutes, the rate at which water flows from a tank is $120 \exp(-t)$ liters per minute. By how many liters has the volume of water in the tank decreased between $t = 2$ and $t = 3$.

2. Suppose that the tank in the preceding exercise initially contained 120 liters of water. After how many minutes is the volume in the tank equal to 60 liters?

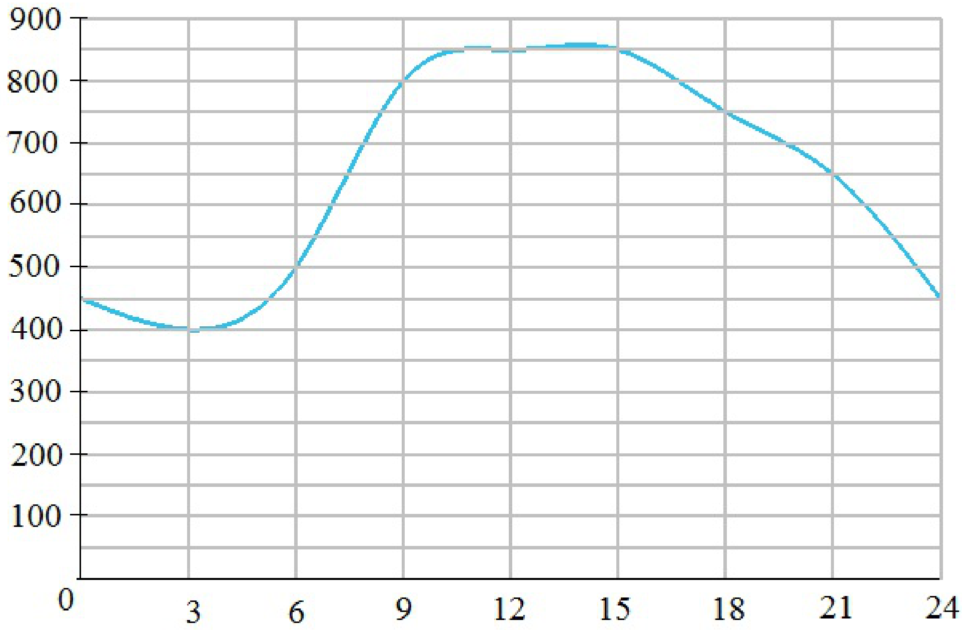
3. At time $t = 0$, Carl, the very hungry caterpillar, encountered a tasty leaf. For the first two minutes thereafter, Carl's rate of consumption was 60 mg/min. At time $t = 2\,\text{min}$, Carl, no longer so very hungry, consumed leafy matter at the rate of $240/t^2$ mg/min. How many mg of leaf did Carl eat in the first 4 min of his feast?

4. The rate of change of energy with respect to time is called *power*. The usual metric system unit of energy is the *Joule*, abbreviated by J. The usual metric system unit of power is the *watt*, abbreviated by w, which is a change of 1 J per second (i.e., $1\,\text{w} = 1\,\text{J/s}$). A megawatt, abbreviated MW, is one million watts (i.e., $1\,\text{MW} = 10^6\,\text{w} = 10^6\,\text{J/s}$. Notice that

$$1\,\text{MW} \times 1\,\text{hr} = 10^6\,\frac{\text{J}}{\text{s}} \times 3600\,\text{s} = 3.6 \times 10^9\,\text{J}.$$

   This quantity is called a *megawatt-hour*, and it is clear from the right side that it is a unit of energy.

   The figure below shows the graph of electric power $P(t)$ (measured in megawatts) as a function of time t (measured in hours) for one day in San Francisco.



(Confession: For your calculating pleasure, the author rounded each of $P(0), P(3), P(6), P(9), P(12), P(15), P(18), P(21), P(24)$ to the nearest multiple of 50. Hands up—who thought the figure is meant to depict a chapeau? An elephant eaten by a snake?) Approximate the electric energy E used in San Francisco that day. Use the *megawatt-hour* as the unit of measurement.

The Substitution Rule (Corresponds to Stewart 5.5)

1. Calculate $\int_0^7 \sqrt[3]{1+9\,x}\,dx$.
2. Calculate $\int_1^4 \sqrt{5-x}\,dx$.
3. Calculate $\int_1^4 x\,\sqrt{5-x}\,dx$.
4. Calculate $\int_1^4 x^2\,\sqrt{5-x}\,dx$.
5. Calculate $\int_0^2 x\,\sqrt{4-x^2}\,dx$.
6. Calculate $\int_0^2 x^3\,\sqrt{4-x^2}\,dx$.
7. Calculate $\int_0^1 \frac{2\,x-1}{x^2+1}\,dx$.
8. Calculate $\int_0^1 \frac{x^2+4\,x+2}{x^2+1}\,dx$.
9. Calculate $\int_0^1 x\,\sqrt[3]{8-7\,x}\,dx$.
10. Calculate $\int_{-1}^0 x^2\,\sqrt[3]{1+x}\,dx$.
11. Calculate $\int_1^2 \frac{\ln(x)}{x}\,dx$.
12. Calculate $\int \frac{1}{x\ln(x)}\,dx$.
13. Calculate $\int_1^e \frac{1}{x}\,\ln\left(\frac{1}{x}\right)\,dx$.

14. Calculate $\displaystyle\int_1^4 \frac{\exp\left(\sqrt{x}\right)}{\sqrt{x}}\,dx$.

15. Calculate $\displaystyle\int_0^{\pi/4} \tan(x)\sec^2(x)\,dx$.

16. Calculate $\displaystyle\int_0^{\pi/3} \tan^2(x)\sec^2(x)\,dx$.

Areas Between Curves (Corresponds to Stewart 6.1)

1. Let R be the region that is bounded above by the horizontal line segment $y = 1, 1 \leq x \leq e$, bounded below by $y = \ln(x), 1 \leq x \leq e$, and bounded on the left by the vertical line segment of $x = 1, 0 \leq y \leq 1$. Express the area of R in two ways: as an integral with respect to x, and as an integral with respect to y. Evaluate the latter. (The method for evaluating the former will be learned in the next chapter.)

2. Calculate the area of the region that is between arcs of $y = \sec(x)$ and $y = \cos(x)$ for $0 \leq x \leq \pi/3$.

3. Calculate the area of the region that is bounded above by a segment of the line $y = x + 1$ and below by an arc of the parabola $y = x^2 + 1$.

4. Calculate the area of the region that is between arcs of the curves $y = x^3 - x + 1$ and $y = 2\,x^2 - 1$ for $-1 \leq x \leq 2$.

5. Calculate the area of the region that is between arcs of the curves $y = \cos(x)$ and $y = \sin(x)$ for $\pi/4 \leq x \leq 5\pi/4$.

6. Let $\mathcal{R}$ be the region with boundary that is composed of a segment of the line $y = 3x$, a segment of the line $y = x + 2$, and an arc of the parabola $y = x^2$. Express the area of $\mathcal{R}$ as a sum of integrals of the form $\int_a^b h(x)\,dx$ and evaluate the area.

7. Let $\mathcal{R}$ be the region described in the preceding exercise. Express the area of $\mathcal{R}$ as a sum of integrals of the form $\int_c^d h(y)\,dy$ and evaluate the area.

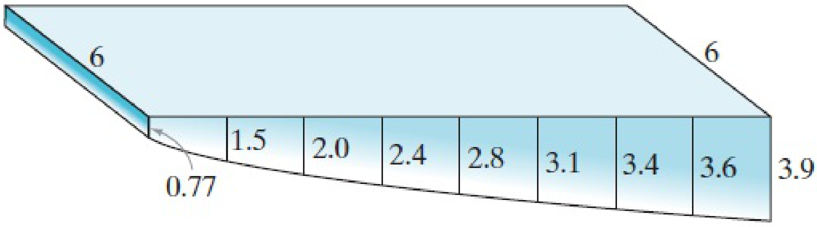
8. Calculate the area of the region that is bounded above by an arc of the graph of $y = \ln(x)$ and below by a segment of the line $y = (x - 1)/(e - 1)$.

9. Let $\mathcal{R}$ be the region that is bounded above by a segment of the horizontal line $y = 1$, below by a segment of the horizontal line $y = -1$, on the left by an arc of the graph of the parabola $x = y^2 - 2$, and on the right by an arc of the graph of $y = \ln(x)$. Express the area of $\mathcal{R}$ as an integral of the form $\int_c^d h(y)\,dy$ and evaluate the area.

10. Let $\mathcal{R}$ be the region described in the preceding exercise. Express the area of $\mathcal{R}$ as a sum of integrals of the form $\int_a^b h(x)\,dx$.

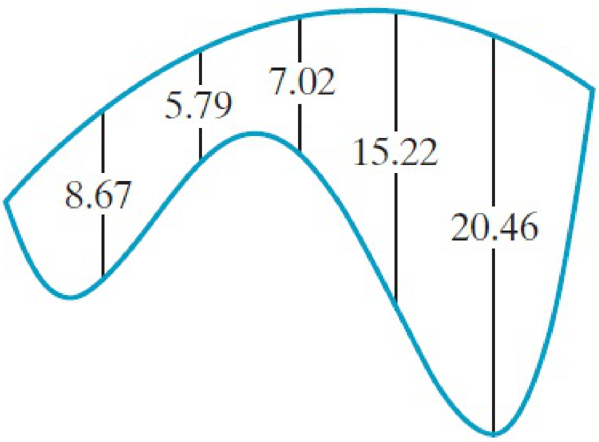
11. Let $\mathcal{R}$ be the region that is inside the circle $x^2 + y^2 = 25$ and above the line segment joining the point $(-4, -3)$ to the point $(3, 4)$. Express the area of $\mathcal{R}$ as a sum of integrals of the form $\int_a^b h(x)\,dx$.

12. Depths of a swimming pool are measured in meters every 2 m.



(i) Use midpoints to estmate the volume of the pool. Upon how many subintervals is the Riemann sum based? What length do the subintervals have? What is the volume estimate? (ii) Use the averages of the left and right endpoints to estimate the volume of the pool. Upon how many subintervals estimate based? What length do the subintervals have? What is the volume estimate?

13. From west to east, a city is 5.1 km wide. North-south measurements in km are given at 0.85 km intervals in the figure.



(i) Use midpoints to estmate the city's area. Upon how many subintervals is the Riemann sum based? What length do the subintervals have? What is the area estimate? (ii) Use left endpoints to estimate the city's area. Upon how many subintervals is the Riemann sum based? What length do the subintervals have? What is the area estimate?

Volumes by Disks and Washers (Corresponds to Stewart 6.2)

1. Let $\mathcal{R}$ be the region bounded above by $y = x^2, \quad 1 \le x \le 2$, below by the x-axis, and laterally by segments of the lines $x = 1$ and $x = 2$. What is the volume of the solid that results when $\mathcal{R}$ is rotated about the x-axis?

2. Let $\mathcal{R}$ be the region bounded above by $y = \sqrt{\cos(x)}, \quad -\pi/2 \le x \le \pi/2$ and below by the x-axis. What is the volume of the solid that results when $\mathcal{R}$ is rotated about the x-axis?

3. Let $\mathcal{R}$ be the region bounded above by $y = \sec(x), \quad 0 \le x \le \pi/4$, below by the x-axis, and laterally by segments of the lines $x = 0$ and $x = \pi/4$. What is the volume of the solid that results when $\mathcal{R}$ is rotated about the x-axis?

4. Let $\mathcal{R}$ be the region bounded above by $y = \sqrt{x}, \quad 4 \le x \le 9$, below by the x-axis, and laterally by segments of the lines $x = 4$ and $x = 9$. What is the volume of the solid that results when $\mathcal{R}$ is rotated about the x-axis?

5. Let $\mathcal{R}$ be the region bounded on the right by $y = \sqrt{x}$, above by $y = 4$, on the left by the y-axis, and below by $y = 3$. What is the volume of the solid that results when $\mathcal{R}$ is rotated about the y-axis?

6. Let $\mathcal{R}$ be the region bounded on the right by $y = x^4$, above by $y = 81$, on the left by the y-axis, and below by $y = 16$. What is the volume of the solid that results when $\mathcal{R}$ is rotated about the y-axis?

7. Let $\mathcal{R}$ be the region bounded on the right by $y = 1/x$, above by $y = 5$, on the left by the y-axis, and below by $y = 1/3$. What is the volume of the solid that results when $\mathcal{R}$ is rotated about the y-axis?

8. Let $\mathcal{R}$ be the region in the first quadrant bounded by $y = x^3$ and $y = 2x^2$. What is the volume of the solid that results when $\mathcal{R}$ is rotated about the x-axis?

9. Let $\mathcal{R}$ be the region bounded above by $y = \exp(x)$, below by $y = \sqrt{x+1}$, and on the right by a segment of the line $x = 1$. What is the volume of the solid that results when $\mathcal{R}$ is rotated about the x-axis?

10. Let $\mathcal{R}$ be the region bounded on the left by $y = 4 - x^2, \quad 0 \le x \le 2$, on the right by $y = (x-2)^2, \quad 2 \le x \le 4$, and on the top by a segment of the line $y = 4$. What is the volume of the solid that results when $\mathcal{R}$ is rotated about the y-axis?

11. Let $\mathcal{R}$ be the region bounded by $y = \sqrt{3\,x}$ and $y = x$. What is the volume of the solid that results when $\mathcal{R}$ is rotated about the y-axis?

12. Let $\mathcal{R}$ be the region bounded on the left by $y = \ln(x)$, below by the x-axis, and on the right by a segment of the line $y = 2 - x/e$. What is the volume of the solid that results when $\mathcal{R}$ is rotated about the y-axis?

13. Let $\mathcal{R}$ be the region bounded by $y = x^2, \quad 1 \le x \le 2$ and $y = 3x - 2, \quad 1 \le x \le 2$. Let V be the volume of the solid that results when $\mathcal{R}$ is rotated about the line $x = -2$. Express V by means of an integral.

14. Let $\mathcal{R}$ be the region bounded by $y = x^2, \quad 1 \le x \le 2$ and $y = 3x - 2, \quad 1 \le x \le 2$. Let V be the volume of the solid that results when $\mathcal{R}$ is rotated about the line $x = 2$. Express V by means of an integral.

15. Let $\mathcal{R}$ be the region bounded by $y = x^2, \quad 1 \le x \le 2$ and $y = 3x - 2, \quad 1 \le x \le 2$. Let V be the volume of the solid that results when $\mathcal{R}$ is rotated about the line $y = 1/2$. Express V by means of an integral.

16. Let $\mathcal{R}$ be the region bounded by $y = x^2, \quad 1 \le x \le 2$ and $y = 3x - 2, \quad 1 \le x \le 2$. Let V be the volume of the solid that results when $\mathcal{R}$ is rotated about the line $y = 6$. Express V by means of an integral.

17. Let $\mathcal{R}$ be the region bounded above by $y = 4 - x^2$ and below by $y = x + 2$. Let V be the volume of the solid that results when $\mathcal{R}$ is rotated about the line $y = 5$. Express V by means of an integral.

18. Let $\mathcal{R}$ be the region bounded on the left by $y = 4 - x^2$, on the right by $y = x + 2$, and above by $y = 3$. Let V be the volume of the solid that results when $\mathcal{R}$ is rotated about the line $x = 5$. Express V by means of an integral.

Volumes by Cylindrical Shells (Corresponds to Stewart 6.3)

1. Let $\mathcal{R}$ be the region bounded by $y = \sqrt{x} - x^2, \quad 0 \le x \le 1$ and the x-axis. What is the volume of the solid that results when $\mathcal{R}$ is rotated about the y-axis?

2. Let $\mathcal{R}$ be the region bounded above by $y = 4/\left(2 + x^2\right)$, by $x = 2$, on the right, by the y-axis on the left, and by the x-axis below. What is the volume of the solid that results when $\mathcal{R}$ is rotated about the y-axis?

3. Let $\mathcal{R}$ be the region bounded above by $y = 4$, bounded on the right by $y = x^2$, and bounded by the y-axis on the left. What is the volume of the solid that results when $\mathcal{R}$ is rotated about the x-axis?

4. Let $\mathcal{R}$ be the region bounded by $y = \sqrt{x}, \quad 1 \le x \le 4$, by $y = 2, \quad 4 \le x \le 5$, by $y = x - 3, \quad 4 \le x \le 5$, and by $y = 1, \quad 1 \le x \le 4$. What is the volume of the solid that results when $\mathcal{R}$ is rotated about the x-axis?

5. Let $\mathcal{R}$ be the region bounded by $y = \sin(x), \quad 0 \le x \le \pi$ and the x-axis. Let V be the volume of the solid that results when $\mathcal{R}$ is rotated about the line $x = 4$. Express V by means of an integral.

6. Let $\mathcal{R}$ be the region bounded by $y = \sin(x), \quad 0 \le x \le \pi/4$, by $y = \cos(x), \quad 0 \le x \le \pi/4$, and by the y-axis. Let V be the volume of the solid that results when $\mathcal{R}$ is rotated about the line $x = -\pi$. Express V by means of an integral.

7. Let $\mathcal{R}$ be the region bounded by $y = x^2, \quad 1 \le x \le 2$ and $y = 3\,x - 2, \quad 1 \le x \le 2$. Let V be the volume of the solid that results when $\mathcal{R}$ is rotated about the line $x = -2$. Express V by means of an integral.

8. Let $\mathcal{R}$ be the region bounded by $y = x^2, \quad 1 \le x \le 2$ and $y = 3\,x - 2, \quad 1 \le x \le 2$. Let V be the volume of the solid that results when $\mathcal{R}$ is rotated about the line $x = 2$. Express V by means of an integral.

9. Let $\mathcal{R}$ be the region bounded by $y = x^2, \quad 1 \le x \le 2$ and $y = 3\,x - 2, \quad 1 \le x \le 2$. Let V be the volume of the solid that results when $\mathcal{R}$ is rotated about the line $y = 1/2$. Express V by means of an integral.

10. Let $\mathcal{R}$ be the region bounded by $y = x^2, \quad 1 \le x \le 2$ and $y = 3\,x - 2, \quad 1 \le x \le 2$. Let V be the volume of the solid that results when $\mathcal{R}$ is rotated about the line $y = 6$. Express V by means of an integral.

Work (Corresponds to Stewart 6.4)

I like work; it fascinates me. I can sit and look at it for hours.
Jerome K. Jerome

1. If a spring with spring constant 8 pounds per inch is stretched 7 inches beyond its equilibrium position, then how much work is done?

2. If a spring with spring constant 20 newtons per centimeter is stretched 12 cm beyond its equilibrium position, then how much work is done?

3. If the amount of work in stretching a spring 0.02 meter beyond its equilibrium position is 8 J, then what force is necessary to maintain the spring at that position?

4. If 5 J work is done in extending a spring 0.2 m beyond its equilibrium position, then how much additional work is required to extend it a further 0.2 m?

5. A spring is stretched 10 cm beyond its equilibrium position. If the force required to maintain it in its stretched position is 240 N, then how much work was done?

6. A force of 280 pounds compresses a spring 4 inches from its natural length. How much work is done compressing it a further 4 inches?

7. If 40 joules of work are done in stretching a spring 8 cm beyond its natural length, then how much work is done stretching a further 4 cm?

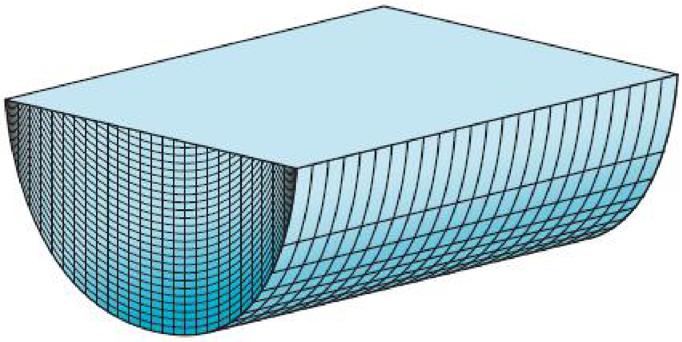
8. The work done in extending a spring at rest two inches beyond its equilibrium position is equal to 2/3 ft-lb. How much force is required to maintain the spring in that position?

9. A tank completely filled with water has square base of side 15 feet. It is 10 feet deep and is being pumped dry. The pump floats on the surface of the water and pumps the water to the top of the pool, at which point the water runs off. How much work is done in emptying the pool?

10. A swimming pool has rectangular base with side lengths of 6 m and 10 m. The pool is 3 m deep but not completely filled: the depth of the water it contains is 2 m. A pump floats on the surface of the water and pumps the water to the top of the pool, at which point the water runs off. How much work is done emptying the pool?

11. A swimming pool has rectangular base with side lengths of 16 and 30 feet. The pool is 10 feet deep but not completely filled: the depth of the water it contains is 8 feet. A pump floats on the surface of the water and pumps the water to the top of the pool, at which point the water runs off. How much work has the pump performed when half the water has been pumped?

12. A semi-cylindrical tank filled with water is 16 feet long. It has rectangular horizontal cross-sections and vertical cross-sections that are semicircular of radius 6 feet. See the figure below. A pump which floats on the surface of the water pumps the water to the top, at which point the water runs off. How much work is done in emptying the tank?



Semi-cylindrical tank

13. A filled reservoir is in the shape of a frustum of a cone. The radius of the circular base of the frustum is 50 ft. The radius of the circular top of the frustum is 25 ft. The depth of the frustum at the center is 50 ft. A pump that floats on the surface of the water pumps water to the top of the reservoir, at which point the water runs off. How much work is done emptying the reservoir? (A frustum of a cone is a cone with the top lopped off by a chop parallel to the base. King Henry VIII once had several cones, but he called out, "Off with their heads." Then he had frustums.)

14. Francis stands at the top of a tall building and pulls a chain up the side of the building. The chain is 50 feet long and weighs 3 pounds per linear foot. How much work does Francis do in pulling the chain to the top?

15. Francis stands at the top of a tall building and pulls a chain up the side of the building. The chain is 50 feet long and weighs 3 pounds per linear foot. How much work does Francis do in pulling up the first 30 feet of the chain to the top of the building?

16. A heavy uniform cable hangs over the side of a building that is 40 m high. From the roof, it is used to lift a 300 kg load from ground level to a height 24 meters above ground level. If the cable weighs 220 newtons per linear meter, then how much work is done?

17. A heavy uniform cable is used to lift a 300 pound load from ground level to the top of a 100 foot high building. If the cable weighs 20 pounds per linear foot then how much work is done?

18. With regard to the preceding exercise, how much work is done in lifting the load from a height 30 feet above ground level to a height 80 feet above ground level?

Average Value (Corresponds to Stewart 6.5)

In each of Exercises 1–7, a function f and an interval $I = [a, b]$ are given. Calculate f_{avg}, the average value of f over the interval I. Find a number c in the interval (a, b) for which $f(c)$ is equal to f_{avg}.

1. $f(x) = \sin(x)$	$I = [0, \pi/3]$
2. $f(x) = 1/x$	$I = [2, 4]$
3. $f(x) = \sqrt{x/4}$	$I = [0, 4]$
4. $f(x) = x^2 + 2x + 2;$	$I = [0, 3]$
5. $f(x) = 60/x^2$	$I = [1, 3]$
6. $f(x) = (x-1)^{1/2}$	$I = [1, 5]$
7. $f(x) = x\sqrt{169 - x^2}$	$I = [5, 12]$

Integration by Parts (Corresponds to Stewart 7.1)

1. Calculate $\int x \, \exp(x) \, dx$
2. Calculate $\int x \, \exp(-x) \, dx$
3. Calculate $\displaystyle\int (2x+5) \, e^{x/3} \, dx$
4. Calculate $\int x \sin(x) \; dx$
5. Calculate $\int (4x+2) \sin(2x) \; dx$
6. Calculate $\int 9x \cos(3x) \; dx$
7. Calculate $\int x \ln(4x) \; dx$
8. Calculate $\int 9 \, x^2 \ln(x) \; dx$
9. Calculate $\displaystyle\int \frac{\ln(x)}{x^2} \, dx$
10. Calculate $\int 16 \, x^3 \ln(2x) \; dx$
11. Calculate $\int_0^2 x e^{-x/2} \, dx$
12. Calculate $\int_0^\pi x \cos(x) \; dx$
13. Calculate $\int_0^{\pi/2} 4x \sin(x/3) \; dx$
14. Calculate $\int_1^e \ln(x) \; dx$
15. Calculate $\int_{1/3}^1 x \, \ln(3x) \; dx$
16. Calculate $\displaystyle\int_1^4 \frac{\ln(x)}{\sqrt{x}} \, dx$
17. Calculate $\int_0^1 x 3^x \, dx$
18. Integrate by parts successively to evaluate $\int x^2 \, e^{-x} \, dx$
19. Integrate by parts successively to evaluate $\int x^2 \cos(x) \; dx$
20. Integrate by parts successively to evaluate $\int 16 \, x^3 \, \ln^2(x) \; dx$

21. Calculate $\int_0^1 2\arctan(x)\ dx$
22. Calculate $\int_0^1 \arcsin(x)\ dx$

Suppose that a is a nonzero constant. In the next two exercises, use the reduction formula

$$\int x^p\, e^{ax}\, dx = \frac{1}{a}\, x^p\, e^{ax} - \frac{p}{a} \int x^{p-1}\, e^{ax}\, dx$$

to evaluate the given integral.

23. $\int_0^1 x^2\, e^{x/2}\, dx$
24. $\int_0^1 4x^2\, e^{-2x}\, dx$
25. Simplify the integrand before integrating by parts: $\int 2\ln\left(\sqrt{x}\right)\ dx$
26. Simplify the integrand before integrating by parts: $\int 4\,x\ \ln\left(x^5\right)\ dx$
27. Simplify the integrand before integrating by parts: $\int\ 16x^3\ \ln(1/x)\ dx$
28. Calculate $\int 9\sqrt{x}\ln(x)\ dx$
29. Calculate $\int x\sec^2(x)\ dx$
30. Evaluate $\int_{-2}^{1} x\,(x+3)^{-1/2}\, dx$. For the first step, integrate by parts with $u = x$.
31. Evaluate $\int_1^2 x\,(x-1)^{1/2} dx$. For the first step, integrate by parts with $u = x$.
32. Make an appropriate substitution before integrating by parts to evaluate $\int \sin\left(\sqrt{x}\right)\ dx$
33. Make an appropriate substitution before integrating by parts to evaluate $\int 2x^3\ \exp\left(x^2\right)\ dx$
34. Make an appropriate substitution before integrating by parts to evaluate $\int 3x^5\sin\left(x^3\right)\ dx$

Trigonometric Integrals (Corresponds to Stewart 7.2)

1. Evaluate $\displaystyle\int_0^{\pi/2} \sin^2(x/2)\ dx$.

2. Evaluate $\displaystyle\int_{\pi/6}^{\pi/4} 48\ \cos^2(2x)\ dx$

3. Evaluate $\displaystyle\int_0^{2\pi/3} 3\ \tan^2(x/2)\ dx$

4. Evaluate $\displaystyle\int \left(\cos^2(x) - \sin^2(x)\right)\ dx$.

5. Evaluate $\displaystyle\int_{\pi/6}^{\pi/4} 48\cos^3(2x+\pi)\ dx$.

6. Evaluate $\displaystyle\int 7\sin^7(x/10)\ dx$

7. Evaluate $\displaystyle\int 5\sin^2\left(\frac{x}{3}\right)\cos^3\left(\frac{x}{3}\right)\ dx$.

8. Evaluate $\displaystyle\int 7\sin^3\left(\frac{2x}{3}\right)\sqrt{\cos\left(\frac{2x}{3}\right)}\,dx$.

9. Evaluate $\displaystyle\int \frac{\cos^3(x)}{\sin(x)}\ dx$.

10. Evaluate $\displaystyle\int \frac{5\cos^3(2x)}{\sqrt{\sin(2x)}}\ dx$

11. Evaluate $\displaystyle\int_0^{\pi} \sin^3(x)\cos^4(x)\ dx$.

12. Evaluate $\displaystyle\int_0^{\pi/2} \sqrt{\sin(x)}\cos^3(x)\ dx$.

13. Evaluate $\displaystyle\int_0^{3\pi/2} 5\sin^2(x/3)\cos^7(x/3)\ dx$.

14. Use the reduction formula

$$\int \sin^n(u)\,du = -\frac{1}{n}\,\cos(u)\,\sin^{n-1}(u) + \frac{n-1}{n}\int \sin^{n-2}(u)\,du$$

to show that

$$\int \sin^4(u)\,du = -\frac{1}{4}\,\sin^3(u)\cos(u) - \frac{3}{8}\,\cos(u)\,\sin(u) + \frac{3}{8}\,u + C.$$

15. Use the reduction formula

$$\int \cos^n(u)\,du = \frac{1}{n}\,\cos^{n-1}(u)\,\sin(u) + \frac{n-1}{n}\int \cos^{n-2}(u)\,du$$

to show that

$$\int \cos^4(u)\,du = \frac{1}{4}\,\cos^3(u)\,\sin(u) + \frac{3}{8}\,\cos(u)\,\sin(u) + \frac{3}{8}\,u + C.$$

16. Use the preceding exercise to evaluate $\displaystyle\int_0^1 \cos^4(\pi x)\,dx$

17. Use either Exercise 14 or Exercise 15 to evaluate $\displaystyle\int_0^{\pi/4} \cos^2(t)\sin^2(t)\,dt$.

18. Use Exercise 15 two times (or three, if you prefer) to evaluate $\displaystyle\int_0^{\pi} \cos^6(x/2)\,dx$.

19. Use Exercise 14 two times (or three, if you prefer) to evaluate $\displaystyle\int_{\pi/6}^{\pi/3} 64\,\cos^2(x)\sin^4(x)\,dx$.

20. Use the identity $\tan^2(x) = \sec^2(x) - 1$ to evaluate $\displaystyle\int \tan^3(x)\,dx$.

21. Use the identity $\tan^2(x) = \sec^2(x) - 1$ and the integral formula

$$\int \sec^3(u)\,du = \frac{1}{2}\,\sec(u)\,\tan(u) + \frac{1}{2}\,\ln(\,|\,\sec(u) + \tan(u)\,|\,) + C$$

to evaluate $\displaystyle\int \tan^2(x)\,\sec(x)\,dx$.

22. Use the identity $\tan^2(x) = \sec^2(x) - 1$ to evaluate $\displaystyle\int \tan^3(x)\,\sec(x)\,dx$.

Trigonometric Substitution (Corresponds to Stewart 7.3)

1. Evaluate $\displaystyle\int_0^{\sqrt{3}/2} \frac{24\,x^2}{\sqrt{1-x^2}}\,dx$.

2. Evaluate $\displaystyle\int_0^3 2\,\sqrt{25-x^2}\,dx$.

3. Evaluate $\displaystyle\int \left(4-x^2\right)^{3/2}\,dx$.

4. Evaluate $\displaystyle\int \frac{2}{\sqrt{1+4x^2}}\,dx$.

5. Evaluate $\displaystyle\int \frac{2x^3}{9+x^2}\,dx$.

6. Evaluate $\displaystyle\int \frac{16\,x^2}{\sqrt{3+4\,x^2}}\,dx$. The integral formula

$$\int \sec^3(\theta)\,d\theta = \frac{1}{2}\,\sec(\theta)\,\tan(\theta) + \frac{1}{2}\,\ln\left(\,|\,\sec(\theta)+\tan(\theta)\,|\,\right) + C$$

might be useful.

7. Evaluate $\displaystyle\int \frac{36}{\left(9+4\,x^2\right)^2}\,dx$.

8. Evaluate $\displaystyle\int \frac{1}{\sqrt{x^2-25}}\,dx$.

9. Evaluate $\displaystyle\int \sqrt{9x^2-4}\,dx$.

10. Calculate $\displaystyle\int \frac{1}{\sqrt{4\,x-x^2}}\,dx$.

11. Calculate $\displaystyle\int_{-4}^{-2} \frac{1}{x^2+8\,x+20}\,dx$.

12. Convert $\displaystyle\int \frac{x^2}{\left(25\,x^2+20\,x+3\right)^{3/2}}\,dx$ to a trigonometric integral by making a substitution of the form $x = a\,t(\theta)+b$ where a and b are nonzero constants, possibly negative, and $t(\theta)$ is a trigonometric function. Do not evaluate the trigonometric integral that results.

Using Partial Fractions to Integrate Rational Functions (Corresponds to Stewart 7.4)

1. Evaluate $\displaystyle\int \frac{x-11}{x^2-x-2}\,dx$.
2. Evaluate $\displaystyle\int \frac{2}{x^3-x}\,dx$.
3. Evaluate $\displaystyle\int \frac{4\,x^2+3\,x+2}{x^3+x^2}\,dx$.
4. Evaluate $\displaystyle\int \frac{6\,x^2+17\,x-8}{(x+4)(x^2+4)}\,dx$.
5. Evaluate $\displaystyle\int \frac{x^3+5\,x^2-x+30}{(x^2+4)\,(x^2+9)}\,dx$.
6. Evaluate $\displaystyle\int \frac{2\,x^3+5\,x-1}{(x^2+1)^2}\,dx$.
7. Evaluate $\displaystyle\int \frac{5\,x^4+6\,x^3+30\,x^2+45}{x^2\,(x^2+3)^2}\,dx$.
8. Evaluate $\displaystyle\int \frac{6}{x^2+6\,x+18}\,dx$.

Numerical Integration (Corresponds to Stewart's Section 7.7)

1. Approximate $\int_1^5 (6/x)\,dx$ by the Trapezoidal Rule, the Midpoint Rule, and Simpson's Rule with $N = 2, 4, 6$.

2. In 2011, the distribution by quintile of pre-tax household income was: lowest quintile 5.13, second quintile 9.45, middle quintile 13.85, fourth quintile 20.33, highest quintile 51.24. Let $L(x)$ be the percentage of pre-tax household income earned by the lowest $x\%$ of households. Use the Trapezoidal Rule to alculate the coefficient of inequality (or Gini coefficient),

$$\gamma = \frac{1}{5000} \int_0^{100} \big(x - L(x)\big)\,dx.$$

3. Refer to the definition of the Gini coefficient γ that was given in the preceding problem. Approximate γ using the Trapezoidal Rule and Simpson's Rule with the following data:

x	10	20	30	40	50	60	70	80	90
$L(x)$	3	7	12	18	24	32	41	54	73

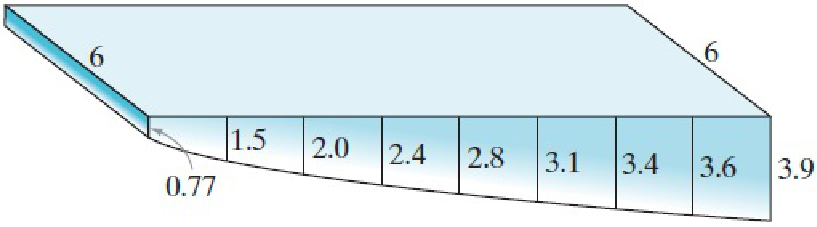
4. After the injection of 5 mg of dye into a heart, readings of the concentration $c(t)$ of ejected dye (in mg/L) were taken at 1.5 second intervals. The equation for cardiac output r is

$$\int_0^T r\,c(t)\,dt = 5$$

where T is a time that is large enough for essentially all of the injected dye to be ejected, but not so long that recirculation occurs. Use Simpson's Rule to estimate cardiac output r, stated in the customary units of L/min, based on the following tabulated readings:

t	0	1.5	3	4.5	6	7.5	9
$c(t)$	0	2.4	6.3	9.7	7.1	2.3	0

5. Depths of a swimming pool are measured in meters every 2 m.



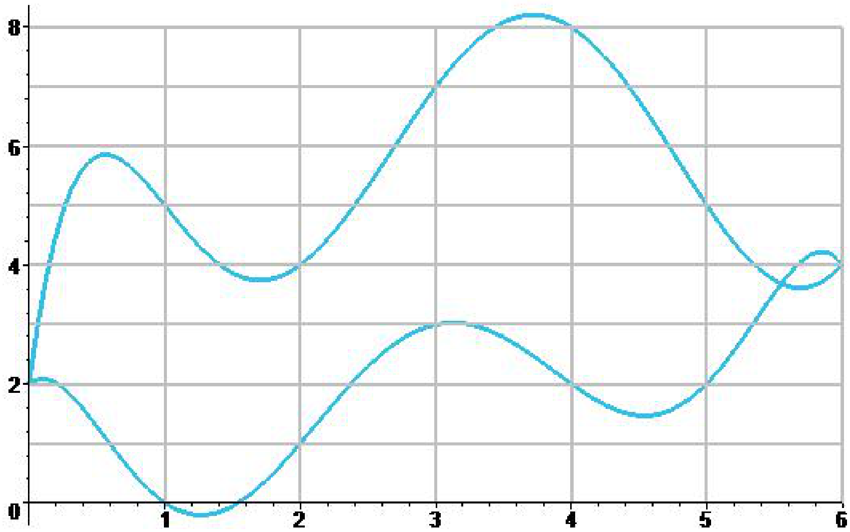
Use Simpson's Rule to estimate the volume of the pool.

6. Speed measurements (in meters per second) of a runner taken at half-second intervals during the first 5 seconds of a sprint are tabulated below.

Time	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
Speed	0	5.26	6.67	7.41	8.33	8.33	9.52	9.52	10.64	10.64	10.87

About how many meters did the athlete run during that 5 second interval? Use Simpson's Rule.

7. Use the Trapezoidal Rule to approximate the area between the curves shown in the figure below. Then find the Simpson Rule aaproximation. (The exact area is, to seven correct decimal places, 22.1710099.)



Improper Integrals (Corresponds to Stewart 7.8)

1. Evaluate $\int_2^4 (4-x)^{-0.9}\,dx$.
2. Evaluate $\int_1^2 (x-2)^{-1/5}\,dx$.
3. Evaluate $\int_{-1}^3 \frac{1}{\sqrt{x+1}}\,dx$.
4. Evaluate $\int_0^1 \frac{x}{(1-x^2)^{1/4}}\,dx$.
5. Evaluate $\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$.
6. Evaluate $\int_0^3 x^{-1/2}(1+x)\,dx$.
7. Evaluate $\int_1^e \frac{1}{x\cdot\ln^{1/3}(x)}\,dx$.
8. Evaluate $\int_0^1 \ln(x)\ dx$.
9. Evaluate $\int_1^{\sqrt{2}} \frac{1}{x\sqrt{x^2-1}}\,dx$.
10. Evaluate $\int_{-2}^4 (x+1)^{-2/3}\,dx$.
11. Evaluate $\int_{1/e}^e \frac{1}{x\ln^{2/7}(x)}\,dx$.
12. Evaluate $\int_{-2}^2 \frac{1}{\sqrt{4-x^2}}\,dx$.
13. Evaluate $\int_0^1 \ln^2(x)\ dx$.
14. Evaluate $\int_0^4 x^{-1/2}\ \ln(x)\,dx$.
15. Evaluate $\int_{-1}^1 \ln\left(1-x^2\right)\ dx$.

16. Evaluate $\int_4^\infty x^{-3/2}\,dx$.

17. Evaluate $\displaystyle\int_0^\infty \frac{1}{1+x^2}\,dx$.

18. Evaluate $\displaystyle\int_0^\infty \frac{x}{(1+x^2)^2}\,dx$.

19. Evaluate $\int_2^\infty e^{-x/2}\,dx$.

20. Evaluate $\displaystyle\int_2^\infty \frac{1}{x\,(x-1)}\,dx$.

21. Evaluate $\displaystyle\int_0^\infty \frac{e^x}{e^{2x}+1}\,dx$.

22. Evaluate $\displaystyle\int_0^\infty \frac{e^x}{(e^x+1)^3}\,dx$.

23. Evaluate $\displaystyle\int_0^\infty \left(\frac{x+1}{x^2+1}\right)^2\,dx$.

24. Evaluate $\int_1^\infty x\,e^{-3x^2}\,dx$.

25. Evaluate $\displaystyle\int_e^\infty \frac{1}{x\ln^2(x)}\,dx$.

26. Evaluate $\int_0^\infty (2/3)^x\,dx$.

27. Evaluate $\int_{-\infty}^{-2} x^{-3}\,dx$.

28. Evaluate $\displaystyle\int_{-\infty}^{-2} \frac{1}{(1+x)^{4/3}}\,dx$.

29. Evaluate $\displaystyle\int_{-\infty}^{1} \frac{x}{(1+x^2)^2}\,dx$.

30. Evaluate $\int_{-\infty}^{0} x\exp(x/2)\,dx$.

31. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{1}{4+x^2}\,dx$.

32. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^2+2x+10}\,dx$.

33. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{x+2}{(x^2+1)^2}dx$.

Arc Length (Corresponds to Stewart 8.1)

1. Calculate the arc length of the graph of $y = \frac{1}{8}\,x^2 - \ln(x),\ 1 \le x \le 2$. Hint: Express the terms inside the square root as one perfect square.

2. *Watch one, do one, create one.* Routine algebraic manipulation in the preceding exercise takes advantage of what may seem from casual observation to be a fortunate accident. In fact, the terms of the preceding exercise were meticulously determined in a math lab. Watch your spelling.

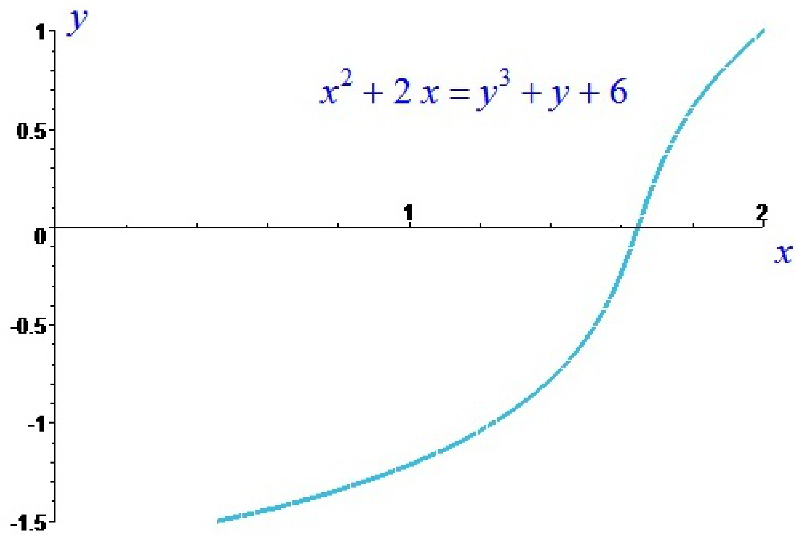
   i) Let p be a constant. For what value of p is the product $\left(\frac{d}{dx}x^3\right)\left(\frac{d}{dx}x^p\right)$ a constant?
   ii) For the remainder of this exercise, let p have the specific value determined in part (i). This value is necessarily negative. Let a and b be nonzero constants and set $f(x) = a\,x^3 + b\,x^p$. Notice that, because p is negative, $\frac{d}{dx}f(x)$ has the form $u - v$ and that $\left(\frac{d}{dx}f(x)\right)^2$ has the form $u^2 - 2uv + v^2$. Calculate $\left(\frac{d}{dx}f(x)\right)^2$, making sure to expand the square, an operation that yields three summands. What relationship must a and b satisfy in order that the cross term $-2uv$ in the expanded square be exactly -1/2?
   iii) For the remainder of this exercise, let $a = 1$ and let b have the value determined by the relationship of part (ii). Then the cross term $-2uv$ of part (ii) is equal to -1/2. It follows that

$$\begin{aligned} 1 + \left(\frac{d}{dx}f(x)\right)^2 &= 1 + (u - v)^2 \\ &= 1 + \left(u^2 - 2uv + v^2\right) \\ &= 1 + \left(u^2 - \frac{1}{2} + v^2\right) \\ &= \left(u^2 + \frac{1}{2} + v^2\right) \\ &= (u + v)^2. \end{aligned}$$

   Use these observations to calculate the arc length of the graph of $y = f(x)$, $1 \le x \le 2$.

3. Approximate the arc length of $y = x^2$, $1/2 \le x \le 1$ using Simpson's Rule with $n = 4$.

4. The plot of the equation $x^2 + 2\,x = y^3 + y + 6$ is shown in the viewing window $[0, 2] \times [-1.5, 1]$.



Express the length of this curve as an unevaluated integral.

5. Let a be a positive constant. The graph of $y = \frac{a}{2}(\exp(x/a) + \exp(-x/a))$ is called a catenary. Calculate its length between the two points with $\pm a$ for abscissas.

Surface Area (Corresponds to Stewart 8.2)

The integral formula

$$\int \sqrt{1+u^2}\,du = \frac{1}{2}\,u\,\sqrt{1+u^2} + \frac{1}{2}\,\ln\left(u + \sqrt{1+u^2}\right) + C$$

can be useful for evaluating a few surface areas.

1. The graph of $y = f(x) = x^3$, $0 \le x \le 1$ is rotated about the x-axis. What is the area of the resulting surface?
2. The graph of $y = f(x) = x^{1/3}$, $1 \le x \le 8$ is rotated about the y-axis. What is the area of the resulting surface?
3. The graph of $y = \exp(-x)$, $0 \le x \le \ln(2)$ is rotated about the x-axis. What is the area of the resulting surface?
4. The graph of $y = \sqrt{25 - x^2}$, $-3 \le x \le 4$ is rotated about the x-axis. What is the area of the resulting surface?
5. The graph of $y = x^2$, $0 \le x \le 2$ is rotated about the y-axis. By integrating with respect to y, find the area of the resulting surface.
6. Redo the previous exercise, this time integrating with respect to x.
7. The graph of $y = f(x) = x^2 - \dfrac{1}{8}\ln(x)$, $1 \le x \le 2$ is rotated about the x-axis. What is the area of the resulting surface? (The first step is to show that $1 + f'(x)^2$ is the square of a rational function $R(x)$. On expanding $f(x)\,R(x)$, you will obtain four summands, of which two are trivial to integrate and two can be handled by techniques learned earlier.)

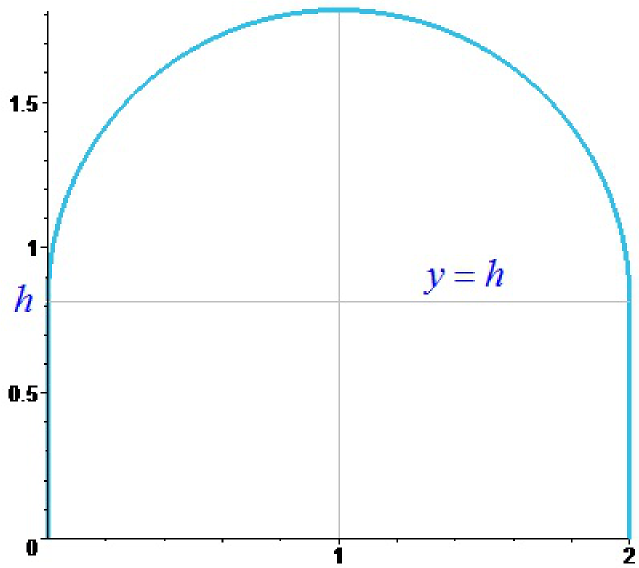
Center of Mass (Corresponds to Stewart 8.3)

1. Find the center of mass of the region in the first quadrant that is bounded above by $y = \ln(x)$ and on the right by $x = e$. (The region has uniform mass density.)

2. Find the center of mass of the region that is bounded above by $y = 2\,x$ and below by $y = x^2$. (The region has uniform mass density.)

3. Find the center of mass of the region that is bounded above by $y = 3\sqrt{x}$, $1 \le x \le 4$ and below by $y = x + 2$, $1 \le x \le 4$. Assume that the mass density of the region is constant.

4. The region shown in the figure is a rectangle surmounted by half a disk. The rectangle has height h, width 2, and vertical central axis $x = 1$. The region has uniform mass density and its center of mass lies on the horizontal diameter of the disk.



What is the value of h? (With some thought and simple geometry, you can do this exercise by calculating only one very easy integral.)

Separable Differential Equations (Corresponds to Stewart 9.3)

1. Solve the IVP
$$(1+x)\,\frac{dy}{dx} = y, \qquad y(3) = 8.$$
What is $y(5)$?

2. If
$$\frac{dy}{dx} = \frac{2x+1}{3y^2+2}, \qquad y(2) = 1,$$
then for what positive value of x is $y(x)$ equal to 0?

3. Solve the initial value problem $y'(x) = (1+2\,x)/y^2$, $y(1) = 2$.

4. Solve the initial value problem $y'(x) = 6\,\sqrt{x\,y}$, $y(1) = 9$.

5. Solve the initial value problem $\dfrac{dy}{dx} = y + \dfrac{1}{y}$, $y\,(0) = 2$

6. A mixing tank initially contains 100 kg of salt dissolved in 1000 L water. At time $t = 0$ an inlet valve and an outlet valve are opened. Salt-free water flows in at the rate of 20 L per minute and is instantly mixed. The salt solution flows out at the rate of 20 L per minute. At what time is the mass of salt in the tank equal to 10 kg?

7. A mixing tank initially contains 5 kg of salt dissolved in 200 L water. At time $t = 0$ an inlet valve and an outlet valve are opened. A salt water solution (brine) of constant concentration 2 kg salt per 10 L water flows in at the rate of 10 L per minute and is instantly mixed. The salt solution flows out of the mixing tank at the rate of 10 L per minute. What is the concentration of salt in the mixing tank one hour after the opening of the valves?

8. A mixing tank contains a brine solution containing 100 kg of salt dissolved in 2000 liters of water. Another brine solution, this one with a concentration of 1 kg salt per 5 liters, flows in at the rate of 10 liters per minute. The well-mixed solution flows out of the tank at the same rate. How long in minutes does it take for the salt concentration in the mixing tank to double? If the inflow and outflow are maintained indefinitely, about how much salt will be dissolved in the mixing tank's solution?

9. The mass $m(t)$ of a radioactive isotope decays according to the Law of Radioactive Decay: $D(m)(t) = -\lambda\, m(t)$ where λ is a positive constant, the *decay constant*, that depends on the particular substance. If m_0 is the mass of the radioactive substance at time $t = 0$, then what is the mass $m(t)$ at an arbitrary time t?

10. (Continuation of Exercise 9.) Let τ be the positive constant that is related to the decay constant by the equatuion $\lambda\tau = \ln(2)$. Show that $m(t+\tau) = m(t)/2$ for every value of t. The constant τ is called the *half-life* of the isotope. If you measure the mass of the substance and then return τ units of time later and measure again, your second measurement will be half the first.

11. (Continuation of Exercise 10.) Use the constant τ to express the Law of Radioactive Decay as exponential decay with base 2.

12. Eight grams of a certain radioactive isotope decay exponentially to six grams in 100 years. After how many more years will only four grams remain? (Use either Exercise 10 or Exercise 11.)

13. (Radiocarbon Dating) Two isotopes of carbon, ^{12}C, which is stable, and ^{14}C, which decays exponentially with a 5700 year half-life, are found in a known fixed ratio in living matter. After death, carbon is no longer metabolized and the amount $m(t)$ of ^{14}C decreases due to radioactive decay. In the analysis of a sample performed T years after death, the mass of ^{12}C, unchanged since death, can be used to determine the mass m_0 of ^{14}C that the sample had at the moment of death. The time T since death can then be calculated from the law of exponential decay and the measurement of $m(T)$. In 1991, radiocarbon dating established that fossils of dwarfed woolly mammoths, then recently found on Wrangel Island, which is located in the Arctic Ocean off the coast of Siberia, were 3700 years old. What percentage of m_0 was the measured amount of ^{14}C in these fossils? (Prior to the discovery of these fossils, mammoths were thought to have become extinct 10,000 years ago.)

14. In 1994, a parka-clad mummified body of a girl was found in a subterranean meat cellar near Barrow, Alaska. Radiocarbon analysis showed that the girl died around CE 1200. What percentage of m_0 was the amount of ^{14}C in the mummy?

15. In recent years, prehistoric cave art has been found in a number of new locations. At Chauvet, France, the amount of ^{14}C in some charcoal samples was determined to be $0.0879 \cdot m_0$. About how old are the cave drawings?

16. Let $y(t)$ and $v(t) = dy/dt$ be the height and velocity of a projectile that has been shot straight up from the surface of Earth with initial velocity v_0. Newton's Law of Gravitation states that

$$\frac{dv}{dt} = -\frac{gR^2}{(R+y)^2},$$

where R is the radius of the Earth and g is the acceleration due to gravity at Earth's surface. Using the Chain Rule to express dv/dt in terms of dv/dy, we have

$$\frac{dv}{dt} = \frac{dv}{dy} \cdot \frac{dy}{dt} = \frac{dv}{dy} \cdot v.$$

By equating this expression for dv/dt with the one given by Newton's Law of Gravitation, we obtain

$$v \cdot \frac{dv}{dy} = -\frac{g\,R^2}{(R+y)^2}.$$

Observe that we are now regarding v as a function of y rather than t. The initial value condition is that $v = v_0$ for $y = 0$. Find the unique solution $v(y)$ to this initial value problem. (You can leave it as a formula for v^2.) If $v_0 < \sqrt{2gR}$, then what is the maximum height attained by the projectile?

Models of Population Growth (Corresponds to Stewart 9.4)

1. Suppose that a population $P(t)$ grows according to the differential equation $D(P)(t) = k\,P(t)$ for some positive constant k. Let T be defined by the equation $k\,T = \ln(2)$. Show that, the equation $P(t+T) = 2\,P(t)$ holds for every value of t. The significance of this equation is that if we take a census and then take a second census T units of time later, the second count is double the first. The constant T is called the *doubling time*.

2. (Continuation of Exercise 1) Show that if $P(t)$ is a population satisfying the differential equation of Exercise 1, and if T is the doubling time for the population, then $P(t) = P(0)\,2^{t/T}$.

3. In a yeast nutrient broth, the population $P(t)$ of a colony of the bacterium Escherichia Coli (E. Coli) grows according to the differential equation $D(P)(t) = k\,P(t)$ for some positive constant k. At 10:00 AM there were 6000 bacterial colonists and at 10:20 AM there were 12 000. How many will there be at 12:30 PM?

4. Day-care centers are germ centers. Little kids coughing through uncovered mouths, little kids sneezing on other little kids, little kids drooling on playmates, little kids touching everything with their unwashed hands. In any given day-care center, at any given time, an infection is spreading among the children doing time there. Let $P(t)$ be the fraction (between 0 and 1) of infected children. It is reasonable that the rate $D(P)(t)$ at which the infection is spreading is proportional to $P(t)$: if one infected child can infect k children, then two infected children can infect $2k$ children. So it may seem reasonable to model the outbreak by the exponential growth differential equation $D(P)(t) = k\,P(t)$. Not so fast! As $P(t)$ grows and becomes closer to 1, there are fewer uninfected children remaining who are capable of becoming infected. The decreasing pool of potential infectees will moderate the rate at which the infection spreads. We therefore emend our model to $D(P)(t) = k\,P(t)\big(1 - P(t)\big)$: the rate at which the infection spreads is jointly proportional to the fraction $P(t)$ of children capable of passing on the infection and the fraction $1 - P(t)$ of children capable of becoming infected. Suppose that P_0 infected children arrive at the day care center Monday morning. What is $P(t)$ in terms of t, k, and P_0?

5. (Continuation) Suppose that 100 children attend a day-care center. Suppose that at 9 AM Monday morning, only one child is infected. Suppose that at 9 AM Tuesday morning three children are infected. By what time will the infection have spread to half the day-care population?

6. Suppose that a and b are positive constants. If a population $P(t)$ satisfies the differential equation $D(P)(t) = a\,P - b\,P^2$, then what is the carrying capacity of the population?

Linear Differential Equations (Corresponds to Stewart 9.5)

1. Solve the initial value problem
$$\frac{dy}{dx} + 2y = 1, \quad y(0) = 2$$
2. Solve the initial value problem
$$\frac{dy}{dx} = 1 + \frac{y}{2}, \quad y(0) = 3$$
3. Solve the initial value problem
$$\frac{dy}{dx} = 15x^2 - \frac{2xy}{1+x^2}, \quad y(1) = 6$$
4. Solve the initial value problem
$$\frac{dy}{dx} = 1 - \frac{y}{x}, \quad y(4) = 5$$
5. Solve the initial value problem
$$\frac{dy}{dx} + \frac{2}{x}y - 7\sqrt{x} = 0, \quad y(1) = 9$$
6. Solve the initial value problem
$$\frac{dy}{dx} + xy = x, \quad y(0) = 4$$
7. Solve the initial value problem
$$\frac{dy}{dx} - 2\,x\,y = 4\,x, \quad y(0) = 1$$
8. Solve the initial value problem
$$\frac{dy}{dx} = 2\,x\,(2 + y), \quad y(0) = 1$$
9. Solve the initial value problem
$$\frac{dy}{dx} - 3y = e^{3x}, \quad y(2) = 0$$
10. Solve the initial value problem
$$\frac{dy}{dx} = \csc(x) + y\,\cot(x)\,, y(\pi/2) = 4$$

11. Solve the initial value problem

$$\frac{dy}{dx} + \frac{y}{x \ln(x)} = 1, \quad y(e) = 2\,e$$

12. A large capacity mixing tank is partially filled with 120 gallons of water in which 80 lbs of a solute is dissolved. Two valves are simultaneously opened. Water containing the same solute as in the mixing tank flows into the tank at the rate of 24 gallons per minute. The concentration of the solute in the water flowing in is 1 lb per gallon. The second open valve allows the thoroughly mixed solution to flow out of the tank at the rate of 16 gallons per minute. Find $y(t)$, the weight of the solute in the mixing tank t minutes after the valves have been opened (and during the time period during which the tank is not filled to capacity). How many pounds of the solute are in the mixing tank one-quarter of an hour after the opening of the valves?

Sequences (Corresponds to Stewart 11.1)

1. What are the first five terms of the sequence $a_n = \dfrac{3^n}{2\,n+1}$, $0 \le n < \infty$?

2. Let $f(x) = x^3 - 4$. Consider the recursive sequence $\{x_n\}_{n=1}^{\infty}$ defined by

$$x_1 = 1, \qquad x_{n+1} = x_n - \frac{f\,(x_n)}{f'\,(x_n)} \quad (1 \le n < \infty)$$

Calculate the first four terms of this sequence exactly as rational numbers. Calculate the fifth term to five decimal places. (This sequence is the Newton-Raphson approximation of the root of $f(x) = 0$. Compare x_5 with $4^{1/3}$.)

3. Find the limit of the sequence $a_n = \dfrac{3 + 5\,n^2}{5 + 3\,n^2}$.

4. Find the limit of the sequence $a_n = \sqrt{\dfrac{n^5 + 9\,n^6}{1 + 4\,n^6}}$.

5. Find the limit of the sequence $a_n = 3^n \pi^{-n}$.

6. Find the limit of the sequence $a_n = \exp(1/\sqrt{n}) + \exp(-1/\sqrt{n})$.

7. Find the limit of the sequence $a_n = \dfrac{(2\,n-1)!\,n^2}{(2\,n+1)!}$.

8. Find the limit of the sequence $a_n = \dfrac{\sin(n)}{n}$. Two intuitive results may help. First, if $|a_n| \to 0$, then $a_n \to 0$. Also, if $a_n \le b_n \le c_n$ for all n and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} a_n = L$. This is sometimes called The Pinching Theorem, sometimes The Squeeze Theorem.

9. Find the limit of the sequence $a_n = n\,\sin(1/n)$.

10. Find the limit of the sequence $a_n = e^{2\,n/(n+3)}$.

11. Find the limit of the sequence $a_n = \dfrac{n^2}{\sqrt{n^5 + 1}}$.

12. Find the limit of the sequence $a_n = \dfrac{\ln(n)}{\sqrt{n}}$.

13. Find the limit of the sequence $a_n = \dfrac{\ln(n)}{\ln(3\,n)}$.

14. Find the limit of the sequence $a_n = \dfrac{\ln(n)}{\ln(n^3)}$.

15. Find the limit of the sequence $a_n = \ln(n+10) - \ln(n+1)$.

16. Find the limit of the sequence $a_n = n^3 \cdot (1.1)^{-n}$.

17. Calculate $\lim\limits_{n\to\infty} \dfrac{2^n + n^5}{2^n + n^2}$.

18. Calculate $\lim\limits_{n\to\infty} \dfrac{3 \times 7^n + 6 \times 11^n}{5^n + 2 \times 11^n}$.

19. Find the limit of the sequence $a_n = \left(1 - \dfrac{1}{n}\right)^n$.

20. Find the limit of the sequence $a_n = \left(\sqrt{1+n^2}\right)^{1/n}$.

21. It can be shown that the recursive sequence

$$a_1 = \sqrt{2}, \qquad a_{n+1} = \sqrt{2\,a_n} \quad (1 \le n < \infty)$$

has a limit L. What is L?

22. It can be shown that the recursive sequence

$$a_1 = 1, \qquad a_{n+1} = 1 + \frac{1}{a_n + 1} \quad (1 \le n < \infty)$$

converges. To what number L does the sequence converge?

Series (Corresponds to Stewart 11.2)

1. Consider the infinite series $\displaystyle\sum_{n=2}^{\infty} \ln\left(\frac{n}{n+2}\right)$.
   (i) Calculate the first five terms $s_1, s_2, s_3, s_4, s_5$of the sequence of partial sums. (Use a property of the logarithm to simplify these terms.)
   (ii) For what integer M is $\displaystyle\sum_{n=2}^{M} \ln\left(\frac{n}{n+2}\right)$ equal to the N^{th} partial sum s_N of the given series?
   (iii) Find a formula for s_N that holds for $N > 1$. (No formal proof needed. Your formula should fit the pattern established in part (i).) (iv) Does $\displaystyle\sum_{n=2}^{\infty} \ln\left(\frac{n}{n+2}\right)$ converge or diverge?
2. For what series $\displaystyle\sum_{n=1}^{\infty} a_n$ is the sequence of partial sums equal to $\displaystyle\left\{\frac{2\,N+1}{N+2}\right\}_{N=1}^{\infty}$? (It might help to first figure out the terms $a_1, a_2, a_3, \ldots$ if you do not immediately see how to find a formula for the general term a_n.) Show that the series you have found is convergent and determine its value.
3. Evaluate $5 - 2.5 + 1.25 - 0.625 + 0.3125 - 0.15625 + \cdots$.
4. Evaluate $\displaystyle\sum_{n=2}^{\infty} \frac{(-2)^{n-1}}{3^n}$.
5. Evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{3^{n+1}}{4^{n-1}}$.
6. Evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{2^{2n+1}}{5^n}$.
7. Use partial fractions to find the sequence of partial sums of the series $\displaystyle\sum_{n=2}^{\infty} \frac{2}{n^2-1}$. Determine the value of the series.
8. The "Test for Divergence" states that if $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges. (We will encounter other tests that can lead to the conclusion "divergent," but the other tests have other names.) To which of the following series can the Test for Divergence be applied.
   a) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$
   b) $\sum_{n=2}^{\infty} \frac{\ln(n)}{n}$
   c) $\sum_{n=1}^{\infty} \frac{1}{n^{1/n}}$
   d) $\sum_{n=1}^{\infty} \frac{1}{\arctan(n)}$
   e) $\sum_{n=1}^{\infty} \left(\frac{1+2/n}{7}\right)^n$

9. In each sentence, use either the word “may” or the word “must” to fill in the blank so that the completed sentence is correct.
(i) A series with summands tending to zero ___ converge.
(ii) A series that converges ___ have summands that tend to zero.
(iii) If a series diverges, then the Test for Divergence ___ succeed in proving the divergence.
(iv) If the partial sums of an infinite series are bounded, then the series ___ converge.
(v) If a series diverges, then its terms ___ diverge.
(vi) If the partial sums of an infinite series diverge, then the series ___ diverge.

The Integral Test (Corresponds to Stewart 11.3)

1. This exercise begins with a bit of (easy) theory. Let K be any fixed integer, which will assume the role of an initial value for an index of summation of a series $S_K = \sum_{n=K}^{\infty} a_n$. Commonly used values of K are 0 and 1, but K is not limited to these two choices. Let M be an integer *greater* than K. It too will be used as the initial value of an index of summation of a series: to be specific, the series $S_M = \sum_{n=M}^{\infty} a_n$ obtained by deleting the terms $a_K, a_{K+1}, a_{K+2}, \ldots, a_{M-1}$ from the series

$$S_K = a_K + a_{K+1} + a_{K+2} + \cdots + a_{M-1} + a_M + a_{M+1} + \cdots .$$

Then we may write S_K as the sum of the finite series

$$F = \sum_{n=K}^{M-2} a_n = a_K + a_{K+1} + a_{K+2} + \cdots + a_{M-1}$$

and the infinite series S_M: Thus, $S_K = F + S_M$. In this context, S_M is said to be a *tail* of the series S_K: a tail of a series is the infinite series that results by dropping a finite number, any finite number no matter how large, of terms from the beginning of an infinite series. A finite series of real numbers always converges—you just add up all the numbers and the process, being finite, comes to a happy end. It follows that, if a tail S_M of a series S_K converges, then so too does the full series $S_K = F + S_M$. Here is the point of this observation: there are series for which a convergence test might be conclusive by applying it to one of its tails even if the test cannot be applied to the full series. This can happen if the hypotheses required for the application of a testhold for the tail but not for the entire series. Consider, for example, the series

$$S_1 = \sum_{n=1}^{\infty} \frac{n^2}{n^3 + 32}.$$

The Integral test cannot be applied to S_1. To understand why, salculate the first three terms in the sum. What hypothesis of the Integral Test is violated? Nevertheless, the Integral Test can be applied to tails

$$S_M = \sum_{n=1}^{\infty} \frac{n^2}{n^3 + 32}$$

of S_1 beginning with what value of M?

2. Does $\sum_{n=1}^{\infty} \frac{n^2}{n^3 + 32}$ converge, or does it diverge? Use the Integral Test.

3. Does $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{(n^3+32)^2}$ converge, or does it diverge? Use the Integral Test.

4. Suppose that r is a positive constant that is less than 1. Use the Integral Test to show that $\displaystyle\sum_{n=1}^{\infty} n\, r^n$ converges.

5. The value of $S = \sum_{n=1}^{\infty} 1/n^2$ is known to be exactly $\pi^2/6$, or 1.644934068 to nine decimal places. The partial sum s_{10} is 1.549767731 to nine decimal places. Calculate the improper integrals $\int_{10}^{\infty} (1/x^2)\, dx$ and $\int_{11}^{\infty} (1/x^2)\, dx$. Use these two calculations together with s_{10} to find the lower estimate ℓ and the upper estimate m of S that arises from the Integral Test. Estimate S by using the midpoint of the interval $[\ell, m]$ containing S.

6. The exact value of $S = \sum_{n=1}^{\infty}(1/n^3)$ is not known. It is not that much work to calculate $\sum_{n=1}^{5}(1/n^3) = 256103/216000 = 1.185662037$ to nine decimal places. Use this sum and the estimates of the Integral Test to find an interval containing the exact value of S. If the midpoint of this interval is used to estimate S, how big can the error be?

Series with Positive Terms (Corresponds to Stewart 11.4 and 11.3)

In each of Exercises 1–12, use the Comparison Test to show that the given series converges. State the series that you use for comparison and the reason for its convergence.

1. $\sum_{n=1}^{\infty} \frac{n}{n^3+1}$

2. $\sum_{n=1}^{\infty} \frac{5n-4}{n^{9/4}}$

3. $\sum_{n=1}^{\infty} \frac{1}{\left(n+\sqrt{2}\right)^2}$

4. $\sum_{n=1}^{\infty} \frac{2+\sin(n)}{n^4}$

5. $\sum_{n=1}^{\infty} \frac{2n-1}{ne^n}$

6. $\sum_{n=1}^{\infty} \frac{n+2}{2n^{5/2}+3}$

7. $\sum_{n=1}^{\infty} \frac{n^2+2n+10}{2n^4}$

8. $\sum_{n=1}^{\infty} \frac{2^n}{n3^n}$

9. $\sum_{n=1}^{\infty} (1/3)^{n^2}$

10. $\sum_{n=1}^{\infty} \frac{1}{2n\sqrt{n}-1}$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{n^{4.01}+1}$

12. $\displaystyle\sum_{n=1}^{\infty} \frac{2}{2^n+1/2^n}$

In each of Exercises 13–20, use the Comparison Test to show that the given series diverges. State the series that you use for comparison and the reason for its divergence.

13. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$

14. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$

15. $\displaystyle\sum_{n=1}^{\infty} \frac{1+\ln(n)}{3n-2}$

16. $\displaystyle\sum_{n=1}^{\infty} \frac{2n+5}{n^2+1}$

17. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{10+n^2}}$

18. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n+1}{3^n+2^n}$

19. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n+n}{\sqrt{n}3^n+1}$

20. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n+1}{2^n+n^2}$

In Exercises 21–36 use the Limit Comparison Test to determine whether the given series $\sum a_n$ converges or diverges.

21. $\displaystyle\sum_{n=1}^{\infty} \frac{3n}{2n^2-1}$

22. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{n^2}$

23. $\displaystyle\sum_{n=1}^{\infty} \frac{(n-1)^2}{(n+1)^4}$

24. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+2}{(n^2+1)^2}$

25. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{2n^2-n}$

26. $\displaystyle\sum_{n=1}^{\infty} \frac{3}{2^n+3}$

27. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n+11}{3^n-1}$

28. $\displaystyle\sum_{n=1}^{\infty} \frac{\arctan(n)}{n^2}$

29. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{1+n^2}}{(1+n^8)^{1/4}}$

30. $\displaystyle\sum_{n=1}^{\infty} \frac{n^{1/3}}{\sqrt{1+n^{5/2}}}$

31. $\displaystyle\sum_{n=1}^{\infty} \sin\left(1/n^2\right)$

(Hint: Use your knowledge of $\displaystyle\lim_{x\to 0} \frac{\sin(x)}{x}$.)

32. $\displaystyle\sum_{n=1}^{\infty} \frac{n+\ln(n)}{n^3}$

33. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+\ln(n)}$

34. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n+n^3}{3^n+n^2}$

35. $\displaystyle\sum_{n=1}^{\infty} \frac{7^n+n^{10}}{10^n}$

36. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n+5}{3n+7}\right)^n$

In Exercises 37–48 use the Integral Test to determine whether the given series converges or diverges. (You may notice that, in several instances, a comparison test would provide a more efficient method of determining the behavior of the series.)

37. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

38. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+4}$

39. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$

40. $\displaystyle\sum_{n=1}^{\infty} n^2 \exp\left(-n^3\right)$

41. $\displaystyle\sum_{n=2}^{\infty} \frac{2n^2}{n^3+4}$

42. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$

43. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln^2(n)}$

44. $\displaystyle\sum_{n=1}^{\infty} \frac{\exp(n)}{(1+\exp(n))^2}$

45. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{10^n}$

46. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$

47. $\displaystyle\sum_{n=2}^{\infty} n \exp(-n/2)$

48. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$

Alternating Series (Corresponds to Stewart Section 11.5)

In each of Exercises 1–6 a convergent alternating series $S = \sum_{n=1}^{\infty} (-1)^{n+1}\, a_n$ is given. It is easy to see that, because the terms a_n decrease to 0, the Alternating Series Test can be applied to each of these series, establishing convergence. Find a value of N such that the partial sum $S_N = \sum_{n=1}^{N} (-1)^{n+1}\, a_n$ approximates the given infinite series to within 0.01. That is, find N so that $|S - S_N| \leq 0.01$ holds. The point of these exercises is to obtain the estimate of S *efficiently*. If a particular value of N works, then any larger value of N also works, but the calculation of the larger partial sum involves more computation. That extra computation is not needed for the desired approximation. Therefore, find the smallest value of N that the Alternating Series Test guarantees will do the job.

1. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{n}}$

2. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^3}$

3. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{4}{2n-1}$

4. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n!}$

5. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2+15n}$

6. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n\,\ln(n+1)}$

In each of Exercises 7–10, an alternating series $\sum_{n=1}^{\infty} (-1)^{n}\, a_n$ is given. For each series, $\lim_{n\to\infty} a_n = 0$. However, $\{a_n\}_{n=1}^{n=\infty}$ is not a decreasing sequence. Nevertheless, in each case, there is an integer M such that $\{a_n\}_{n=M}^{n=\infty}$ *is* a decreasing sequence. Therefore, the Alternating Series Test may be applied to the tail $\sum_{n=M}^{\infty} (-1)^{n}\, a_n$, establishing that the given series converges. Find the smallest value of M,

7. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2-20n+101}$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4 - 32n^2 + 400}$

9. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^5}{2^n}$

10. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\ln\left(n^3\right)}{\sqrt{n}}$

In each of Exercises 11 and 12 calculate the given alternating series to three decimal places. (That means that the truncation error should not exceed 5×10^{-4}, or 0.0005.)

11. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!}$

12. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{2 - 1/n}{10^n}\right)$

The Ratio Test and the Root Test (Corresponds to Stewart Section 11.6)

In each of Exercises 1–8 use the Ratio Test to determine the convergence or divergence of the given series.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{e^n}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^3}$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{10^n}{n!}$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{n^{100}}{n!}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^5 \cdot 7^n}$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n + n}{2^n + n^3}$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{\sqrt{n!}}$

In each of Exercises 9–18, verify that the Ratio Test yields no information about the convergence of the given series. Use other methods to determine whether the series converges absolutely, converges conditionally, or diverges.

9. $\displaystyle\sum_{n=2}^{\infty} (-1)^n \frac{\ln(n)}{n}$

10. $\sum_{n=2}^{\infty} (-1)^n \frac{1}{\ln(n^2)}$

11. $\sum_{n=1}^{\infty} (-1)^n \frac{\ln(n)}{n^2}$

12. $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{n+1}{n^3+1}$

13. $\sum_{n=1}^{\infty} (-1)^n \cdot \left(1 + \frac{1}{n}\right)$

14. $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+4}$

15. $\sum_{n=1}^{\infty} (-1)^n \frac{e^n + \ln(n)}{e^n \cdot n^2}$

16. $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{n!}{n! + 2^n}$

17. $\sum_{n=1}^{\infty} (-1)^n \arctan(n)$

18. $\sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{1}{n}\right)$

In each of Exercises 19–25, use the Root Test to determine the convergence or divergence of the given series.

19. $\sum_{n=1}^{\infty} n^{-n/2}$

20. $\sum_{n=1}^{\infty} \frac{2^{3n}}{3^{2n}}$

21. $\sum_{n=1}^{\infty} \frac{n}{2^n}$

22. $\sum_{n=1}^{\infty} \frac{10^n}{n^{10}}$

23. $\sum_{n=2}^{\infty} \frac{n^7}{\ln^n(n)}$

24. $\sum_{n=1}^{\infty} \left(\frac{\ln(n+4)}{\ln(n^2+4)}\right)^n$

25. $\sum_{n=1}^{\infty} \left(\frac{37}{n}\right)^n$

In each of Exercises 26–40 determine whether the series converges absolutely, converges conditionally, or diverges. You may use any test.

26. $\sum_{n=1}^{\infty} (-1)^n \frac{(2n)!}{(3n)!}$

27. $\sum_{n=1}^{\infty} (-1)^n \frac{(n!)^2}{(2n)!}$

28. $sum_{n=1}^{\infty} (-1)^n \frac{n+3^n}{n^3+2^n}$

29. $\sum_{n=1}^{\infty} (-1)^n \left(n^{1/n} + 1/2\right)^n$

30. $\sum_{n=1}^{\infty} \frac{2^n}{1+\ln^n(n)}$

31. $\sum_{n=1}^{\infty} \frac{\exp(n)}{\ln^n(n)}$

32. $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n+10}}$

33. $\sum_{n=1}^{\infty} (-1)^n \frac{n!}{3^n}$

34. $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2-11}$

35. $\sum_{n=2}^{\infty} (-1)^n \frac{1}{n \ln^3(n)}$

36. $\sum_{n=1}^{\infty} (-1)^n \frac{n^2+2n+2}{3n^3+7}$

37. $\sum_{n=1}^{\infty} (-1)^n \frac{e^{1/n}}{n^e}$

38. $\sum_{n=1}^{\infty} (-1)^n \ln(1+1/n)$

39. $\sum_{n=1}^{\infty} (-1)^n \frac{1}{(1+1/n)^n}$

40. $\displaystyle\sum_{n=1}^{\infty}(-1)^n\,\frac{\ln(n)}{\sqrt{n}}$

Intervals of Convergence (Corresponds to Stewart 11.8)

In Exercises 1–16, determine the interval of convergence of the given power series $\sum_{n=0}^{\infty} a_n (x-c)^n$. You can complete the calculations by carrying out the following four step algorithm (which reduces to two steps if the interval of convergence is either a single point or the entire real axis).

Step 1 Identify the sequence of coefficients $\{a_n\}$ and the base point c. This may require rewriting the power series so that it is in *exactly* the generic form written above. This means that the coefficient of the variable in parentheses is 1, the operation between the variable and the base point is subtraction, and the power to which the difference of the variable and the base pointbase point is raised is the index of summation. For example, $(3x-2)^n$ should be rewritten as $3^n(x-2/3)^n$, $(x+2)^n$ should be rewritten as $\big(x-(-2)\big)^n$, and $(x-5)^{2n}$ should be rewritten as u^n with $u=(x-5)^2$ replacing x as the variable of the power series.

Step 2 Calculate the radius of convergence R using either of the following two formulas, $R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$ or $R = \lim_{n\to\infty} \frac{1}{|a_n|^{1/n}}$, whichever seems easiest to use. Notice that the quantities in the limits are the *reciprocals* of the fractions used in the Ratio and Root Tests. At this stage, if $R=0$ or $R=\infty$, the calculation is finished. If $R=0$, then the interval is $[c,c]$, consisting of only one point: the base point. If $R=\infty$, then the interval is $(-\infty,\infty)$, which is to say the entire real line.

Step 3 Check to see if the left endpoint of the interval of convergence belongs to the interval. Do so by substituting $x=c-R$ into the power series and use any suitable test for convergence. (However, neither the Ratio Test nor he Root Test will yield a conclusion for an endpoint of an interval of convergence, so there is no point trying them.) For an endpoint, the type of convergence, conditional or absolute, is *immaterial.*

Step 4 Check to see if the right endpoint of the interval of convergence belongs to the interval. Do so by substituting $x=c+R$ into the power series and use any suitable test for convergence. As in Step 3, do not bother with either the Ratio Test or the Root Test here. Again, the type of convergence, conditional or absolute, is immaterial.

1. $\sum_{n=0}^{\infty} \left(\frac{-x}{3}\right)^n$

2. $\sum_{n=1}^{\infty} n\, x^n$

3. $\displaystyle\sum_{n=0}^{\infty} \frac{10}{n+1} x^n$

4. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(10x)^n}{\sqrt{n}}$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{(4x)^n}{n^2}$

6. $\displaystyle\sum_{n=0}^{\infty} \frac{n}{4^n} (2x+1)^n$

7. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{-x+1}{n}\right)^n$

8. $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{3}{\sqrt{n^2+1}} x^n$

9. $\displaystyle\sum_{n=0}^{\infty} \frac{n^2}{n^3+1} \cdot \left(x - \frac{1}{2}\right)^n$

10. $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{n^3+1} (x+2)^n$

11. $\displaystyle\sum_{n=0}^{\infty} \frac{1+2^n}{1+3^n} \cdot \left(x + \frac{1}{2}\right)^n$

12. $\displaystyle\sum_{n=1}^{\infty} \frac{(x+6)^n}{\sqrt{n}}$

13. $\displaystyle\sum_{n=1}^{\infty} \frac{(\pi - x)^n}{n}$

14. $\displaystyle\sum_{n=0}^{\infty} \frac{(x+4)^n}{n^5+1}$

15. $\sum_{n=0}^{\infty} (3x+2)^n$

16. $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{n+1}{3n+1} (2x+5)^n$

In Exercises 17 and 18, the powers that appear in the given power series are not the indices of summations of the series. To use the algorithm as it is given in the instructions to Exercises 1–16, you must make a change of variable. Change $\displaystyle\sum_{n=0}^{\infty} a_n\,(x-c)^{2n}$ to $\displaystyle\sum_{n=0}^{\infty} a_n\,u^n$ and $\displaystyle\sum_{n=0}^{\infty} a_n\,(x-c)^{2n+1}$ to $(x-c)\displaystyle\sum_{n=0}^{\infty} a_n\,u^n$ where $u = (x-c)^2$. Treat u as the variable of the power series and use the algorithm with u replacing x. Find an interval of convergence for u and unwind what it means for x.

17. $\displaystyle\sum_{n=0}^{\infty} \frac{4^n}{n+1}\,(x+3)^{2n}$

18. $\displaystyle\sum_{n=0}^{\infty} \frac{4^n+n^2}{(n+1)9^n}\,(x-1)^{2n+1}$

Series Related to $1/(1 \pm x)$ (Corresponds to Stewart 11.8)

In each of Exercises 1–6 express the given function as a power series in x with base point (center of convergence) 0. Instead of a brute force computation, use the formula

$$\sum_{n=0}^{\infty} u^n = \frac{1}{1-u} \qquad (-1 < u < 1)$$

and report terms through at least order 4.

1. $\dfrac{1}{2+x}$
2. $\dfrac{1}{1-2x}$
3. $\dfrac{x}{4+3x}$
4. $\dfrac{1}{4+x^2}$
5. $\dfrac{1+x^2}{1-x^2}$
6. $\dfrac{x}{2+3\,x^2}$

Exercises 7–12 all use the power series for $1/(2+x)$ found in Exercise 1. The techniques for working with series include multiplication, division, differentiation, and integration.

7. Express $1/(2+x)^2$ as a power series with base point 0. For the method of calculation, multiply the series for $1/(2+x)$, which you found in Exercise 1, by itself. Carry out the multiplication term-by-term and collect all terms through x^3.
8. Express $1/(2+x)^2$ as a power series with base point 0, but use a method of calculation that differs from that of the preceding problem. This time, observe that $\dfrac{1}{(2+x)^2} = -\dfrac{d}{dx}\left(\dfrac{1}{2+x}\right)$. Use the series for $1/(2+x)$, which you found in Exercise 1, and differentiate term-by-term all terms through x^4.

9. Factor $2 - x - x^2$. One of the factors is $2 + x$. Then use the method of Exercise 7 together with the calculation of Exercise 1 to calculate the power series of $1/\left(2 - x - x^2\right)$ with base point 0. Report all terms through x^3.

10. Express $\ln(2 + x)$ as a power series with base point 0. Report terms through x^5.

11. The power series $\sum_{n=1}^{\infty} n\,x^n$ converges for $-1 < x < 1$. Let $f(x)$ be the sum of this series for x in the interval of convergence. It can be shown that $f(x) = x/(x-1)^2$ but this exercise does *not* rely on this evaluation of $f(x)$. Calculate the power series of $q(x) = \dfrac{f(x)}{2+x}$. Do the calculation in two different ways. First, multiply the given series for $f(x)$ term-by-term with the series for $1/(2+x)$. Report all terms through x^4. For the second calculation, write $q(x) = a_0 + a_1\,x + a_2\,x^2 + a_3\,x^3 + a_4\,x^4 + \cdots$. Then

$$\sum_{n=1}^{\infty} n\,x^n = f(x) = (2+x)\,\left(a_0 + a_1\,x + a_2\,x^2 + a_3\,x^3 + a_4\,x^4 + \cdots\right).$$

Multiply the two factors on the right term-by-term. Collect all terms through x^4. Equate the constant terms of the series on each side to find a_0. It will be 0. Then equate the coefficients of x on each side to find a_1. Continue with x^2, x^3, and x^4.

12. The power series $2 + \sum_{n=1}^{\infty} n\,x^n$ converges for $-1 < x < 1$. Let $f(x)$ be the sum of this series for x in the interval of convergence. It can be shown that $f(x) = \left(2x^2 - 3x + 2\right)/(x-1)^2$ but this exercise does *not* rely on this evaluation of $f(x)$. Calculate the power series of $q(x) = \dfrac{2+x}{f(x)}$. To do so, write $q(x) = a_0 + a_1\,x + a_2\,x^2 + a_3\,x^3 + \cdots$. Then

$$2 + x = \left(2 + \sum_{n=1}^{\infty} n\,x^n\right)\,\left(a_0 + a_1\,x + a_2\,x^2 + a_3\,x^3 + a_4\,x^4 + \cdots\right).$$

Multiply the two factors on the right term-by-term. Collect all terms through x^4. Equate like powers of x on each side of the equation through x^3. Successively solve for a_0, a_1, a_2 and a_3.

In Exercises 13 and 14, use the power series

$$\ln(1+u) = u - \frac{1}{2}u^2 + \frac{1}{3}u^3 - \frac{1}{4}u^4 + \cdots \qquad (-1 < u < 1)$$

to express the given function as a power series in x with base point 0. Report terms through at least order 4.

13. $\ln(2 - x)$

14. $x\,\ln(3 + 4x^2)$

Taylor Polynomials (Corresponds to Stewart 11.9)

1. Calculate the order 3 Taylor polynomial of $f(x) = \sqrt{x}$ with base point 4.
2. Calculate the order 4 Taylor polynomial of $f(x) = 1/x$ with base point $1/2$.
3. Calculate the order 4 Taylor polynomial of $f(x) = \sin(x)$ with base point π.
4. Calculate the order 4 Taylor polynomial of $f(x) = \exp(1 + x/2)$ with base point -2.
5. Calculate the order 3 Taylor polynomial of $f(x) = (4 + x)^{-3/2}$ with base point -3.
6. Calculate the order 2 Taylor polynomial of $f(x) = \dfrac{\sqrt{3+x}}{x}$ with base point 1.
7. Calculate the orders 1, 2, and 3 Taylor polynomials of $f(x) = \cos(x)$ with base point $\pi/3$.

Taylor and Maclaurin Series

In each of Exercises 1–4 use the Maclaurin series

$$\sin(u) = u - \frac{1}{3!}u^3 + \frac{1}{5!}u^5 - \cdots \qquad (\infty < u < \infty)$$

for $\sin(u)$,

$$\cos(u) = 1 - \frac{1}{2!}u^2 + \frac{1}{4!}u^4 - \cdots \qquad (\infty < u < \infty)$$

for $\cos(u)$, and/or

$$\exp(u) = 1 + u + \frac{1}{2!}u^2 + \frac{1}{3!}u^3 + \frac{1}{4!}u^4 + \cdots \qquad (\infty < u < \infty)$$

for $\exp(u)$ to obtain Maclaurin series for the given function. Report terms through at least order 4.

1. $\dfrac{\sin\left(x^2\right)}{x}$
2. $\dfrac{1 - \cos(2x)}{x}$
3. $\dfrac{x}{\exp\left(x^2\right)}$
4. $\exp\left(x^2 + 2x\right)$

In Exercises 5 and 6, use Newton's Binomial Series,

$$(1+u)^\alpha = 1 + \alpha u + \frac{\alpha(\alpha-1)}{2!}u^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}u^3 + \frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{4!}u^4 + \cdots$$

to find the first few terms of the Maclaurin series of the given functions.

5. $\dfrac{1}{\sqrt{1+2x^2}}$
6. $(8 + x^3)^{1/3}$

In Exercises 7 and 8, use a Taylor polynomial to calculate the given integral with an error less than 10^{-5}.

7. $\displaystyle\int_0^{0.2} \sin\left(x^2\right)\, dx$

8. $\displaystyle\int_0^{0.2} \frac{1}{1+x^3}\, dx$

Polar Coordinates (Corresponds to Stewart 10.3)

1. The polar coordinates of three points are given. State the Cartesian coordinates of each point. Then find two other pairs of polar coordinates of each point: one with $r > 0$ and one with $r < 0$.

   $P = (3, \pi/4), \quad Q = (-4, 3\pi/2), \quad R = (10, -\pi/3)$

2. The Cartesian coordinates of three points are given. For each point, find two pairs of polar coordinates (r, θ): one pair with $r > 0$ and $0 \leq \theta < 2\pi$ and one pair with $r < 0$ and $0 \leq \theta < 2\pi$:

   $P = (-3, 3), \quad Q = (2, 2\sqrt{3}), \quad R = (-\sqrt{3}, 1)$

3. Find the distance between the points with polar coordinates $(2, -\pi/6)$ and $(-6, \pi/3)$.

4. Sketch the region in the plane that consists of points with polar coordinates that satisfy the inequalities $1 \leq r \leq 2$ and $\pi/4 \leq \theta \leq 2\pi/3$.

5. Identify the curves with the given polar equations:
   (i) $r = 5\ \sec(\theta)$
   (ii) $r^2\ \sin(2\theta) = 6$.

6. Sketch the curve with polar equation $r = -3\cos(\theta)$.

7. Sketch the curve with polar equation $r = (1.2)^\theta, \quad 0 \leq \theta \leq 7\,\pi/2$.

8. Sketch the curve with polar equation $r = 3 + 2\sin(\theta)$.

9. Sketch the curve with polar equation $r = 2\cos(3\theta)$.

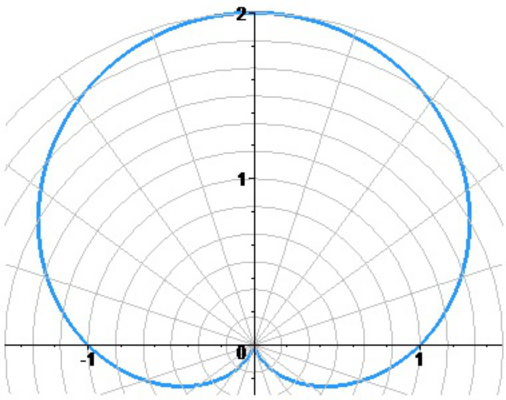
10. Sketch the curve with polar equation $r = \cos(\theta/2)$.

11. Sketch the curve with polar equation $r = 1 + 2\sin(\theta)$.

12. Sketch the curve with polar equation $r = \cos(2\theta + \pi/4)$.

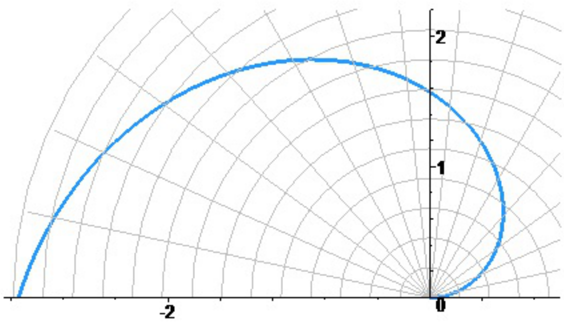
Areas in Polar Coordinates (Corresponds to Stewart 10.4)

1. Calculate the area of the region inside the cardioid $r = 1 + \sin(\theta)$.



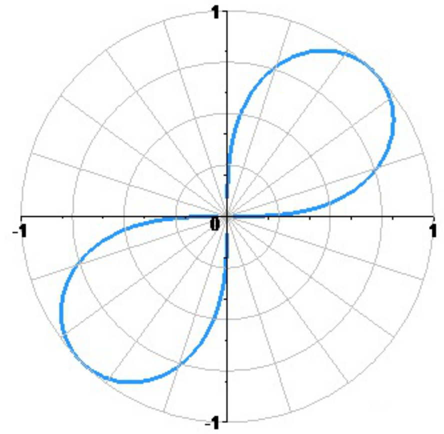
$r = 1 + \sin(\theta)$

2. Calculate the area of the region between the x-axis and the polar curve $r = \theta,\ 0 \leq \theta \leq \pi$.



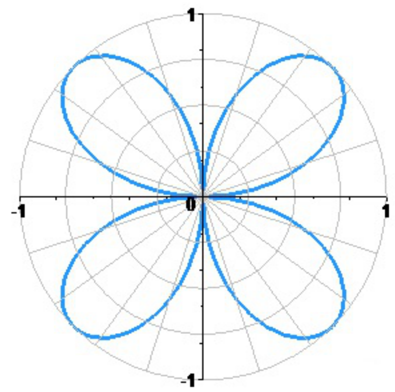
$r = \theta, \quad 0 \leq \theta \leq \pi$

3. Find the area inside the polar curve $r = \sqrt{\sin(2\,\theta)}$.



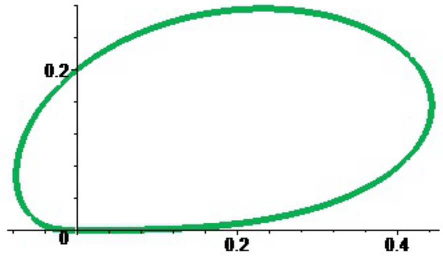
$r = \sqrt{\sin(2\theta)}$

4. Find the area inside the polar curve $r = \sin(2\,\theta)$.



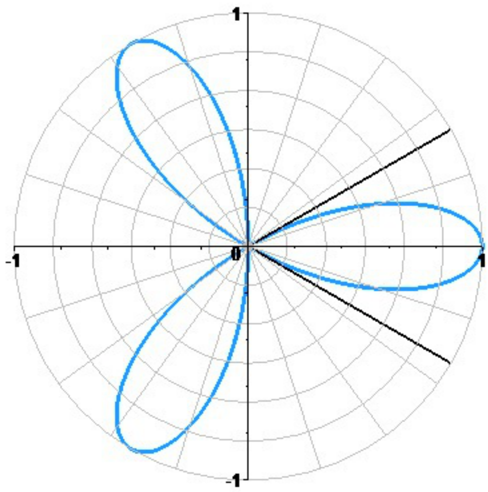
$r = \sin(2\theta)$

5. The figure below shows a cross-section of a lump of green kryptonite, a once-rare mineral first discovered on Earth by Jerry Siegel in the 1940s, but which has become increasingly common thanks to meteor showers and the proliferation of superhero films. The boundary curve is given in polar coordinates by $r = \sqrt{\sin(\theta)}/\big(5 - 4\,\cos(\theta)\big)$. What is the area of the enclosed region?



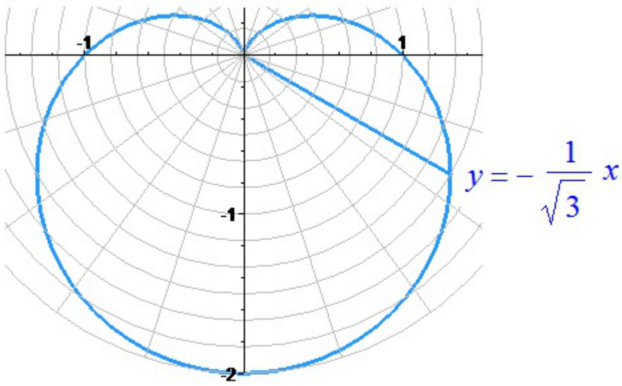
$$r = \frac{\sqrt{\sin(\theta)}}{5 - 4\cos(\theta)}$$

6. Find the area of the region inside one leaf of the three-leaf rose $r = \cos(3\,\theta)$.



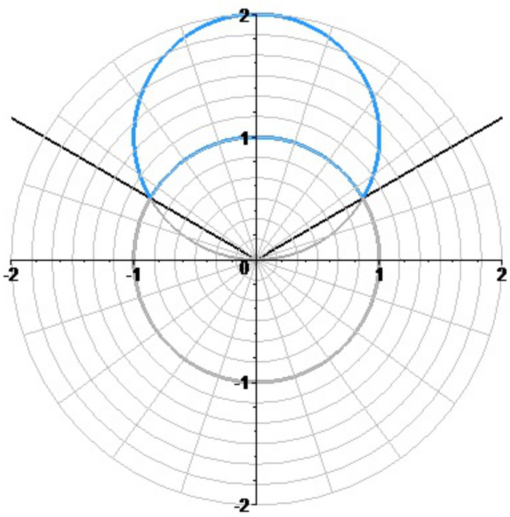
$$r = \cos(3\,\theta)$$

7. Find the area of the region inside the cardioid $r = 1 - \sin(\theta)$ and above the line $y = -x/\sqrt{3},\ 0 \le x \le 3\sqrt{3}/4$.



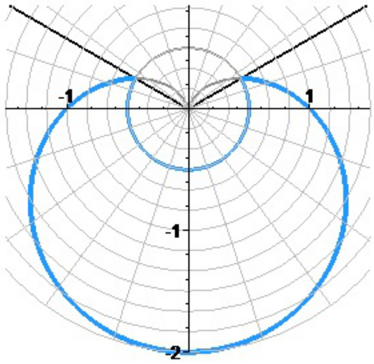
$r = 1 - \sin(\theta)$

8. Find the area of the region inside the circle $r = 2\ \sin(\theta)$ and outside the circle $r = 1$.



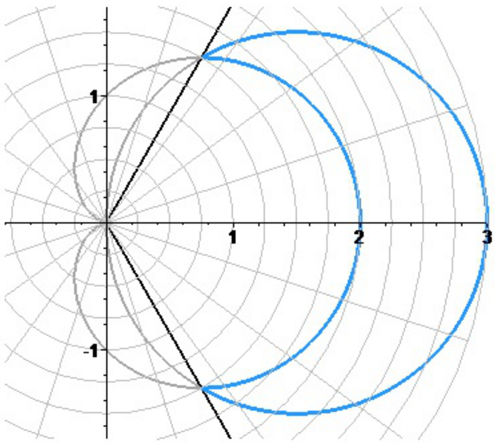
The circles $r = 2\ \sin(\theta)$ and $r = 1$

9. Find the area of the region inside the cardioid $r = 1 - \sin(\theta)$ and outside the circle $r = 1/2$.



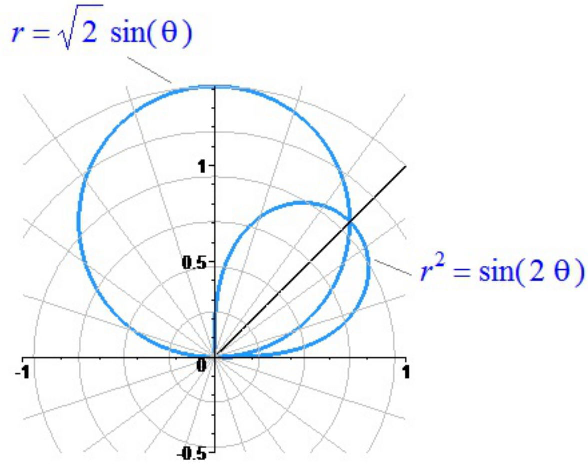
The cardioid $r = 1 - \sin(\theta)$ and the circle $r = \frac{1}{2}$

10. Find the area of the region inside the circle $r = 3\,\cos(\theta)$ and outside the cardioid $r = 1 + \cos(\theta)$.

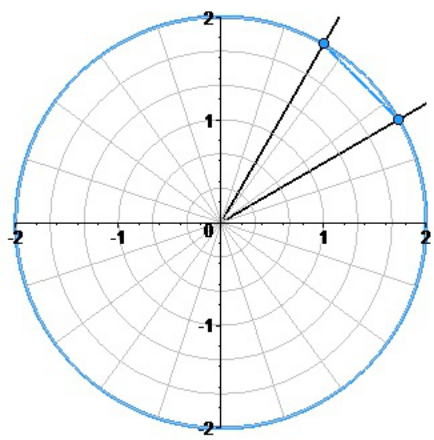


The cardioid $r = 1 + \cos(t)$ and the circle $r = 3\,\cos(t)$

11. The figure below shows the circle with polar equation $r = \sqrt{2}\,\sin(\theta)$ and the curve with polar equation $r^2 = \sin(2\,\theta)$, $0 \le \theta \le \pi/2$. Find the area of the region inside both curves.



12. The figure below shows the circle with polar equation $r = 2$ and the secant line segment between the two points $(1, \sqrt{3})$ and $(\sqrt{3}, 1)$ on the circle.
(a) Show that the line segment has the rectangular equation $y = -x + 1 + \sqrt{3}$ for x in the interval $\left[1, \sqrt{3}\right]$.
(b) Show that the line segment has the polar equation $r = \dfrac{1+\sqrt{3}}{\cos(\theta)+\sin(\theta)}$ for θ in the interval $[\pi/6, \pi/3]$.
c) Show that the line segment has the polar equation $r = \dfrac{1+\sqrt{3}}{\sqrt{2}} \csc\left(\theta + \dfrac{\pi}{4}\right)$ for θ in the interval $[\pi/6, \pi/3]$.
d) Using polar coordinates, calculate the area of the sliver of the illustrated circle that lies above the illustrated secant.
e) (Optional. Only for those who wish to prolong the enjoyment of this exercise set) It is a fact known to few humans that
$$\cot\left(\frac{5\pi}{12}\right) = 2 - \sqrt{3}.$$
Most of the handful of humans who have derived this formula have obtained it from trigonometric identities. Join the select few who have espied this cotangent evaluation by redoing the calculation of part (d) using Cartesian coordinates and comparing the outcomes.



Solutions: Areas and Distances (Corresponds to Stewart 5.1)

1. i) Let Δx denote the width of each rectangle. Then $\Delta x = 1$. From left to right, the heights of the rectangles are $f(0) = 2, f(1) = 4, f(2) = 5, f(3) = 7, f(4) = 8, f(5) = 7$. The approximate area is the sum of the areas of the rectangles: $(2 + 4 + 5 + 7 + 8 + 7)\ \Delta x$, or 33.

   ii) Let Δx denote the width of each rectangle. Then $\Delta x = 1$. From left to right, the heights of the rectangles are $f(1) = 4, f(2) = 5, f(3) = 7, f(4) = 8, f(5) = 7, f(6) = 4$. The approximate area is the sum of the areas of the rectangles: $(4 + 5 + 7 + 8 + 7 + 4)\ \Delta x$, or 35.

   iii) Let Δx denote the width of each rectangle. Then $\Delta x = 2$. From left to right, the heights of the rectangles are $f(1) = 4, f(3) = 7, f(5) = 7$. The approximate area is the sum of the areas of the rectangles: $(4 + 7 + 7)\ \Delta x$, or 36.

2. i) Let Δx denote the width of each rectangle. Then $\Delta x = 1/4$. From left to right, the heights of the rectangles are $f(1) = 5, f(5/4) = 4, f(3/2) = 7, f(7/4) = 8, f(2) = 6, f(9/4) = 6, f(5/2) = 5, f(11/4) = 3$. The approximate area is the sum of the areas of the rectangles: $(5 + 4 + 7 + 8 + 6 + 6 + 5 + 3)\ \Delta x$, or 11.

   ii) Let Δx denote the width of each rectangle. Then $\Delta x = 1/4$. From left to right, the heights of the rectangles are $f(5/4) = 4, f(3/2) = 7, f(7/4) = 8, f(2) = 6, f(9/4) = 6, f(5/2) = 5, f(11/4) = 3, f(3) = 2$. The approximate area is the sum of the areas of the rectangles: $(4 + 7 + 8 + 6 + 6 + 5 + 3 + 2)\ \Delta x$, or 10.25.

   iii) Let Δx denote the width of each rectangle. Then $\Delta x = 1/2$. From left to right, the heights of the rectangles are $f(5/4) = 4, f(7/4) = 8, f(9/4) = 6, f(11/4) = 3$. The approximate area is the sum of the areas of the rectangles: $(4 + 8 + 6 + 3)\ \Delta x$, or 10.5.

3. i) $(54 + 61 + 65)\ \frac{\text{ft}}{\text{s}} \times 20\,\text{s}$, or 3600 ft.

   ii) $(61 + 65 + 62)\ \frac{\text{ft}}{\text{s}} \times 20\,\text{s}$, or 3760 ft.

   iii) $(57 + 66 + 63)\ \frac{\text{ft}}{\text{s}} \times 20\,\text{s}$, or 3720 ft.

   iv) $(54 + 57 + 61 + 66 + 65 + 63)\ \frac{\text{ft}}{\text{s}} \times 10\,\text{s}$, or 3660 ft.

v) $(57 + 61 + 66 + 65 + 63 + 62)\ \frac{\text{ft}}{\text{s}} \times 10\,\text{s}$, or 3740 ft.

vi) One can start from scratch and average the left and right endpoints of all six subintervals. However, we can use the calculations already performed in parts (iv) and (v). The answer is the average of the estimates found in those two parts: $(3660 + 3740)/2$ ft, or 3700 ft.

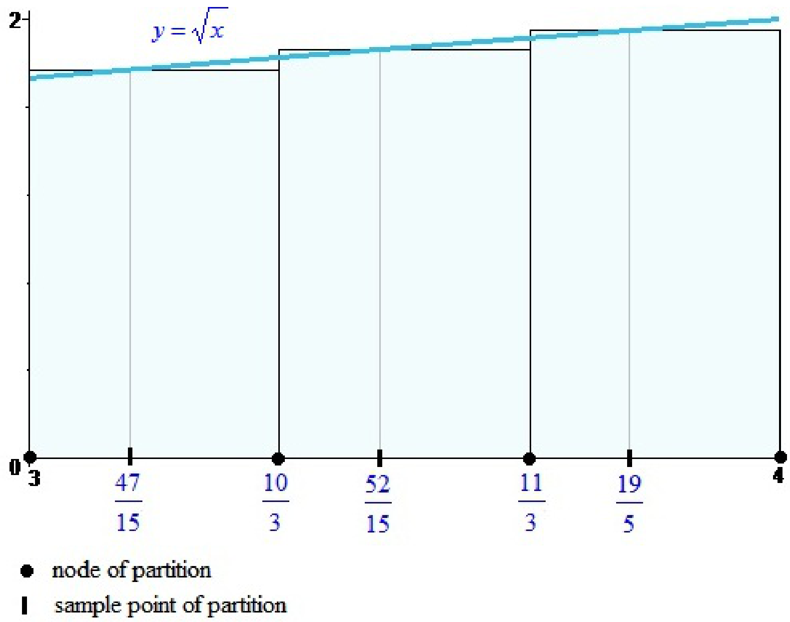
Solutions: Riemann Sums, Riemann Integrals, Definite Integrals, Midpoint Rule (Corresponds to Stewart 5.2)

1. i) Here, $\Delta x = (4-3)/3 = 1/3$. The nodes are $x_0 = 3, x_1 = x_0 + \Delta x = 10/3, x_2 = x_1 + \Delta x = 11/3, x_3 = 4$. The sample points are $s_1 = x_0 = 3, s_2 = x_1 = 10/3, s_3 = x_2 = 11/3$. We calculate $f(s_1) = 1.73205, f(s_2) = 1.82574, f(s_3) = 1.91485$. The requested Riemann sum is $(\, f(s_1) + f(s_2) + f(s_3)\,)\ \Delta x = 1.8242$.

   ii) The sample points are now $s_1 = x_1 = 10/3, s_2 = x_2 = 11/3, s_3 = x_3 = 4$. We calculate $f(s_1) = 1.82574, f(s_2) = 1.91485, f(s_3) = 2$. The requested Riemann sum is $(\, f(s_1) + f(s_2) + f(s_3)\,)\ \Delta x = 1.9135$.

   iii) The sample points are now $s_1 = x_0 + (\Delta x)/2 = 19/6, s_2 = x_1 + (\Delta x)/2 = 7/2, s_3 = x_2 + (\Delta x)/2 = 23/6$. We calculate $f(s_1) = 1.77952, f(s_2) = 1.87083, f(s_3) = 1.95789$. The requested Riemann sum is $(\, f(s_1) + f(s_2) + f(s_3)\,)\ \Delta x = 1.8694$.

   iv) The sample points are now $s_1 = 47/15, s_2 = s_1 + \Delta x = 52/15, s_3 = s_2 + \Delta x = 19/5$. We calculate $f(s_1) = 1.77012, f(s_2) = 1.86189, f(s_3) = 1.94936$. The requested Riemann sum is $(\, f(s_1) + f(s_2) + f(s_3)\,)\ \Delta x = 1.8605$. The rectangles corresponding to this Riemann sum are shown in the figure below.



2. Here, $\Delta x = (4-1)/4 = 3/4$. The nodes are $x_0 = 1, x_1 = x_0 + \Delta x = 7/4, x_2 = x_1 + \Delta x = 5/2, x_3 = x_2 + \Delta x = 13/4 x_4 = x_3 + \Delta x = 4$. The sample points are $s_1 = x_0 + (\Delta x)/3 = 5/4, s_2 = x_1 + (\Delta x)/3 = 2, s_3 = x_2 + (\Delta x)/3 = 11/4, s_4 = x_3 + (\Delta x)/3 = 7/2$. We calculate $f(s_1) = 0.80000, f(s_2) = 0.50000, f(s_3) = 0.36364, f(s_4) = 0.28571$. The requested Riemann sum is $(\,f(s_1) + f(s_2) + f(s_3) + f(s_4)\,)\ \Delta x = 1.4620$.

3. The width of each subinterval is $\Delta x = (2-1)/5 = 0.2$. The nodes of the partition are $x_0 = 1, x_1 = x_0 + \Delta x = 1.$ The sample points arising from the Midpoint Rule are $s_1 = x_0 + \frac{1}{2}\Delta x = 1 + 0.1 = 1.1, s_2 = x_1 + \frac{1}{2}\Delta x = 1.2 +$ The Midpoint Rule approximation is $3\left((1.1)^2 + (1.3)^2 + (1.5)^2 + (1.7)^2 + (1.9)^2\right) \times 0.2$ or 6.99. With $n = 100$, the approximation is 6.999975, which is quite accurate for an extremely modest amount of computer calculation.

4. i) The area of the region under the graph of f and above the interval $[1, 2]$ of the x-axis is $5/12$. The area of the region above the graph of f and under the interval $[2, 4]$ of the x-axis is $8/3$. Use the given information to evaluate $\int_1^4 f(x)\,dx$.

   ii) Estimate the definite integral $\int_1^2 f(x)\,dx$ using the Midpoint Rule with $n = 3$.

   iii) Estimate the definite integral $\int_2^4 f(x)\,dx$ using the Midpoint Rule with $n = 6$.

   iv) Estimate the definite integral $\int_1^4 f(x)\,dx$ using the Midpoint Rule with $n = 9$.

   i) $\displaystyle\int_1^4 f(x)\,dx = \frac{5}{12} - \frac{8}{3} = -\frac{9}{4}.$

   ii) Here, $\Delta x = (2-1)/3 = 1/3$. The nodes are $x_0 = 1, x_1 = 4/3, x_2 = 5/3, x_3 = 2$. The sample points are $s_1 = x_0 + (\Delta x)/2 = 7/6, s_2 = x_1 + (\Delta x)/2 = 9/6, s_3 = x_2 + (\Delta x)/2 = 11/6$. We calculate $f(s_1) = 0.39351, f(s_2) = 0.62500, f(s_3) = 0.30092$. The Midpoint Rule approximation is

$(f(s_1) + f(s_2) + f(s_3))\ \Delta x = 0.4398.$

iii) Here, $\Delta x = (4-2)/6 = 1/3$. The nodes are $x_0 = 2, x_1 = 7/3, x_2 = 8/3, x_3 = 3, x_4 = 10/3, x_5 = 11/3, x_6 = 4$. The sample points are $s_1 = x_0 + (\Delta x)/2 = 13/6, s_2 = x_1 + (\Delta x)/2 = 15/6, s_3 = x_2 + (\Delta x)/2 = 17/6, s_4 = x_3 + (\Delta x)/2 = 19/6, s_5 = x_4 + (\Delta x)/2 = 21/6, s_6 = x_5 + (\Delta x)/2 = 23/6$. We calculate $f(s_1) = -0.35648, f(s_2) = -1.12500, f(s_3) = -1.78240, f(s_4) = -2.10648, f(s_5) = -1.87500, f(s_6) = -0.86574$. The Midpoint Rule approximation is $(f(s_1) + f(s_2) + f(s_3) + f(s_4) + f(s_5) + f(s_6))\ \Delta x = -2.7037$.

iv) We can calculate the requested Midpoint Rule approximation from scratch, but that would duplicate all the work already done. Notice that, for the requested Midpoint Rule, $\Delta x = (4-1)/9 = 1/3$. Because $\Delta x = 1/3$ in both parts (ii) and (iii), we can combine the two calculations: the nodes found in part (ii) partition the subinterval $[1, 2]$ into 3 sub-subintervals and the nodes found in part (iii) partition the subinterval $[2, 4]$ into 6 sub-subintervals—altogether there are 9 equal-width subintervals of $[1, 4]$. We find that the Midpoint Rule approximation is $0.4398 + (-2.7037)$, or -2.2639.

5. The local extrema can be located by the first derivative test. We calculate $f'(x) = 2\,\left(12\,x^2 - 43\,x + 35\right)$. If you are skilled at factoring, you can write this as $f'(x) = 2\,(4x-5)\,(3x-7)$, which shows that $5/4$ and $7/3$ are the roots of f'. Without factoring, you can determine these roots using the Quadratic Formula. With one possible exception, we can now determine the sample points for both parts (i) and (ii). The possible exception is the sample point s_2 in the subinterval $[1, 3/2]$ for the lower Riemann sum. We calculate $f(1) = 14$ and $f(1/2) = 57/4 = 14 + 1/4 > f(1)$. The sample points for the upper Riemann sum are $s_1 = 1, s_2 = 5/4, s_3 = 3/2, s_4 = 2, s_5 = 3$. Because $\Delta x = (3-1/2)/5 = 1/2$, the upper Riemann sum R^* is:

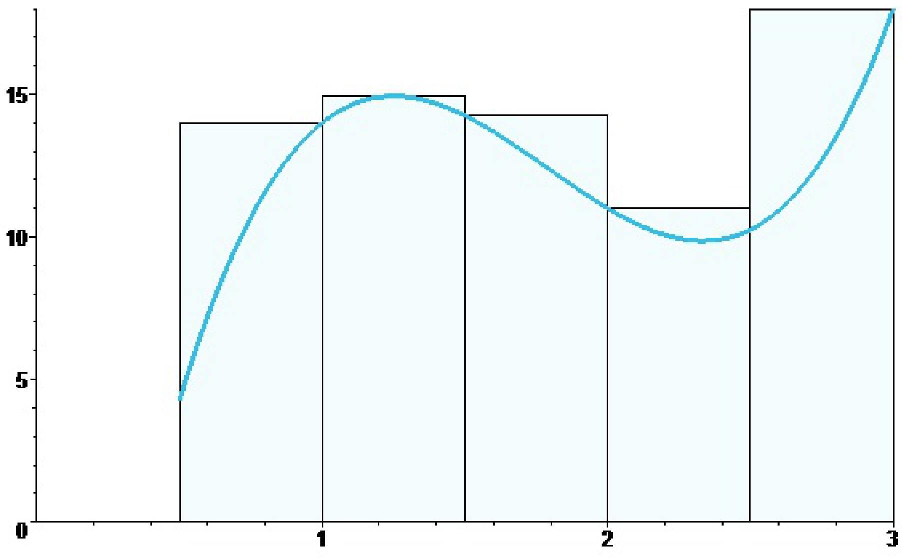
$$R^* = \left(f(1) + f\left(\frac{5}{4}\right) + f\left(\frac{3}{2}\right) + f(2) + f(3) \right)\ \Delta x = \left(14 + \frac{239}{16} + \frac{57}{4} + 11 + 18 \right) \frac{1}{2} = \frac{1155}{32} = 36.09375.$$

The sample points for the lower Riemann sum are $s_1 = 1/2, s_2 = 1, s_3 = 2, s_4 = 7/3, s_5 = 5/2$. The lower Riemann sum R_* is:

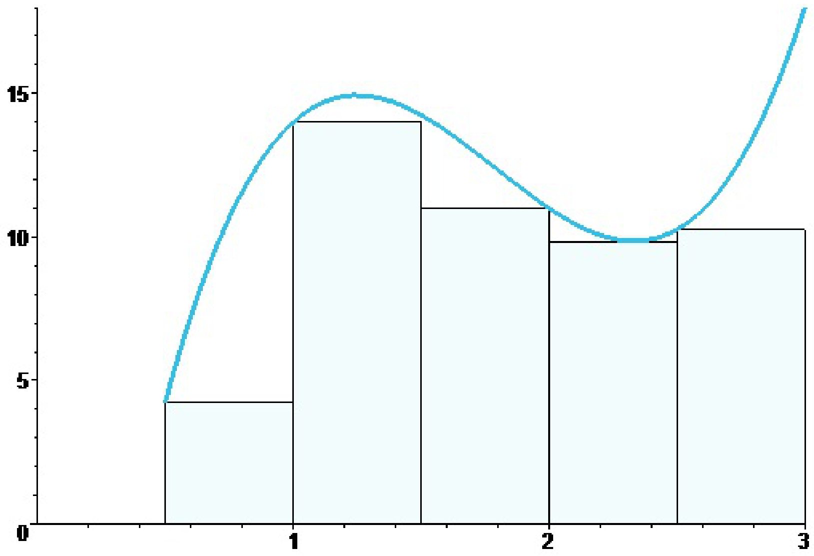
$$R_* = \left(f\left(\frac{1}{2}\right) + f(1) + f(2) + f\left(\frac{7}{3}\right) + f\left(\frac{5}{2}\right) \right)\ \Delta x = \left(\frac{17}{4} + 14 + 11 + \frac{266}{27} + \frac{41}{4} \right) \frac{1}{2} = \frac{2665}{108} \approx 24.67593.$$

The exact evaluation of the integral is $\displaystyle\int_{1/2}^{3} f(x)\,dx = \frac{365}{12}$, just in case you ever wondered about the significance of the ratio of the number of days in a year to the number of months. This ratio, about 30.41666667, is between the upper and lower Riemann sums, of which we could be certain even if we could not determine its exact value.

The upper and lower Riemann sums are illustrated in the two figures that follow



Upper Riemann Sum



Lower Riemann Sum

6. For every j from 1 to N, the point s_j is the midpoint of the j^{th} subinterval. All we require for this problem is that s_j is a point in the j^{th} subinterval. Had s_j been an endpoint of the subinterval or a randomly selected point inside the subinterval, there would have been no change whatsoever in the solution that follows.

We must identify $\displaystyle\sum_{j=1}^{N} \frac{4\,s_j^2 - 1}{N}$ as $\displaystyle\sum_{j=1}^{N} f\left(s_j\right)\Delta x$ for some function f and width Δx. The latter quantity follows easily from the given information $a = 1$ and $b = 5$:

$$\Delta x = \frac{b-a}{N} = \frac{5-1}{N} = \frac{4}{N}.$$

We express the given sum with the factor Δx appearing explicitly:

$$\sum_{j=1}^{N} \frac{4\,s_j^2 - 1}{N} = \sum_{j=1}^{N} \left(s_j^2 - \frac{1}{4}\right)\frac{4}{N} = \sum_{j=1}^{N} \left(s_j^2 - \frac{1}{4}\right)\Delta x.$$

If we define $f(x) = x^2 - 1/4$, then we have

$$\sum_{j=1}^{N} \frac{4\,s_j^2 - 1}{N} = \sum_{j=1}^{N} f\left(s_j\right)\Delta x.$$

Thus,

$$\begin{aligned}
\lim_{N\to\infty} \sum_{j=1}^{N} \frac{4\,s_j^2 - 1}{N} &= \lim_{N\to\infty} \sum_{j=1}^{N} f\left(s_j\right)\Delta x \\
&= \int_1^5 f(x)\,dx \\
&= \int_1^5 \left(x^2 - \frac{1}{4}\right)\,dx \\
&= \left(\frac{x^3}{3} - \frac{x}{4}\right)\Bigg|_1^5 \\
&= \frac{121}{3}.
\end{aligned}$$

7. Students possessing a very good facility for working with algebraic expressions might not need the first step of this written solution, but even they may benefit from a reminder that studying a particular case can provide an insight into the general situation. We will write the sum explicitly for $N = 4$. Why this particular choice of N? There is nothing special about $N = 4$, but it makes for a Goldilocks choice: 4 subintervals are not so few as to inhibit detection of a pattern nor so large that calculation is tedious. In other words, the number N of subintervals is just right when $N = 4$. For this value of N, we have $\Delta x = (b-a)/N = (3-1)/4 = 1/2$. The nodes x_0, x_1, x_2, x_3, x_4 of the partition are $x_0 = a = 1, x_1 = x_0 + \Delta x = 1 + 1/2 = 3/2, x_2 = x_1 + \Delta x = 3/2 + 1/2 = 2, x_3 = x_2 + \Delta x = 2 + 1/2 = 5/2, x_4 = x_3 + \Delta x = 5/2 + 1/2 = 3$. The subintervals $[x_0, x_1], [x_1, x_2], [x_2, x_3], [x_3, x_4]$ of the partition are therefore $[1, 3/2], [3/2, 2], [2, 5/2], [5/2, 3]$. We will be looking for 4 sample points s_1, s_2, s_3, s_3, s_4 with s_1 in the first subinterval of the partition, s_2 in the second subinterval of the partition, s_3 in the third, and s_4 in the fourth. To identify these sample points, we will rewrite the sum explicitly, without sigma

notation, making sure that the subinterval width Δx, in this case $1/2$, appears as a common factor:

$$\begin{aligned}
\sum_{j=1}^{N}\left(1+\frac{2j-1}{N}\right)^2\frac{6}{N} &= \left(1+\frac{2-1}{4}\right)^2\frac{6}{4}+\left(1+\frac{4-1}{4}\right)^2\frac{6}{4}+\left(1+\frac{6-1}{4}\right)^2\frac{6}{4}+\left(1+\frac{8-1}{4}\right)^2\frac{6}{4}\\
&= \left(\frac{5}{4}\right)^2\frac{6}{4}+\left(\frac{7}{4}\right)^2\frac{6}{4}+\left(\frac{9}{4}\right)^2\frac{6}{4}+\left(\frac{11}{4}\right)^2\frac{6}{4}\\
&= 3\left(\frac{5}{4}\right)^2\frac{1}{2}+3\left(\frac{7}{4}\right)^2\frac{1}{2}+3\left(\frac{9}{4}\right)^2\frac{1}{2}+3\left(\frac{11}{4}\right)^2\frac{1}{2}.
\end{aligned}$$

Let us now observe that $5/4$, $7/4$, $9/4$, and $11/4$ are points that are in, respectively, the subintervals $[1, 3/2], [3/2, 2], [2, 5/2], [5/2, 3]$. Indeed, the four listed points are the midpoints of the listed subintervals. It appears that we have our sample points $s_1 = 5/4, s_2 = 7/4, s_3 = 9/4, s_4 = 11/4$, and

$$\sum_{j=1}^{N}\left(1+\frac{2j-1}{N}\right)^2\frac{6}{N} = 3\,(s_1)^2\,\Delta x + 3\,(s_2)^2\,\Delta x + 3\,(s_3)^2\,\Delta x + 3\,(s_4)^2\,\Delta x.$$

We need a function f such that $f(s_1) = 3\,s_1^2, f(s_2) = 3\,s_2^2, f(s_3) = 3\,s_3^2, f(s_4) = 3\,s_4^2$. Clearly $f(x) = 3\,x^2$ does the job. In summary, for this function f, for $N = 4$, and for the midpoints of the subintervals as the choice of sample points, we have

$$\sum_{j=1}^{N}\left(1+\frac{2j-1}{N}\right)^2\frac{6}{N} = \sum_{j=1}^{N} f(s_j)\,\Delta x.$$

Now we know what we're looking for. Let us proceed to the general case. Letting $a = 1$ and $b = 3$, the width of the entire interval is $b - a$, or 2. Therefore, the subinterval width Δx is given by $\Delta x = 2/N$. We can rewrite the given sum in terms of Δx as follows:

$$\begin{aligned}
\sum_{j=1}^{N}\left(1+\frac{2j-1}{N}\right)^2\frac{6}{N} &= \sum_{j=1}^{N} 3\left(1+\left(j-\frac{1}{2}\right)\frac{2}{N}\right)^2\frac{2}{N}\\
&= \sum_{j=1}^{N} 3\left(1+\left(j-\frac{1}{2}\right)\Delta x\right)^2\Delta x\\
&= \sum_{j=1}^{N} 3\left(1+\frac{1}{2}\,\Delta x+(j-1)\Delta x\right)^2\Delta x.
\end{aligned}$$

For $j = 1$, we see that

$$s_1 = 1 + \frac{1}{2}\,\Delta x + (1-1)\Delta x = a + \frac{1}{2}\,\Delta x,$$

which is in the first subinterval. For $j = 2$, we see that

$$s_2 = 1 + \frac{1}{2}\,\Delta x + (2-1)\Delta x = 1 + \frac{1}{2}\,\Delta x + \Delta x = (a+\Delta x) + \frac{1}{2}\,\Delta x = x_1 + \frac{1}{2}\,\Delta x,$$

which is in the second subinterval. For $j = 3$, we see that

$$s_3 = 1 + \frac{1}{2}\,\Delta x + (3-1)\Delta x = a + \frac{1}{2}\,\Delta x + 2\,\Delta x = (a+2\,\Delta x) + \frac{1}{2}\,\Delta x = x_2 + \frac{1}{2}\,\Delta x,$$

which is in the third subinterval. And so on. For $f(x) = 3\,x^2$ and for midpoints as the choice of sample points, we have

$$\sum_{j=1}^{N} \left(1 + \frac{2\,j-1}{N}\right)^2 \frac{6}{N} = \sum_{j=1}^{N} f\left(s_j\right)\,\Delta x.$$

Thus,

$$\lim_{N\to\infty} \sum_{j=1}^{N} \left(1 + \frac{2\,j-1}{N}\right)^2 \frac{6}{N} = \lim_{N\to\infty} \sum_{j=1}^{N} f\left(s_j\right)\,\Delta x = \int_1^3 f(x)\,dx = \int_1^3 3\,x^2\,dx = 26.$$

Solutions: The Fundamental Theorem of Calculus (Corresponds to Stewart 5.3)

1. 13
2. 38/3
3. 3/2
4. 38/3
5. 32/15
6. $e^2/(e+1)$
7. 4
8. $2\sqrt{3}/3$
9. $3\sqrt{2} - 2\sqrt{3}$
10. $\pi/4$
11. 2 ln(2)
12. $\pi/6$
13. We have
$$\frac{3x^2-1}{x^2+1} = \frac{\left(3x^2+3\right)-4}{x^2+1} = \frac{3\left(x^2+1\right)-4}{x^2+1} = 3 - \frac{4}{x^2+1}.$$
Therefore,
$$\int_0^1 \frac{3x^2-1}{x^2+1}\,dx = \int_0^1 \left(3 - \frac{4}{x^2+1}\right)\,dx = \left(3\,x - 4\,\arctan(x)\right)\Big|_{x=0}^{x=1} = 3 - \pi.$$
14. The identity $1 + \tan^2(\alpha) = \sec^2(\alpha)$ tells us that $\tan^2(\alpha) = \sec^2(\alpha) - 1$. Therefore,
$$\int_{\pi/6}^{\pi/3} \tan^2(\alpha)\,d\alpha = \int_{\pi/6}^{\pi/3} \left(\sec^2(\alpha) - 1\right)\,d\alpha = \int_{\pi/6}^{\pi/3} \sec^2(\alpha)\,d\alpha - \int_{\pi/6}^{\pi/3} 1\,d\alpha = \frac{2\sqrt{3}}{3} - \frac{\pi}{6}.$$

15. We begin by expanding the exponential:
$$\int_0^1 3^{2+t}\,dt = \int_0^1 3^2\,3^t\,dt = 9\int_0^1 3^t\,dt = 9\,\frac{3^t}{\ln(3)}\bigg|_{t=0}^{t=1} = \frac{18}{\ln(3)}.$$

16. $F'(x) = \sqrt{1+t^3}\big|_{t=x} = \sqrt{1+x^3}$ and $F'(2) = 3$

17. $G'(s) = \dfrac{19+u^4}{1+u^2}\bigg|_{u=s} = \dfrac{19+s^4}{1+s^2}$ and $G'(3) = 100/10 = 10$

18. First observe that $F(x) = -\displaystyle\int_{2\pi}^{x} \sqrt{17+16\sin(s)}\,ds$. Therefore
$$D(F)(x) = -\sqrt{17+16\sin(s)}\Big|_{s=x} = -\sqrt{17+16\sin(x)}$$
and $D(F)(\pi/6) = -\sqrt{17+16\sin(\pi/6)} = -\sqrt{17+8} = -5$.

19. For $u = x^2$, we have
$$\begin{aligned}
D(F)(x) &= \frac{d}{dx}\int_1^{x^2} \sqrt{1+\sqrt{t}}\,dt \\
&= \frac{d}{dx}\int_1^{u} \sqrt{1+\sqrt{t}}\,dt \\
&= \left(\frac{d}{du}\int_1^{u} \sqrt{1+\sqrt{t}}\,dt\right)\left(\frac{du}{dx}\right) \\
&= \left(\sqrt{1+\sqrt{t}}\Big|_{t=u}\right)(2x) \\
&= \left(\sqrt{1+\sqrt{u}}\right)(2x) \\
&= \left(\sqrt{1+\sqrt{x^2}}\right)(2x) \\
&= 2\,x\,\sqrt{1+|\,x\,|}.
\end{aligned}$$
and $D(F)(3) = 2\cdot 3\cdot\sqrt{4} = 12$.

20. Calculate $G'(x)$ and $G'(2)$ for $G(x) = \displaystyle\int_{3x}^{2\pi} \sin\left(\frac{\pi}{t}\right)\,dt$. For $u = 3x$, we have
$$\begin{aligned}
G'(x) &= \frac{d}{dx}\int_{3x}^{2\pi} \sin\left(\frac{\pi}{t}\right)\,dt \\
&= \frac{d}{dx}\int_{u}^{2\pi} \sin\left(\frac{\pi}{t}\right)\,dt \\
&= -\frac{d}{dx}\int_{2\pi}^{u} \sin\left(\frac{\pi}{t}\right)\,dt \\
&= -\left(\frac{d}{du}\int_{2\pi}^{u} \sin\left(\frac{\pi}{t}\right)\,dt\right)\left(\frac{du}{dx}\right) \\
&= -\left(\sin\left(\frac{\pi}{t}\right)\Big|_{t=u}\right)(3) \\
&= -3\sin\left(\frac{\pi}{u}\right) \\
&= -3\sin\left(\frac{\pi}{3x}\right)
\end{aligned}$$

and $G'(2) = -3\sin(\pi/6) = -3/2$.

21. Let $u = x^2$ and $w = x^3$. Choose any constant, such as 7. Ultimately we are interested in the value $x = 2$. It so happens that when $x = 2$, we have $u = 4$, $w = 8$, and 7 is between 4 and 8. However, had we chosen a different constant not between 4 and 8, a constant such as -13 or 100000, the calculation would be exactly the same and the answer would be the same. We have

$$F(x) = \int_{x^2}^{x^3} \exp\left(\frac{16}{z}\right)\, dz = \int_{x^2}^{7} \exp\left(\frac{16}{z}\right)\, dz + \int_{7}^{x^3} \exp\left(\frac{16}{z}\right)\, dz = \int_{7}^{x^3} \exp\left(\frac{16}{z}\right)\, dz - \int_{7}^{x^2} \exp\left(\frac{16}{z}\right)\, dz.$$

Therefore

$$\begin{aligned} F'(x) &= \frac{d}{dx}\int_7^{x^3} \exp\left(\frac{16}{z}\right) dz - \frac{d}{dx}\int_7^{x^2} \exp\left(\frac{16}{z}\right) dz \\ &= \frac{d}{dx}\int_7^{w} \exp\left(\frac{16}{z}\right) dz - \frac{d}{dx}\int_7^{u} \exp\left(\frac{16}{z}\right) dz \\ &= \left(\frac{d}{dw}\int_7^{w} \exp\left(\frac{16}{z}\right) dz\right)\left(\frac{dw}{dx}\right) - \left(\frac{d}{du}\int_7^{u} \exp\left(\frac{16}{z}\right) dz\right)\left(\frac{du}{dx}\right) \\ &= \left(\exp\left(\frac{16}{w}\right)\right)(3x^2) - \left(\exp\left(\frac{16}{u}\right)\right)(2x) \\ &= \left(\exp\left(\frac{16}{x^3}\right)\right)(3x^2) - \left(\exp\left(\frac{16}{x^2}\right)\right)(2x) \end{aligned}$$

and

$$F'(2) = \left(\exp\left(\frac{16}{8}\right)\right)(12) - \left(\exp\left(\frac{16}{4}\right)\right)(4) = 12\exp(2) - 4\exp(4).$$

Solutions: The So-Called Net Change Theorem (Corresponds to Stewart 5.4)

1. By the Net Change Theorem, the decrease in volume is

$$\int_2^3 120\,\exp(-t)\,dt = \; -120\,\exp(-t)\Big|_2^3 = -120\,\exp(-3) + 120\,\exp(-2) = 120\,e^{-3}\,(e-1)\,.$$

2. The question asked, in other words, is, *For what value of T is the decrease of volume during the time interval $[0,T]$ equal to 120 - 60, or 60, liters?* That is, we must solve the equation

$$\int_0^T 120\,\exp(-t)\,dt = 60$$

for T. The left side is $-\;120\,\exp(-t)\Big|_0^T$, or $-120\,\exp(-T)+120$. Thus, $-120\,\exp(-T)+120 = 60$, or $120\,\exp(-T) = 60$, or $\exp(-T) = 1/2$. Applying the natural logarithm to each side, we obtain $-T = \ln(1/2)$, or $T = -\ln\left(2^{-1}\right)$. Using the identity $\ln\left(u^p\right) = p\,\ln(u)$ with $u = 2$ and $p = -1$, we arrive at the simplified value, $T = \ln(2)$.

3. Let $f(t)$ be the rate of consumption at time t. Then

$$f(t) = \begin{cases} 60 & \text{if} \quad 0 \le t \le 2 \\ \frac{240}{t^2} & \text{if} \quad 2 \le t \end{cases}$$

The mass of leaf consumption in mg was

$$\int_0^4 f(t)\,dt = \int_0^2 60\,dt + \int_2^4 \frac{240}{t^2}\,dt = 120 + \left(-\frac{240}{t}\Big|_2^4\right) = 120 + 60 = 180.$$

4. By the Net Change Theorem, $E = \int_0^{24} P(t)\,dt$. Notice that if $P(t)$ is measured in megawatts and t is measured in hors, then $P(t)\,dt$ and hence E is measured in megawatt-hours. No directions for estimating the integral were given, so several answers are acceptable. If the Midpoint Rule is to be used, then we can either use $\Delta\,t = 3$ and estimate the values $P(1.5), P(4.5), P(7.5), \ldots, P(22.5)$, or we can use $\Delta\,t = 6$ and read off the values $P(3), P(9), P(15), P(21)$. We will opt for the latter:

$$E \approx (400 + 800 + 850 + 650) \times 6 = 16200.$$

If we use left endpoints for the sample points, then we set $\Delta\,t = 3$ and obtain the estimate

$$E \approx (450 + 400 + 500 + 800 + 850 + 850 + 750 + 650) \times 3 = 15750.$$

Because $P(0) = P(24)$, right endpoints for the sample points lead to the same estimate, 15750, as do left endpoints.

As it happens, the actual value of E for the function graphed in the figure is 15789.433, to three decimal places. That function will not be given here for space reasons: it is a "cubic spline" made up of 8 different degree 3 polynomials used on different subintervals.

Solutions: The Substitution Rule (Corresponds to Stewart 5.5)

1. Letting $u = 1 + 9\,x, du = 9\,dx$, we have
$$\int_0^7 \sqrt[3]{1+9\,x}\,dx = \int_1^{64} \frac{1}{9}\,\sqrt[3]{u}\,du = \left.\frac{1}{12}\,u^{4/3}\right|_{u=1}^{u=64} = \frac{85}{4}.$$

2. Letting $u = 5 - x, du = -dx$, we have
$$\int_1^4 \sqrt{5-x}\,dx = \int_1^4 \sqrt{u}\,du = \left.\frac{2}{3}\,u^{3/2}\right|_{u=1}^{u=4} = \frac{14}{3}.$$

3. Letting $u = 5 - x, du = -dx$, we have
$$\begin{aligned} \int_1^4 x\,\sqrt{5-x}\,dx &= \int_1^4 (5-u)\,\sqrt{u}\,du \\ &= \int_1^4 \left(5\,u^{1/2} - u^{3/2}\right)\,du \\ &= \left.\left(\frac{10}{3}\,u^{3/2} - \frac{2}{5}\,u^{5/2}\right)\right|_{u=1}^{u=4} \\ &= \left.\left(\frac{2}{15}\,u^{3/2}\,(25-3\,u)\right)\right|_{u=1}^{u=4} \\ &= \frac{164}{15}. \end{aligned}$$

4. Letting $u = 5 - x, du = -dx$, we have
$$\begin{aligned} \int_1^4 x^2\,\sqrt{5-x}\,dx &= \int_1^4 (5-u)^2\,\sqrt{u}\,du \\ &= \int_1^4 \left(25\,\sqrt{u} - 10\,u^{3/2} + u^{5/2}\right)\,du \\ &= \left.\frac{50}{3}\,u^{3/2} - 4\,u^{5/2} + \frac{2}{7}\,u^{7/2}\right|_{u=1}^{u=4} \\ &= \left.\frac{1}{21}\,u^{3/2}\,\left(350 - 84\,u + 6\,u^2\right)\right|_{u=1}^{u=4} \\ &= \frac{608}{21}. \end{aligned}$$

5. Letting $u = 4 - x^2, du = -2\,x\,dx$, we have
$$\int_0^2 x\,\sqrt{4-x^2}\,dx = -\frac{1}{2}\int_4^0 \sqrt{u}\,du = \frac{8}{3}.$$

6. Letting $u = 4 - x^2, du = -2\,x\,dx$, we have $x^2 = 4 - u$ and
$$\int_0^2 x^3\,\sqrt{4-x^2}\,dx = \int_0^2 x^2\,\sqrt{4-x^2}\,x\,dx = -\frac{1}{2}\int_4^0 (4-u)\,\sqrt{u}\,du = -\frac{1}{2}\int_4^0 \left(4\,u^{1/2} - u^{3/2}\right)\,du = \frac{64}{15}.$$

7. Letting $u = x^2 + 1, du = 2\,x\,dx$, we have
$$\begin{aligned}\int_0^1 \frac{2\,x-1}{x^2+1}\,dx &= \int_0^1 \left(\frac{2\,x}{x^2+1} - \frac{1}{x^2+1}\right)\,dx\\ &= \int_0^1 \frac{2\,x}{x^2+1}\,dx - \int_0^1 \frac{1}{x^2+1}\,dx\\ &= \int_1^2 \frac{1}{u}\,du - \int_0^1 \frac{1}{x^2+1}\,dx\\ &= \ln(u)\Big|_{u=1}^{u=2} - \arctan(x)\Big|_{x=0}^{x=1}\\ &= \ln(2) - \frac{\pi}{4}.\end{aligned}$$

8. We begin with the observation
$$\frac{x^2+4\,x+2}{x^2+1} = \frac{\left(x^2+1\right)+4\,x+1}{x^2+1} = 1 + \frac{4\,x}{x^2+1} + \frac{1}{x^2+1}.$$
Therefore, letting $u = x^2 + 1, du = 2\,x\,dx$, we have
$$\begin{aligned}\int_0^1 \frac{x^2+4\,x+2}{x^2+1}\,dx &= \int_0^1 \left(1 + \frac{4\,x}{x^2+1} + \frac{1}{x^2+1}\right)\,dx\\ &= \int_0^1 1\,dx + 2\int_0^1 \frac{2\,x}{x^2+1}\,dx + \int_0^1 \frac{1}{x^2+1}\,dx\\ &= 1 + 2\int_1^2 \frac{1}{u}\,du + \int_0^1 \frac{1}{x^2+1}\,dx\\ &= 1 + 2\,\ln(u)\Big|_{u=1}^{u=2} - \arctan(x)\Big|_{x=0}^{x=1}\\ &= 1 + 2\,\ln(2) + \frac{\pi}{4}.\end{aligned}$$

9. Letting $u = 8 - 7\,x, du = -7\,dx$, we have
$$\int_0^1 x\,\sqrt[3]{8-7\,x}\,dx = -\frac{1}{49}\int_8^1 (8-u)\,u^{1/3}\,du = \frac{1}{49}\int_1^8 \left(8\,u^{1/3} - u^{4/3}\right)\,du = \frac{249}{343}.$$

10. 27/140

11. Letting $u = \ln(x), du = (1/x)\,dx$, we have
$$\int_1^2 \frac{\ln(x)}{x}\,dx = \int_0^{\ln(2)} u\,du = \frac{1}{2}\,(\ln(2))^2.$$

12. Letting $u = \ln(x), du = (1/x)\,dx$, we have

$$\int \frac{1}{x\ln(x)}\,dx = \int \frac{1}{u}\,du = \ln(|\,u\,|) + C = \ln(|\,\ln(x)\,|) + C.$$

13. Letting $u = \ln(x), du = (1/x)\,dx$, we have

$$\int_1^e \frac{1}{x}\,\ln\left(\frac{1}{x}\right)\,dx = \int_1^e \frac{1}{x}\,\ln\left(x^{-1}\right)\,dx = -\int_1^e \frac{1}{x}\,\ln(x)\,dx = -\int_0^1 u\,du = -\frac{1}{2}.$$

14. Letting $u = \sqrt{x}, du = \big(1/(2\sqrt{x})\big)\,dx$, we have

$$\int_1^4 \frac{\exp(\sqrt{x})}{\sqrt{x}}\,dx = 2\int_1^2 \exp(u)\,du = 2\,e\,(e-1).$$

15. Letting $u = \tan(x), du = \sec^2(x)\,dx$, we have

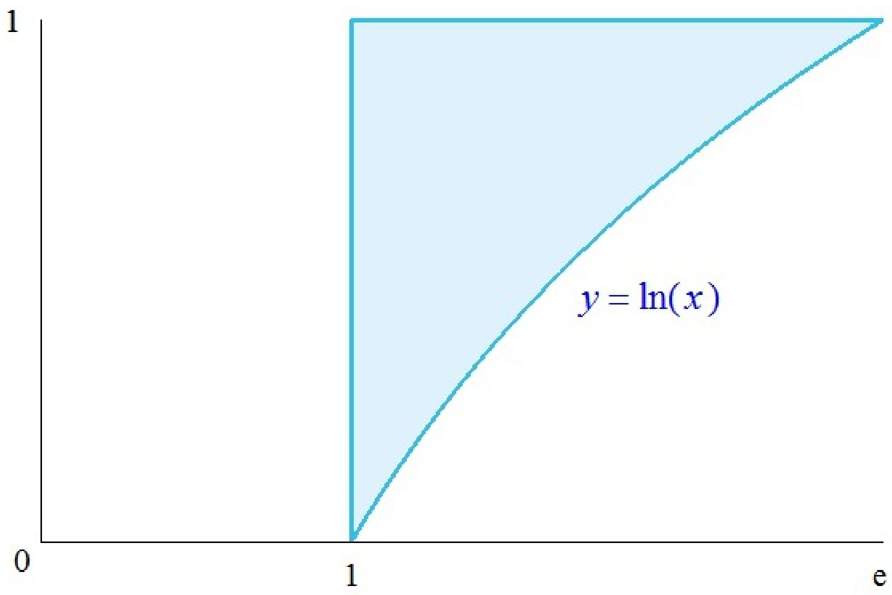
$$\int_0^{\pi/4} \tan(x)\sec^2(x)\,dx = \int_0^1 u\,du = \frac{1}{2}.$$

16. Letting $u = \tan(x), du = \sec^2(x)\,dx$, we have

$$\int_0^{\pi/3} \tan^2(x)\sec^2(x)\,dx = \int_0^{\sqrt{3}} u^2\,du = \sqrt{3}.$$

Solutions: Areas Between Curves (Corresponds to Stewart 6.1)

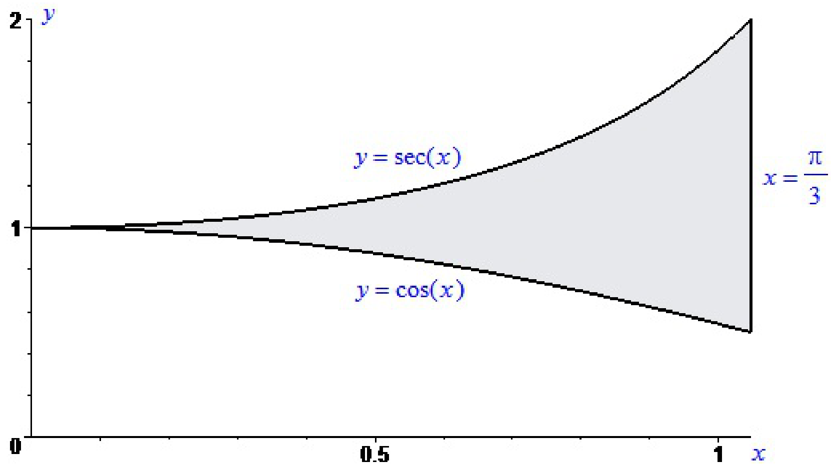
1. The region is shown in the figure.



The area is

$$\int_1^e \big(1 - \ln(x)\big)\, dx = \int_0^1 \big(\exp(y) - 1\big)\, dy = e - 2.$$

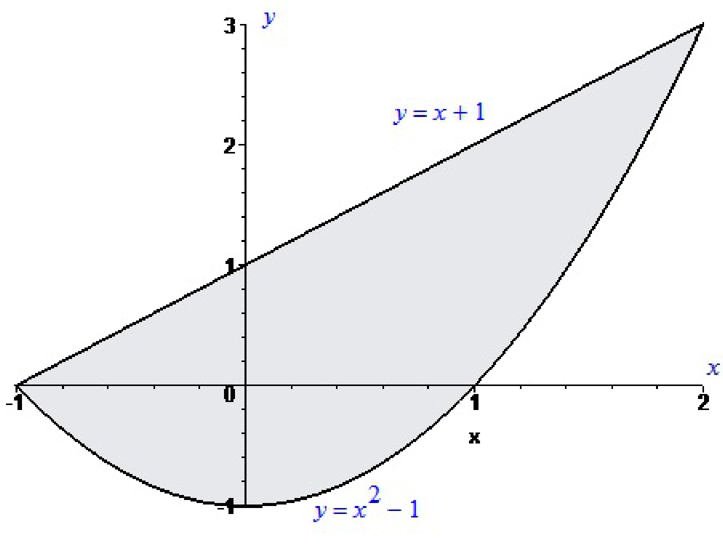
2. The region is shown in the figure.



The area is

$$\int_0^{\pi/3} \sec(x) - \cos(x)\, dx = \ln(|\sec(x) + \tan(x)|) - \sin(x)\Big|_0^{\pi/3} = \ln\left(2 + \sqrt{3}\right) - \frac{\sqrt{3}}{2}.$$

3. The region is shown in the figure.



The area is

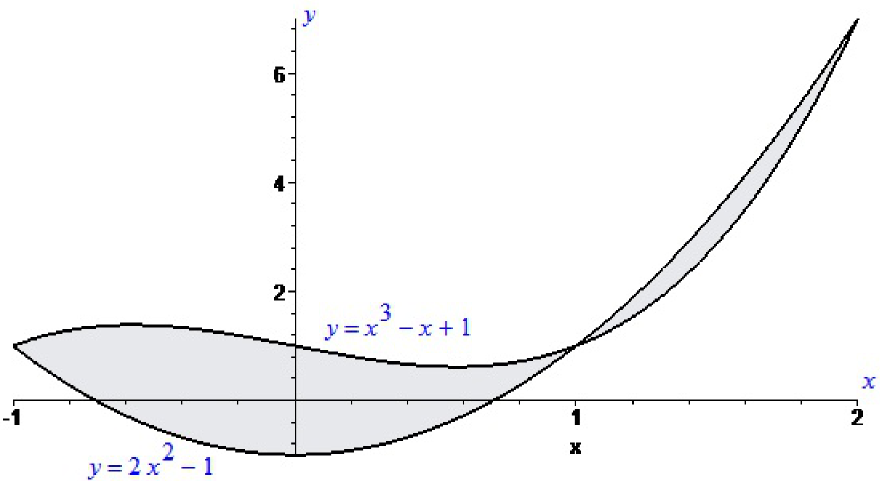
$$\int_{-1}^{2} \left((x+1) - \left(x^2 - 1\right)\right) dx = \frac{1}{2}x^2 + 2x - \frac{1}{3}x^3\Big|_{-1}^{2} = \frac{9}{2}.$$

4. The two given curves intersect at the point $(-1, 1)$ for the left value $x = -1$ of the region and at the point $(2, 7)$ for the right value $x = 2$ of the region. Because a cubic may intersect a quadratic in as

many as three points, we must investigate whether there is an additional point of intersection between the two already observed. We can answer this question by determining if there is a linear term $x - c$ such that

$$x^3 - x + 1 - \left(2\,x^2 - 1\right) = x^3 - 2x^2 - x + 2 = (x+1)(x-2)(x-c).$$

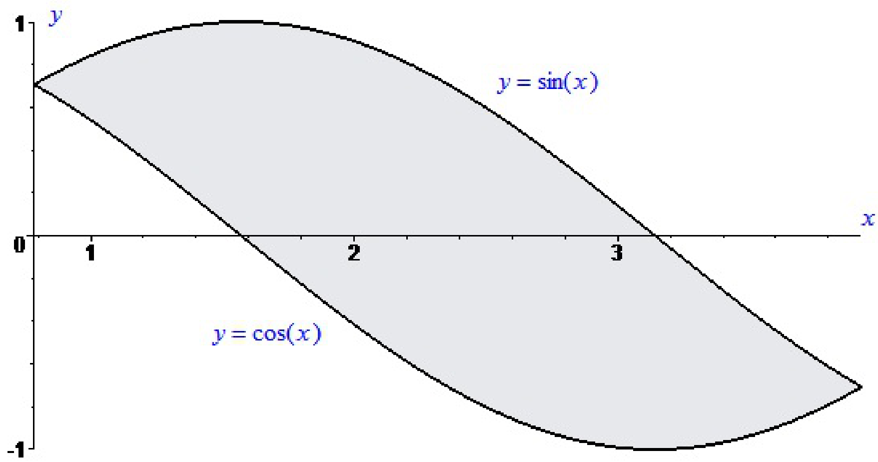
The constant term of the middle expression is 2, and the constant term of the expression on the right is $(1)(-2)(-c)$, or $2c$. We see that, if there is such a factoring, then c must be 1. We can then verify by expansion that the factoring does indeed hold for $c = 1$. It follows that $(1, 1)$ is another point of intersection of the two curves. It is where we will break up the integration. The region is shown in the figure.



The area is

$$\int_{-1}^{1}\left(\left(x^3 - x + 1\right) - \left(2\,x^2 - 1\right)\right)dx + \int_{1}^{2}\left(\left(2\,x^2 - 1\right) - \left(x^3 - x + 1\right)\right)dx = \frac{37}{12}.$$

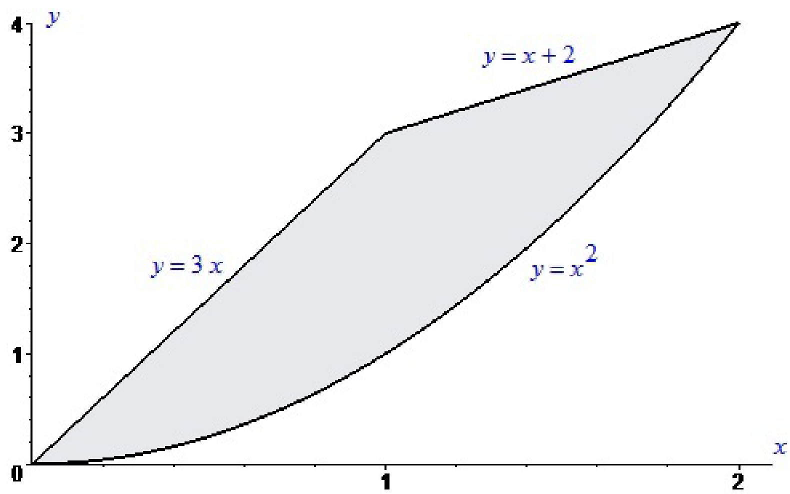
5. The region is shown in the figure.



The area is

$$\int_{\pi/4}^{5\pi/4} \big(\sin(x) - \cos(x)\big)\, dx = 2\sqrt{2}.$$

6. The region is shown in the figure.



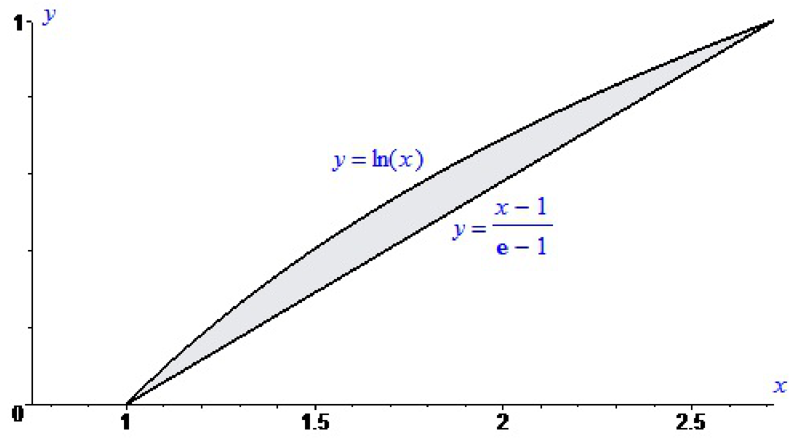
The area is

$$\int_0^1 \left(3\,x - x^2\right)\, dx + \int_1^2 \left((x+2) - x^2\right)\, dx = \frac{7}{3}.$$

7. The area is

$$\int_0^3 \left(\sqrt{y} - \frac{y}{3}\right)\, dy + \int_3^4 \left(\sqrt{y} - (y-2)\right)\, dy = \frac{7}{3}.$$

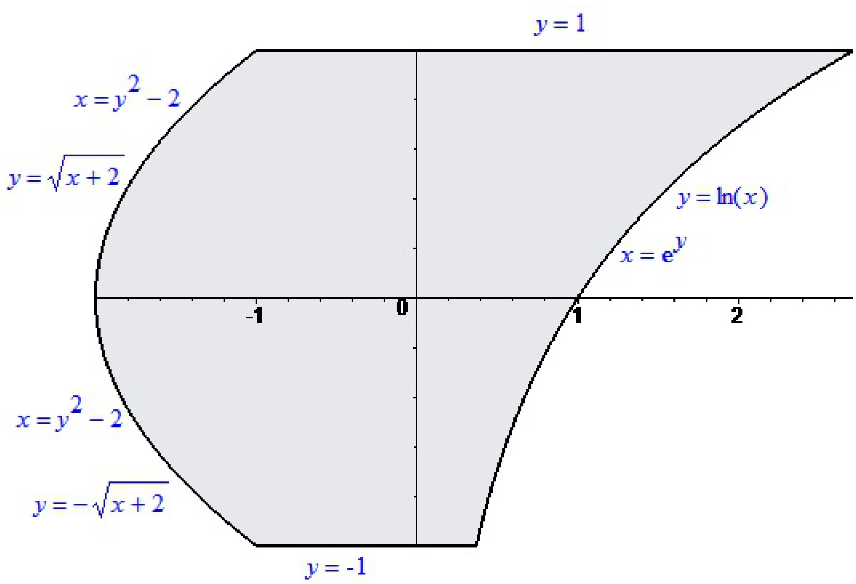
8. The region is shown in the figure.



The area is $\int_1^e \left(\ln(x) - \frac{x-1}{e-1}\right)\,dx$. If you know the aniderivative of $\ln(x)$, and it is not assumed that you do at this point, then you can calculate the area using this integral formula. The area is $(3-e)/2$. Otherwise, you can calculate the area this way:

$$\int_0^1 \Big(\big((e-1)\,y+1\big) - \exp(y)\Big)\,dy = \left(\frac{1}{2}\,(e-1)\,y^2 + y - e^y\right)\Bigg|_{y=0}^{y=1} = \frac{1}{2}\,(3-e).$$

9. The region is shown in the figure.



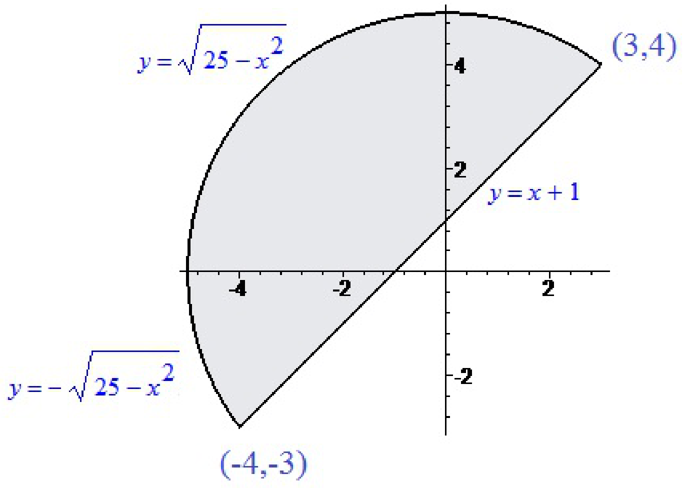
The area is

$$\int_{-1}^{1}\left(e^y-(y^2-2)\right)dy=\frac{10}{3}+e-\frac{1}{e}.$$

10. The area is

$$\int_{-2}^{-1}\left(\sqrt{x+2}-(-\sqrt{x+2})\right)dx+\int_{-1}^{1/e}(1-(-1))\,dx+\int_{1/e}^{e}(1-\ln(x))\,dx.$$

11. The region is shown in the figure.



The area is

$$\int_{-5}^{-4}\left(\sqrt{25-x^2}-\left(-\sqrt{25-x^2}\right)\right)dx+\int_{-4}^{3}\left(\sqrt{25-x^2}-(x+1)\right)dx.$$

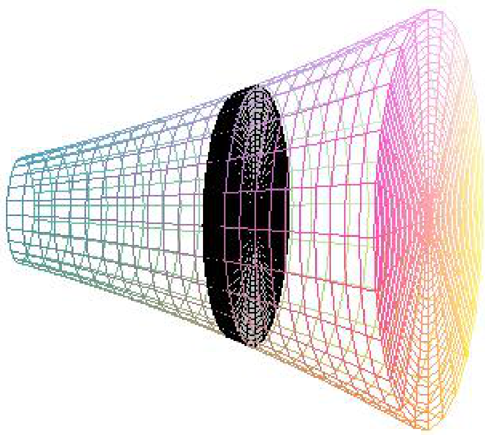
12. (i) 4, 4, 254.4; (ii) 8, 2, 253.62

13. (i) 3, 1.7, 61.455; (ii) 6, 0.85, 48.586

Solutions: Volumes by Disks and Washers (Corresponds to Stewart 6.2)

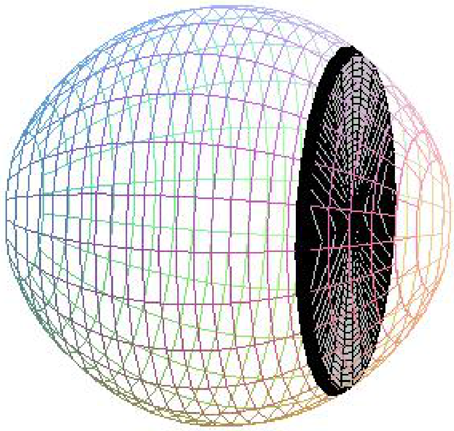
1. The requested volume V is

$$V = \int_1^2 \pi \left(x^2\right)^2 \, dx = \frac{31}{5}\,\pi.$$



2. The requested volume V is

$$V = \int_{-\pi/2}^{\pi/2} \pi \left(\sqrt{\cos(x)}\right)^2 \, dx = 2\,\pi.$$



3. The requested volume V is

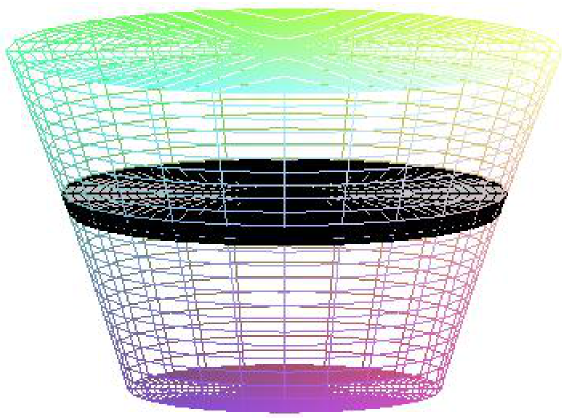
$$V = \int_0^{\pi/4} \pi \, (\sec(x))^2 \, dx = \pi.$$

4. The requested volume V is

$$V = \int_4^9 \pi \, \left(\sqrt{x}\right)^2 \, dx = \frac{65}{2} \, \pi.$$

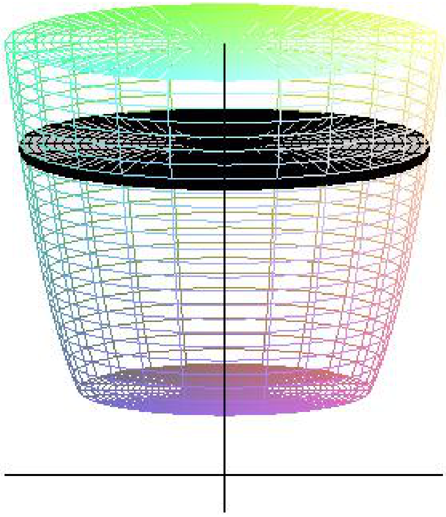
5. The requested volume V is

$$V = \int_3^4 \pi \, \left(y^2\right)^2 \, dy = \frac{781}{5} \, \pi.$$



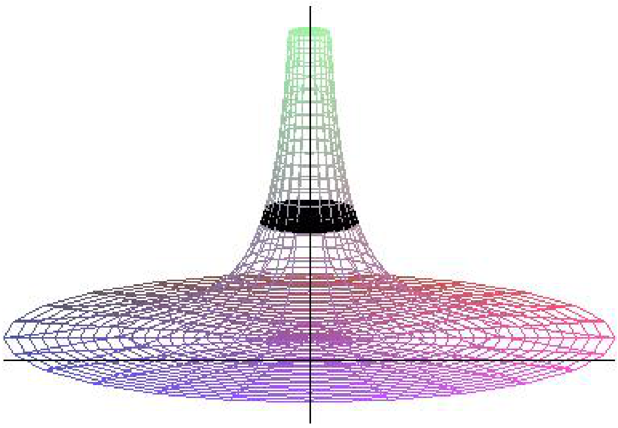
6. The requested volume V is

$$V = \int_{16}^{81} \pi \, \left(y^{1/4}\right)^2 \, dy = \frac{1330}{3} \, \pi.$$



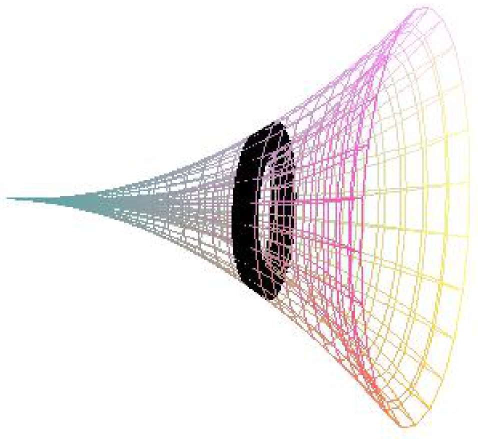
7. The requested volume V is

$$V = \int_{1/3}^{5} \pi \left(\frac{1}{y}\right)^2 dy = \frac{14}{5}\pi.$$



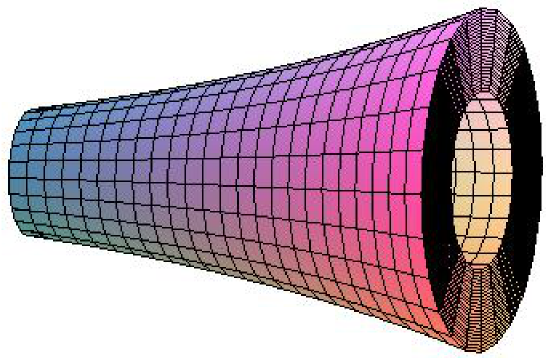
8. The requested volume V is

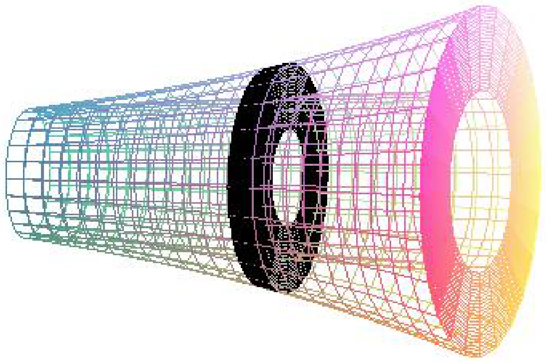
$$V = \int_{0}^{2} \pi \left(2x^2\right)^2 - \pi \left(x^3\right)^2 dx = \frac{256}{35}\pi.$$



9. The requested volume V is

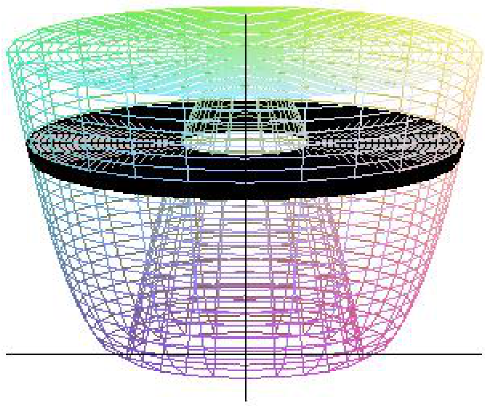
$$V = \int_0^1 \pi\ (e^x)^2 - \pi\ \left(\sqrt{x+1}\right)^2\ dx = \frac{1}{2}\,\pi\ \left(e^2 - 4\right).$$





10. Notice that $y = 4 - x^2$ is, for $0 \le x \le 2$, equivalent to $x = \sqrt{4-y}$, $0 \le y \le 4$ and $y = (x-2)^2$ is equivalent to $x = 2 + \sqrt{y}$. The requested volume V is

$$V = \int_0^4 \pi\ (2+\sqrt{y})^2 - \pi\ \left(\sqrt{4-y}\right)^2\ dy = \frac{112}{3}\,\pi.$$



11. Notice that $y = \sqrt{3\,x}$ is equivalent to $x = y^2/3$. The requested volume V is

$$V = \int_0^3 \left(\pi\, y^2 - \pi\ \left(\frac{y^2}{3}\right)^2\right)\ dy = \frac{18}{5}\,\pi.$$

12. Notice that $y = \ln(x)$ is equivalent to $x = \exp(y)$ and $y = 2 - x/e$ is equivalent to $x = e\,(2-y)$. The requested volume V is

$$V = \int_0^1 \left(\pi\big(e\,(2-y)\big)^2 - \pi\ \exp(2y)\right)\ dy = \frac{1}{6}\,\pi\ \left(11\,e^2 + 3\right).$$

13. $\displaystyle\int_1^4 \left(\pi\ (\sqrt{y}+2)^2 - \pi\ \left(\frac{1}{3}\,(y+2)+2\right)^2\right)\,dy$

14. $\displaystyle\int_1^4 \left(\pi\ \left(2-\frac{1}{3}\,(y+2)\right)^2 - \pi\ (2-\sqrt{y})^2\right)\,dy$

15. $\displaystyle\int_1^2 \left(\pi\ \left((3\,x-2)-\frac{1}{2}\right)^2 - \pi\ \left(x^2-\frac{1}{2}\right)^2\right)\,dx$

16. $\displaystyle\int_1^2 \left(\pi\ \left(6-x^2\right)^2 - \pi\ (6-(3\,x-2))^2\right)\,dx$

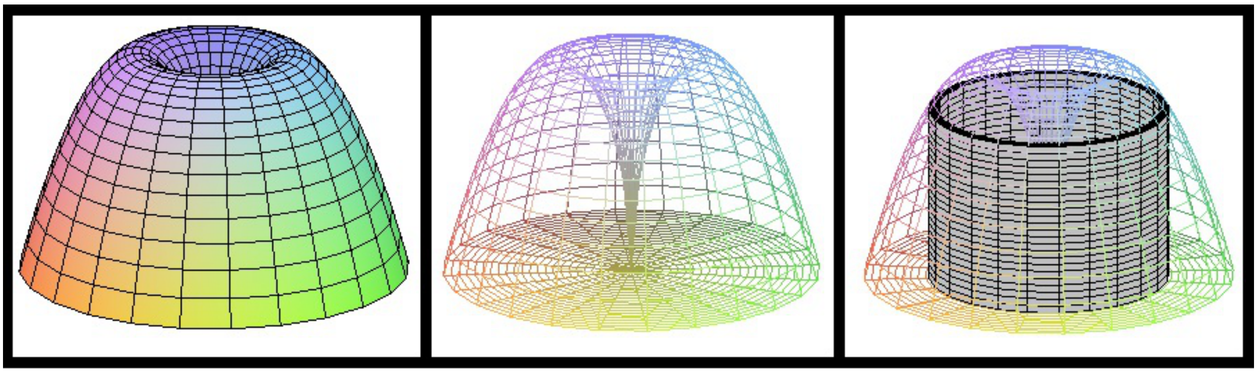
17. $\displaystyle\int_{-2}^1 \left(\pi\left(5-(x+2)\right)^2 - \pi\ \left(5-\left(4-x^2\right)\right)^2\right)\,dx$

18. $\displaystyle\int_0^3 \left(\pi\ \left(5+\sqrt{4-y}\right)^2 - \pi\ (5-(y-2))^2\right)\,dy$

Solutions: Volumes by Cylindrical Shells (Corresponds to Stewart 6.3)

1. The requested volume V is

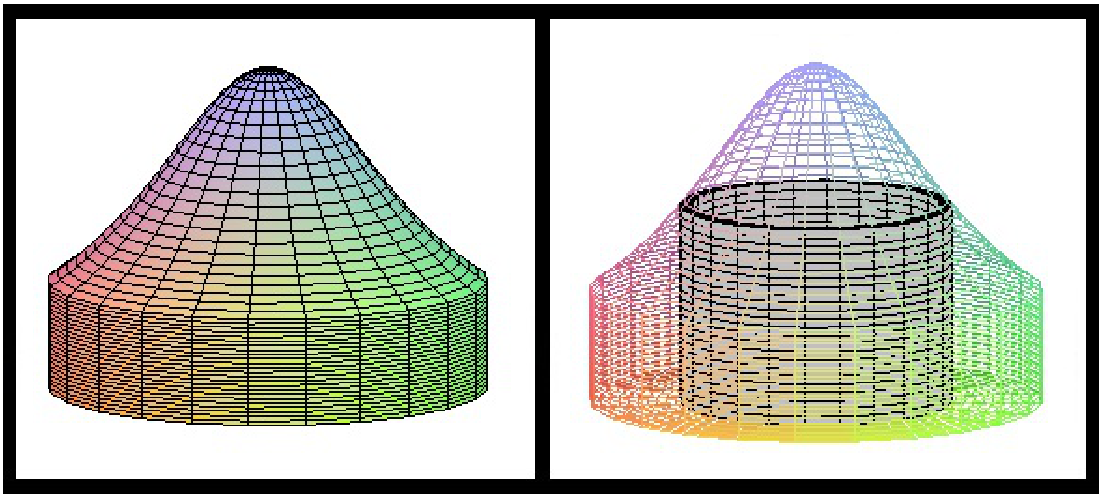
$$V = \int_0^1 2\,\pi\,x\,\left(\sqrt{x} - x^2\right)\,dx = \frac{3}{10}\,\pi.$$



Left panel: Solid of revolution; Middle panel: See-through solid; Right panel: See-through solid with shell

2. The requested volume V is

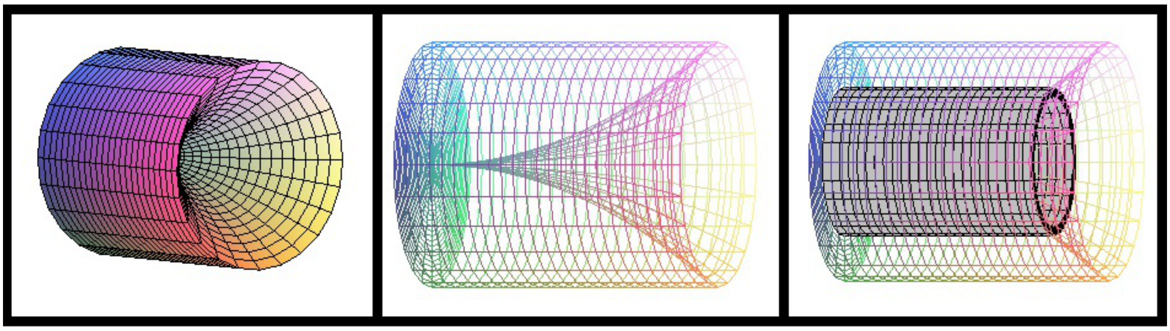
$$V = \int_0^2 2\,\pi\,x\,\frac{4}{2+x^2}\,dx = 4\,\pi\,\ln(3).$$



Left panel: Solid of revolution; Right panel: See-through solid with shell

3. The requested volume V is

$$V = \int_0^4 2\,\pi\, y\,\sqrt{y}\,dy = \frac{128}{5}\,\pi.$$



Left panel: Solid of revolution; Middle panel: See-through solid; Right panel: See-through solid with shell

4. First note that $y = x - 3$ is equivalent to $x = y + 3$ and $y = \sqrt{x}$ is equivalent to $x = y^2$. The requested volume V is

$$V = \int_1^2 2\,\pi\, y\,\left((y+3) - y^2\right)\,dy = \frac{37}{6}\,\pi.$$

5. $\displaystyle\int_0^{\pi} 2\,\pi\,(4 - x)\,\sin(x)\,dx$

6. $\displaystyle\int_0^{\pi/4} 2\,\pi\,(x + \pi)\,(\cos(x) - \sin(x))\,dx$

7. $\displaystyle\int_1^2 2\,\pi\,(x + 2)\left((3\,x - 2) - x^2\right)dx$

8. $\displaystyle\int_1^2 2\,\pi\,(2-x)\left((3\,x-2)-x^2\right)dx$

9. $\displaystyle\int_1^4 2\,\pi\,\left(y-\frac{1}{2}\right)\,\left(\sqrt{y}-\frac{1}{3}(y+2)\right)\,dy$

10. $\displaystyle\int_1^4 2\,\pi\,\,(6-y)\,\left(\sqrt{y}-\frac{1}{3}(y+2)\right)\,dy$

Solutions: Work (Corresponds to Stewart 6.4)

1. The work done is $\int_0^{7\,\text{in}} 8\frac{\text{lb}}{\text{in}}\, x\, dx$, or 196 ft-in, or 49/3 ft-lb.

2. The work done is $\int_0^{12\,\text{cm}} 20\frac{\text{N}}{\text{cm}}\, x\, dx$, or 1440 N-cm, or 14.4 J.

3. The given information allows us to solve for the spring constant:

$$8\text{J} \ = \int_{0\,\text{m}}^{0.02\,\text{m}} kx\, dx = \frac{1}{2}kx^2\Big|_{x=0\,\text{m}}^{x=0.02\,\text{m}} = 0.0002k\ \text{m}^2,$$

   which implies that $k = 40000$ N/m. The force necessary to maintain the spring at $x = 0.02\,\text{m}$ is $k \times 0.02\,\text{m}$, or $40000 \times 0.02\,\text{N}$, or 800 N.

4. The given information allows us to solve for the spring constant:

$$5\text{J} \ = \int_{0\,\text{m}}^{0.2\,\text{m}} kx\, dx = \frac{1}{2}kx^2\Big|_{x=0\,\text{m}}^{x=0.2\,\text{m}} = 0.02k\ \text{m}^2$$

   or

$$k = \frac{5}{0.02}\,\text{J}\cdot\text{m}^{-2} = 250\,\text{N}\cdot\text{m}\cdot\text{m}^{-2} = 250\,\text{N}\cdot\text{m}^{-1}.$$

   The required work W is calculated as follows:

$$W = \int_{0.2\,\text{m}}^{0.4\,\text{m}} kx\, dx = \frac{1}{2}kx^2\Big|_{x=0.2\,\text{m}}^{x=0.4\,\text{m}} = \frac{1}{2}k\left(0.4^2\,\text{m}^2 - 0.2^2\,\text{m}^2\right) = 0.06k\,\text{m}^2.$$

   The answer can now be obtained by substituting in the value for the spring constant:

$$W = (0.06)\,(250\,\text{N}\cdot\text{m}^{-1})\,\text{m}^2 = 15\,\text{N}\cdot\text{m} = 15\,\text{J}.$$

5. The equation

$$240\text{N} = F = k\,x = k\,(10)\,\text{cm} = k\,\left(\frac{1}{10}\right)\,\text{m}$$

   gives us $k = 2400\,\text{N/m}$. The requested work W is

$$W = \int_0^{10\,\text{cm}} k\,x\,dx = 2400\,\frac{\text{N}}{\text{m}}\int_0^{1/10\,\text{m}} x\,dx = 1200\,\frac{\text{N}}{\text{m}}\,x^2\Big|_{x=0}^{1/10\,\text{m}} = 12\,\frac{\text{N}}{\text{m}}\,\text{m}^2 = 12\,\text{J}.$$

6. The equation

$$280\text{lb} = F = k\,x = k\,(4)\,\text{in} = k\left(\frac{4}{12}\right)\text{ft}$$

gives us $k = 840\,\text{lb/ft}$. The requested work W is

$$W = \int_{4\,\text{in}}^{8\,\text{in}} k\,x\,dx = 840\,\frac{\text{lb}}{\text{ft}}\int_{4/12\,\text{ft}}^{8/12\,\text{ft}} x\,dx = 420\,\frac{\text{lb}}{\text{ft}}\,x^2\Big|_{1/3\,\text{ft}}^{2/3\,\text{ft}} = 420\times\frac{2^2-1^2}{3^2}\,\frac{\text{lb}}{\text{ft}}\,\text{ft}^2 = 140\,\text{ft lb}.$$

7. The given information can be expressed by the equation

$$40\,\text{J} = \int_{0\,\text{cm}}^{8\,\text{cm}} k\,x\,dx = \frac{1}{2}\,k\,x^2\Big|_{0\,\text{m}}^{8/100\,\text{m}} = \frac{64}{2}\,10^{-4}\,k\,\text{m}^2,$$

or $k = 12500\ \text{N/m}$. The work W done stretching the spring from 8 cm to 12 cm is given by

$$W = \int_{8\,\text{cm}}^{12\,\text{cm}} k\,x\,dx = \frac{1}{2}\,k\,x^2\Big|_{8/100\,\text{m}}^{12/100\,\text{m}} = \frac{1}{2}\,12500\,\frac{\text{N}}{\text{m}}\,\frac{12^2-8^2}{10^4}\,\text{m}^2 = 50\,\text{J}.$$

8. The given information can be expressed by the equation

$$\frac{2}{3}\,\text{ft lb} = \int_{0\,\text{in}}^{2\,\text{in}} k\,x\,dx = \frac{1}{2}\,k\,x^2\Big|_{0\,\text{ft}}^{2/12\,\text{ft}} = \frac{1}{72}\,k\,\text{ft}^2,$$

or $k = 48\ \text{lb/ft}$. The force F required to maintain the spring at $x = 2\,\text{in}$ is

$$F = k\times 2\,\text{in} = 48\,\frac{\text{lb}}{\text{ft}}\times 2\,\text{in} = 48\,\frac{\text{lb}}{\text{ft}}\times\frac{1}{6}\,\text{ft} = 8\,\text{lb}.$$

9. The work W done is

$$\begin{aligned} W &= 62.42796\frac{\text{lb}}{\text{ft}^3}\int_{0\,\text{ft}}^{10\,\text{ft}} 15^2\text{ft}^2\,y\,dy \\ &= \frac{62.42796\times 15^2}{2}\,\frac{\text{lb}}{\text{ft}}\,y^2\Big|_{y=0\,\text{ft}}^{y=10\,\text{ft}} \\ &= 62.42796\times 15^2\times 50\,\text{ft lb} \\ &= 702314.55\,\text{ft lb}. \end{aligned}$$

10. The work W done is

$$\begin{aligned} W &= 9.80665\,\frac{\text{m}}{\text{s}^2}\,10^3\,\frac{\text{kg}}{\text{m}^3}\int_{1\,\text{m}}^{3\,\text{m}} 6\times 10\text{m}^2\,y\,dy \\ &= \frac{9806.65\times 6\times 10}{2}\,\frac{\text{kg}}{\text{s}^2}\,y^2\Big|_{y=1\,\text{m}}^{y=3\,\text{m}} \\ &= 9806.65\times\times 30\times\left(3^2-1^2\right)\,\frac{\text{kg}\,\text{m}^2}{\text{s}^2} \\ &= 2353596\,\text{J}. \end{aligned}$$

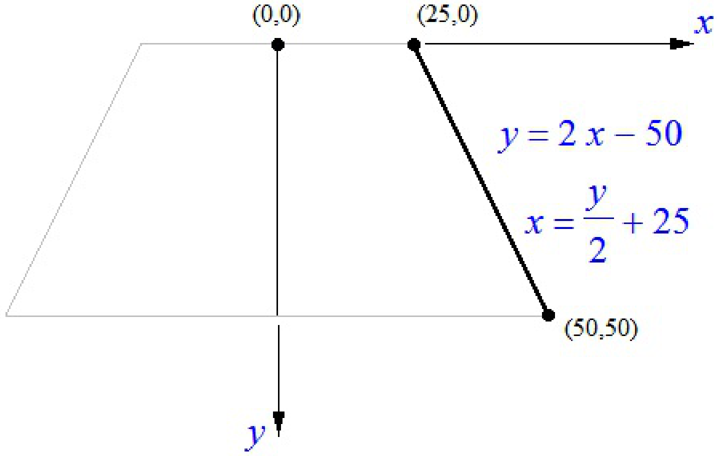
11. The work W done is

$$\begin{aligned} W &= 62.42796\frac{\text{lb}}{\text{ft}^3}\int_{2\,\text{ft}}^{2+8/2\,\text{ft}} 16\times 30\text{ft}^2\, y\,dy \\ &= \frac{62.42796\times 480}{2}\,\frac{\text{lb}}{\text{ft}}\,y^2\Big|_{y=2\,\text{ft}}^{y=6\,\text{ft}} \\ &= 62.42796\times 240\left(6^2-2^2\right)\text{ ft lb} \\ &= 479446.7\,\text{ft lb}. \end{aligned}$$

12. The work W done is

$$\begin{aligned} W &= 62.42796\frac{\text{lb}}{\text{ft}^3}\int_{0\,\text{ft}}^{6\,\text{ft}} 16\,\text{ft}\times 2\times\sqrt{36-y^2}\,y\,dy \\ &= -\,665.89824\,\left(36-y^2\right)^{3/2}\,\frac{\text{lb}}{\text{ft}^2}\Big|_{y=0\,\text{ft}}^{y=6\,\text{ft}} \\ &= 665.89824\times 6^3\text{ ft lb} \\ &= 143834\,\text{ft lb}. \end{aligned}$$

13.



The work W done is

$$W = 62.42796\frac{\text{lb}}{\text{ft}^3}\int_{0\,\text{ft}}^{50\,\text{ft}} \pi\times\left(\frac{y}{2}+25\,\text{ft}\right)^2\,y\,dy = 62.42796\,\pi\,\left(\frac{1}{16}\,y^4+\frac{25}{3}\,y^3+\frac{625}{2}\,y^2\right)\Big|_{y=0}^{y=50} = 434126920\,\text{ft lb}.$$

14. The work W done is given by

$$W = \int_{0\,\text{ft}}^{50\,\text{ft}} 3\,\frac{\text{lb}}{\text{ft}}\,y\,dy = \frac{3}{2}\,\frac{\text{lb}}{\text{ft}}y^2\Big|_{0\,\text{ft}}^{50\,\text{ft}} = \frac{3}{2}\,\frac{\text{lb}}{\text{ft}}50^2\,\text{ft}^2 = 3750\,\text{ft lb}.$$

15. The 20 ft closest the ground weighs 60 lb and is lifted 30 ft. Therefore, the work done on the 20 ft closest the ground is (60)(30) ft lb. The total work W done is given by

$$W = (60)(30)\text{ ft lb} + \int_{0\text{ ft}}^{30\text{ ft}} 3\,\frac{\text{lb}}{\text{ft}}\,y\,dy = 1800\text{ ft lb} + \frac{3}{2}\frac{\text{lb}}{\text{ft}}y^2\Big|_{0\text{ ft}}^{30\text{ ft}} = 1800\text{ ft lb} + \frac{3}{2}\frac{\text{lb}}{\text{ft}}30^2\text{ ft}^2 = 3150\text{ ft lb}.$$

16. There are three components to the total work: the work done lifting the load, the work done lifting the 40 - 24, or 16, meters of cable closest the ground, and the work done lifting the 24 meters of cable closest to the roof. The total work W done is given by

$$W = 300 \times 9.80665 \times 24\text{J} + 220 \times 16 \times 24\text{J} + \int_{0\text{ m}}^{24\text{ m}} 220\,\frac{\text{N}}{\text{m}}\,y\,dy = 218447.88\text{ J}.$$

(Make sure you understand why one summand has the factor 9.80665 but the other two do not.)

17. The total work W done is given by

$$W = (300)(100)\text{ ft lb} + \int_{0\text{ ft}}^{100\text{ ft}} 20\,\frac{\text{lb}}{\text{ft}}\,y\,dy = 130\,000\text{ ft lb}.$$

18. When the lift begins, there are 100 − 30, or 70, feet of cable hanging over the side of the building. The load and the lowest 20 feet of cable are lifted 50 feet. There are three components to the total work: the work done lifting the load, the work done lifting the 20 feet of cable closest the ground, and the work done lifting the 50 feet of cable closest to the roof. The total work W done is given by

$$W = 300 \times 50\text{ft lb} + 20\text{ ft} \times 20\,\frac{\text{lb}}{\text{ft}} \times 50\text{ ft} + \int_{0\text{ m}}^{50\text{ ft}} 20\,\frac{\text{lb}}{\text{ft}}\,y\,dy = 60\,000\text{ ft-lb}.$$

Solutions: Average Value (Corresponds to Stewart 6.5)

1. $f_{\text{avg}} = 3/(2\,\pi)$, $\quad c = \arcsin(3/(2\,\pi)) \approx 0.497767$.
2. $f_{\text{avg}} = \dfrac{1}{2}\,\ln(2)$, $\quad c = \dfrac{2}{\ln(2)} \approx 2.88539$.
3. $f_{\text{avg}} = 2/3$, $\quad c = 4/9$.
4. $f_{\text{avg}} = 8$, $\quad c = \sqrt{7} - 1 \approx 1.64575$.
5. $f_{\text{avg}} = 20$, $\quad c = \sqrt{3}$.
6. $f_{\text{avg}} = 4/3$, $\quad c = 25/9$.
7. Making the substitution $u = 169 - x^2, du = -2\,x\,dx$, we have

$$f_{\text{avg}} = \frac{1}{12-5}\int_5^{12} x\,\sqrt{169 - x^2}\,dx = -\frac{1}{7}\cdot\frac{1}{2}\cdot\int_{144}^{25} u^{1/2}\,du = \frac{229}{3}.$$

We seek a value c between 5 and 12 such that $c\,\sqrt{169 - c^2} = 229/3$, or $c^2\left(169 - c^2\right) = (229/3)^2$, or $9\left(c^2\right)^2 - (9)(169)\,c^2 + (229)^2 = 0$. This is a quadratic equation with respect to the variable c^2. The quadratic formula gives $c^2 = \dfrac{169}{2} \pm \dfrac{7}{6}\sqrt{965}$. Both of these values give rise to a c in the interval $(5, 12)$. Therefore, we have

$$c = \sqrt{\frac{169}{2} \pm \frac{7}{6}\sqrt{965}}$$

with both values of c in the interval $(5, 12)$ and satisfying $f(c) = f_{\text{avg}}$. Twins! The values of c are, approximately, 6.946808 for the minus sign and 10.98826 for the plus sign.)

Solutions: Integration by Parts (Corresponds to Stewart 7.1)

1. Let $u = x$, $dv = \exp(x)\,dx$. Then $du = dx$, $v = \exp(x)$ and

$$\int x\ \exp(x)\,dx = u\,v - \int v\,du = x\ \exp(x) - \int \exp(x)\,dx = x\ \exp(x) - \exp(x) + C = (x-1)\ e^{x} + C.$$

2. Let $u = x$, $dv = \exp(-x)\,dx$. Then $du = dx$, $v = -\exp(-x)$ and

$$\int x\ \exp(-x)\,dx = u\,v - \int v\,du = -x\ \exp(-x) + \int \exp(-x)\,dx = -x\ \exp(-x) - \exp(-x) + C = -\,(x+1)\ e^{-x} + C.$$

3. Let $u = 2x+5$, $dv = \exp(x/3)\,dx$. Then $du = 2\,dx$, $v = 3\ \exp(x/3)$ and

$$\begin{aligned}\int (2x+5)\,e^{x/3}\,dx &= u\,v - \int v\,du \\ &= 3\ (2x+5)\ \exp(x/3) - (3)(2)\int \exp(x/3)\,dx \\ &= 3\ (2x+5)\ \exp(x/3) - 18\ \exp(x/3) + C \\ &= 3\ (2x-1)\ \exp(x/3) + C.\end{aligned}$$

4. Let $u = x$, $dv = \sin(x)\,dx$. Then $du = dx$, $v = -\cos(x)$ and

$$\int x \sin(x)\ dx = u\,v - \int v\,du = -x\ \cos(x) + \int \cos(x)\,dx = -x\ \cos(x) + \sin(x) + C.$$

5. Let $u = 4\,x + 2$, $dv = \sin(2x)\,dx$. Then $du = 4\,dx$, $v = -(1/2)\ \cos(2x)$, and

$$\int (4\,x+2)\ \sin(2\,x)\,dx = u\,v - \int v\,du = -\frac{1}{2}\,(4\,x+2)\ \cos(2x) + 4\,\frac{1}{2}\int \cos(2x)\,dx = -(2\,x+1)\ \cos(2x) + \sin(2x) + C.$$

6. Let $u = x$, $dv = 9\cos(3x)\,dx$. Then $du = dx$, $v = 3\sin(3x)$, and

$$\int 9x \cos(3x)\ dx = u\,v - \int v\,du = 3x\ \sin(3x) - 3\int \sin(3x)\,dx = 3x\ \sin(3x) + \cos(3x) + C.$$

7. Let $u = \ln(4x) = \ln(4) + \ln(x)$, $dv = x\,dx$. Then $du = (1/x)\,dx$, $v = (1/2)\,x^2$ and

$$\begin{aligned}\int x^2 \ln(4x)\;dx &= u\,v - \int v\,du \\ &= \frac{1}{2}x^2\;\ln(4x) - \frac{1}{2}\int \frac{x^2}{x}\,dx \\ &= \frac{1}{2}x^2\;\ln(4x) - \frac{1}{2}\int x\,dx \\ &= \frac{1}{2}x^2\;\ln(4x) - \frac{1}{4}x^2 + C \\ &= \frac{1}{4}x^2\big(2\ln(4x) - 1\big) + C.\end{aligned}$$

Actually, this calculation could have been done in much greater generality. Suppose that a and p are constants with $a \neq 0$ and $p \neq -1$. Setting $u = \ln(ax)$, $dv = x^p\,dx$, we have $du = (1/x)\,dx$, $v = (1/(p+1))\,x^{p+1}$, and

$$\begin{aligned}\int x^p \ln(a\,x)\;dx &= u\,v - \int v\,du \\ &= \frac{1}{p+1}\,x^{p+1}\;\ln(a\,x) - \frac{1}{p+1}\int \frac{x^{p+1}}{x}\,dx \\ &= \frac{1}{p+1}\,x^{p+1}\;\ln(a\,x) - \frac{1}{p+1}\int x^p\,dx \\ &= \frac{1}{p+1}\,x^{p+1}\;\ln(a\,x) - \frac{1}{(p+1)^2}\,x^{p+1} + C \\ &= \frac{1}{p+1}\,x^{p+1}\left(\ln(a\,x) - \frac{1}{p+1}\right) + C.\end{aligned}$$

8. Let $u = \ln(x)$, $dv = 9x^2\,dx$. Then $du = (1/x)\,dx$, $v = 3\,x^3$ and

$$\int 9\,x^2 \ln(x)\;dx = x^3\;(3\ln(x) - 1) + C,$$

which agrees with the result of the general calculation with which the solution to Exercise 7 concludes.

9. Let $u = \ln(x)$, $dv = x^{-2}\,dx$. Then $du = (1/x)\,dx$, $v = -x^{-1}$, and

$$\int \frac{\ln(x)}{x^2}\,dx = -\frac{(1 + \ln(x))}{x} + C,$$

which agrees with the result of the general calculation with which the solution to Exercise 7 concludes.

10. Setting $a = 2$ and $p = 3$, follow the general calculation with which the solution to Exercise 7 concludes: $\int 16\,x^3 \ln(2x)\;dx = x^4\;(4\ln(2x) - 1) + C$.

11. With $u = x$ and $dv = e^{-x/2}$, we have $du = dx$, $v = -2\,e^{-x/2}$, and

$$\int_0^2 xe^{-x/2}\,dx = -2x\,e^{-x/2}\;\Big|_{x=0}^{x=2} + 2\int_0^2 e^{-x/2}\,dx = \Big(-2x\,e^{-x/2} \quad - 4\,e^{-x/2}\Big)\Big|_{x=0}^{x=2} = 4 - \frac{8}{e}.$$

12. With $u = x$ and $dv = \cos(x)$, we have $du = dx$, $v = \sin(x)$, and

$$\int_0^\pi x\cos(x)\;dx = x\;\sin(x)\;\Big|_{x=0}^{x=\pi} - \int_0^\pi \sin(x)\,dx = \Big(x\;\sin(x) \quad + \cos(x)\Big)\Big|_{x=0}^{x=\pi} = -1 - 1 = -2.$$

13. With $u = x$ and $dv = 4\sin(x/3)$, we have $du = dx$, $v = -12\cos(x/3)$, and

$$\begin{aligned}\int_0^{\pi/2} 4\,x\,\sin\left(\frac{x}{3}\right)\,dx &= -12\,x\,\cos\left(\frac{x}{3}\right)\ \Big|_{x=0}^{x=\pi/2} + 12\int_0^{\pi/2}\cos\left(\frac{x}{3}\right)\,dx \\ &= \left(-12\,x\,\cos\left(\frac{x}{3}\right)\ + 36\,\sin\left(\frac{x}{3}\right)\right)\Big|_{x=0}^{x=\pi/2} \\ &= -12\,\frac{\pi}{2}\,\cos\left(\frac{\pi}{6}\right) + 36\,\sin\left(\frac{\pi}{6}\right) \\ &= 18 - 3\sqrt{3}\,\pi.\end{aligned}$$

14. Integrate by parts with $u = \ln(x)$ and $dv = dx$. Then

$$\int_1^e \ln(x)\,dx = x\,\ln(x)\ \Big|_1^e - \int_1^e x\,\frac{1}{x}\,dx = \left(x\,\ln(x)\ \ - x\right)\Big|_1^e = (e-e)-(0-1) = 1.$$

15. Integrating by parts with $u = \ln(3x)$ and $dv = x\,dx$, we have

$$\begin{aligned}\int_{1/3}^1 x\,\ln(3x)\,dx &= \frac{1}{2}\,x^2\,\ln(3x)\Big|_{1/3}^1 - \frac{1}{2}\int_{1/3}^1 x^2\frac{1}{x}\,dx \\ &= \left(\frac{1}{2}\,x^2\,\ln(3x) - \frac{1}{4}\,x^2\right)\Big|_{1/3}^1 \\ &= \frac{1}{2}\,\ln(3) - \frac{1}{4} + \frac{1}{4}\cdot\frac{1}{9} \\ &= \frac{1}{2}\,\ln(3) - \frac{2}{9}.\end{aligned}$$

$\ln(3)/2 - 2/9$

16. Let $u = \ln(x)$ and $dv = (1/\sqrt{x})\,dx$. Then, $du = (1/x)\,dx$, $v = 2\sqrt{x}\,dx$, and

$$\int \frac{\ln(x)}{\sqrt{x}}\,dx = 2\sqrt{x}\,\ln(x) - 2\int \frac{1}{x}\sqrt{x}\,dx = 2\sqrt{x}\,\ln(x) - 2\int x^{-1/2}\,dx = 2\sqrt{x}\,(\ln(x)-2)\,.$$

It follows that

$$\int_1^4 \frac{\ln(x)}{\sqrt{x}}\,dx = 2\sqrt{4}\,(\ln(4)-2) - 2\sqrt{1}\,(\ln(1)-2) = 8\,\ln(2) - 4.$$

17. Integrate by parts with $dv = 3^x\,dx$. Then $v = (3^x)/\ln(3)$ and

$$\begin{aligned}\int_0^1 x\,3^x\,dx &= \frac{1}{\ln(3)}\,x\,3^x\Big|_{x=0}^{x=1} - \frac{1}{\ln(3)}\int_0^1 3^x\,dx \\ &= \left(\frac{1}{\ln(3)}\,x\,3^x - \frac{1}{\ln^2(3)}\,3^x\right)\Big|_{x=0}^{x=1} \\ &= \left(\frac{3}{\ln(3)} - \frac{3}{\ln^2(3)}\right) - \left(0 - \frac{1}{\ln^2(3)}\right) \\ &= \frac{1}{\ln^2(3)}\,(3\ln(3) - 2)\,.\end{aligned}$$

18. We integrate by parts twice. In each application we take $dv = e^{-x}\,dx$:

$$\begin{aligned}\int x^2\,e^{-x}\,dx &= -x^2\,e^{-x} + 2\int x\,e^{-x}\,dx \\ &= -x^2\,e^{-x} + 2\left(-x\,e^{-x} + \int e^{-x}\,dx\right) \\ &= -x^2\,e^{-x} + 2\left(-x\,e^{-x} - e^{-x}\right) + C \\ &= -\left(x^2 + 2x + 2\right)\,e^{-x} + C.\end{aligned}$$

19. In the first integration by parts we set $u = x^2$. In the second, we set $u = x$:

$$\begin{aligned}\int x^2\cos(x)\,dx &= x^2\sin(x) - 2\int x\,\sin(x)\,dx \\ &= x^2\sin(x) - 2\left(-x\,\cos(x) + \int \cos(x)\,dx\right) \\ &= x^2\sin(x) - 2\,\left(-x\,\cos(x) + \sin(x)\right) + C \\ &= \left(x^2 - 2\right)\,\sin(x) + 2\,x\,\cos(x) + C.\end{aligned}$$

20. In the first integration by parts we set $u = \ln^2(x)$. In the second, we set $u = \ln(x)$:

$$\begin{aligned}\int 16\,x^3\,\ln^2(x)\,dx &= 4\,x^4\,\ln^2(x) - 8\int x^3\,\ln(x)\,dx \\ &= 4\,x^4\,\ln^2(x) - 8\left(\frac{1}{4}\,x^4\,\ln(x) - \frac{1}{4}\int x^3\,dx\right) \\ &= 4\,x^4\,\ln^2(x) - 8\left(\frac{1}{4}\,x^4\,\ln(x) - \frac{1}{16}\,x^4\right) + C \\ &= x^4\left(4\,\ln^2(x) - 2\,\ln(x) + \frac{1}{2}\right) + C.\end{aligned}$$

21. Let $u = 2\arctan(x)$, $dv = dx$. Then $du = 2/\left(1 + x^2\right)\,dx$, $v = x$, and

$$\begin{aligned}\int_0^1 2\,\arctan(x)\,dx &= 2\,x\,\arctan(x)\Big|_{x=0}^{x=1} - \int_0^1 \frac{2x}{1+x^2}\,dx \\ &= \left(2x\,\arctan(x) - \ln\left(1 + x^2\right)\right)\Big|_{x=0}^{x=1} \\ &= 2\arctan(1) - \ln(2) - (0 - 0) \\ &= \frac{\pi}{2} - \ln(2).\end{aligned}$$

22. Integrate by parts with $u = \arcsin(x)$, $dv = dx$. Then $v = x$, $du = \left(1/\sqrt{1-x^2}\right)\,dx$, and

$$\int_0^1 \arcsin(x)\,\sqrt{1-x^2}\,dx = x\,\arcsin(x)\,\Big|_{x=0}^{x=1} - \int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx = \left(x\,\arcsin(x) + \left(1 - x^2\right)^{1/2}\right)\Big|_{x=0}^{x=1} = \frac{\pi}{2} - 1.$$

23. We use the reduction formula twice, setting $a = 1/2$ in both applications of the formula. We use $p = 2$

in the first application and $p = 1$ in the second:

$$\begin{aligned}\int x^2\, e^{x/2}\, dx &= \frac{1}{1/2}\, x^2\, e^{x/2} - \frac{2}{1/2} \int x^1\, e^{x/2}\, dx \\ &= 2\, x^2\, e^{x/2} - 4 \int x^1\, e^{x/2}\, dx \\ &= 2\, x^2\, e^{x/2} - 4 \left(\frac{1}{1/2}\, x^1\, e^{x/2} - \frac{1}{1/2} \int x^0\, e^{x/2}\, dx \right) \\ &= 2\, x^2\, e^{x/2} - 4 \left(2\, x\, e^{x/2} - 4\, e^{x/2} \right) + C \\ &= 2\, \left(x^2 - 4x + 8 \right)\, e^{x/2} + C.\end{aligned}$$

Therefore,

$$\int_0^1 x^2\, e^{x/2}\, dx = 2\, (1 - 4 + 8)\, e^{1/2} - 2\, (8) = 10\sqrt{e} - 16.$$

24. We use the reduction formula twice, setting $a = -2$ in both applications of the formula. We use $p = 2$ in the first application and $p = 1$ in the second:

$$\begin{aligned}\int_0^1 4\, x^2\, e^{-2x}\, dx &= 4 \cdot \frac{1}{(-2)}\, x^2\, e^{-2x} - 4 \cdot \frac{2}{(-2)} \int x^1\, e^{-2x}\, dx \\ &= -2\, x^2\, e^{-2x} + 4 \int x^1\, e^{-2x}\, dx \\ &= -2\, x^2\, e^{-2x} + 4 \left(\frac{1}{(-2)}\, x\, e^{-2x} - \frac{1}{(-2)} \int e^{-2x}\, dx \right) \\ &= -2\, x^2\, e^{-2x} - 2\, x\, e^{-2x} + 2 \int e^{-2x}\, dx \\ &= -2\, x^2\, e^{-2x} - 2\, x\, e^{-2x} - e^{-2x} + C \\ &= -\left(2\, x^2 + 2\, x + 1 \right)\, e^{-2x} + C.\end{aligned}$$

Therefore,

$$\int_0^1 4x^2\, e^{-2x}\, dx = -\,(2 + 2 + 1)\, e^{-2} + (0 + 0 + 1)\, e^0 = 1 - 5\, e^{-2}.$$

25.

$$\begin{aligned}\int 2\, \ln\left(\sqrt{x}\right)\, dx &= \int 2\, \ln\left(x^{1/2}\right)\, dx \\ &= \int 2\, \frac{1}{2}\, \ln(x)\, dx \\ &= x\, (\ln(x) - 1) + C.\end{aligned}$$

26. We first use a basic property of the logarithm:

$$\int 4x\, \ln\left(x^5\right)\, dx = 20 \int x\, \ln(x)\, dx.$$

Now set $a = 1$ and $p = 1$, and follow the general calculation with which the solution to Exercise 7 concludes:

$$\int 4x\, \ln\left(x^5\right)\, dx = 20\, \frac{1}{2}\, x^2 \left(\ln(x) - \frac{1}{4} \right) + C = 5x^2\, \left(2\ln(x) - 1 \right) + C.$$

27. We first use a basic property of the logarithm:

$$\int 16x^3 \ln\left(\frac{1}{x}\right)\, dx = -16 \int x^3 \ln(x)\, dx.$$

Now set $a = 1$ and $p = 3$, and follow the general calculation with which the solution to Exercise 7 concludes:

$$\int 16x^3 \ln\left(\frac{1}{x}\right)\, dx = -4\, x^4 \left(\ln(x) - \frac{1}{4}\right) + C = x^4\ (1 - 4\ln(x)) + C.$$

28. Set $a = 1$ and $p = 1/2$, and follow the general calculation with which the solution to Exercise 7 concludes:

$$\int 9\sqrt{x} \ln(x)\, dx = \frac{9}{3/2}\, x^{3/2} \left(\ln(x) - \frac{1}{3/2}\right) + C = 2\, x^{3/2}\ (3\ln(x) - 2) + C.$$

29. Integrating by parts with $u = x$, $dv = \sec^2(x)\, dx$, we have $du = dx$, $v = \tan(x)$, and

$$\begin{aligned}\int x \sec^2(x)\, dx &= x\tan(x) - \int \tan(x)\, dx\\ &= x\tan(x) - \ln(|\sec(x)|) + C.\end{aligned}$$

30. Integrating by parts with $u = x$, $dv = (x+3)^{-1/2}\, dx$, we have $du = dx$, $v = 2\,(x+3)^{1/2}$, and

$$\begin{aligned}\int x\,(x+3)^{-1/2}\, dx &= 2\,x\,(x+3)^{1/2} - 2\int (x+3)^{1/2}\, dx\\ &= 2\,x\,(x+3)^{1/2} - \frac{4}{3}\,(x+3)^{3/2} + C\\ &= \frac{2}{3}\,(x+3)^{1/2}\ (x-6) + C.\end{aligned}$$

Therefore,

$$\int_{-2}^{1} x\,(x+3)^{-1/2}\, dx = \frac{2}{3}\,(2)\ (-5) - \frac{2}{3}\,(1)\ (-8) = \frac{2}{3}(-10+8) = -\frac{4}{3}.$$

31. Integrating by parts with $u = x$, $dv = (x-1)^{1/2}\, dx$, we have $du = dx$, $v = (2/3)\,(x-1)^{3/2}$, and

$$\begin{aligned}\int x\,(x-1)^{1/2}\, dx &= \frac{2}{3}\,x\,(x-1)^{3/2} - \frac{2}{3}\int (x-1)^{3/2}\, dx\\ &= \frac{2}{3}\,x\,(x-1)^{3/2} - \frac{2}{3}\left(\frac{2}{5}\,(x-1)^{5/2}\right) + C\\ &= \frac{2}{15}\,(x-1)^{3/2}\ (3x+2) + C.\end{aligned}$$

Therefore,

$$\int_{1}^{2} x\,(x-1)^{1/2}\, dx = \frac{2}{15}\ (6+2) - 0 = \frac{16}{15}.$$

32. Let $y = \sqrt{(x)} = x^{1/2}$. Then $dy = (1/2)\, x^{-1/2}\, dx$. It follows that $dx = 2\, y\, dy$. Then, integration by parts is applied with $u = y$ and $dv = \sin(y)\, dy$. Thus,

$$\begin{aligned}\int \sin\left(\sqrt{x}\right)\, dx &= 2\int y\, \sin(y)\, dy \\ &= -2y\, \cos(y) + 2\int, \cos(y)\, dy \\ &= -2y\, \cos(y) + 2\sin(y) + C \\ &= -2\sqrt{x}\, \cos\left(\sqrt{x}\right) + 2\sin\left(\sqrt{x}\right) + C.\end{aligned}$$

33. Begin with the substitution $y = x^2$, $dy = 2x\, dx$. From these two equations we see that $2\, x^3\, dx = x^2\, 2x\, dx = y\, dy$. After making this subsitution, use integration by parts with $u = y$, $dv = \exp(y)\, dy$. Then $du = dy\, v = \exp(y)$. Thus,

$$\begin{aligned}\int 2x^3\, \exp\left(x^2\right)\, dx &= \int y\, \exp(y)\, dy \\ &= y\, \exp(y) - \int, \exp(y)\, dy \\ &= y\, \exp(y) - \exp(y) + C \\ &= \left(x^2 - 1\right)\, \exp\left(x^2\right) + C\end{aligned}$$

34. Begin with the substitution $y = x^3$, $dy = 3x^2\, dx$. From these two equations we see that $3\, x^5\, dx = x^3\, 3\, x^2\, dx = y\, dy$. After making this subsitution, use integration by parts with $u = y$, $dv = \sin(y)\, dy$. Then $du = dy\, v = -\cos(y)$. Thus,

$$\begin{aligned}\int 3x^5 \sin\left(x^3\right)\, dx &= \int y\, \sin(y)\, dy \\ &= -y\, \cos(y) + \int, \cos(y)\, dy \\ &= -y\, \cos(y) + \sin(y) + C \\ &= -x^3\, \cos\left(x^3\right) + \sin\left(x^3\right) + C\end{aligned}$$

Solutions: Trigonometric Integrals (Corresponds to Stewart 7.2)

1. First make the change of variable $u = x/2, du = (1/2)\,dx$. The result is $\int_0^{\pi/4} 2\,\sin(u)^2\,du$. The antiderivative of the integrand is $u - \cos(u)\,\sin(u) + C$. The answer is $(\pi - 2)/4$.

2. First make the change of variable $u = 2\,x, du = 2\,dx$. The result is $\int_{\pi/3}^{\pi/2} 24\,\cos(u)^2\,du$. The antiderivative of the integrand is $12\,u + 12\,\cos(u)\,\sin(u) + C$. The answer is $2\pi - 3\sqrt{3}$.

3. First make the change of variable $u = x/2, du = (1/2)\,dx$. The result is $\int_0^{\pi/3} 6\,\tan(u)^2\,du$, or $\int_0^{\pi/3} 6\,\left(\sec(u)^2 - 1\right)\,du$. The antiderivative of the integrand is $6\tan(u) - 6u + C$ and the definite integral evaluates to $6\sqrt{3} - 2\,\pi$.

4. The integrand may be expanded and each term integrated. Easier is to use the identity $\cos^2(x) - \sin^2(x) = \cos(2x)$. The answer is $C + \sin(2x)/2$, or $\cos(x)\sin(x) + C$.

5. First make the change of variable $t = 2x + \pi, dt = 2\,dx$. The result is $\int_{4\pi/3}^{3\pi/2} 24\,\cos(t)^3\,dt$, or $\int_{4\pi/3}^{3\pi/2} 24\,\left(1 - \sin(t)^2\right)\,\cos(t)\,dt$. The change of variable $u = \sin(t), du = \cos(t)\,dt$ results in $\int_{-1}^{-\sqrt{3}/2} 24\,\left(u^2 - 1\right)\,du$, which evaluates to $9\sqrt{3} - 16$.

6. Making the changes of variable $u = x/10, du = (1/10)\,dx$ first and $w = \cos(u), dw = -\sin(u)\,du$ later,

we have

$$\begin{aligned}
\int 7\sin^7\left(\frac{x}{10}\right)\,dx &= 70\int \sin^7(u)\,du \\
&= 70\int \left(\sin^2(u)\right)^3\,\sin(u)\,du \\
&= 70\int \left(1-\cos^2(u)\right)^3\,\sin(u)\,du \\
&= -70\int \left(1-w^2\right)^3\,dw \\
&= -70\int \left(w-3\,w^2+3\,w^4-w^6\right)\,dw \\
&= C-70\left(w-w^3+\frac{3}{5}\,w^5-\frac{1}{7}w^7\right) \\
&= C-70\,\cos(u)+70\,\cos(u)^3-42\,\cos(u)^5+10\,\cos(u)^7 \\
&= C-70\,\cos\left(\frac{x}{10}\right)+70\,\cos\left(\frac{x}{10}\right)^3-42\,\cos\left(\frac{x}{10}\right)^5+10\,\cos\left(\frac{x}{10}\right)^7.
\end{aligned}$$

7. Making the changes of variable $u=x/3, du=(1/3)\,dx$ first and $w=\sin(u), dw=\cos(u)\,du$ later, we have

$$\begin{aligned}
\int 5\sin^2\left(\frac{x}{3}\right)\cos^3\left(\frac{x}{3}\right)\,dx &= 15\int \sin^2(u)\cos^3(u)\,du \\
&= 15\int \sin^2(u)\cos^2(u)\,\cos(u)\,du \\
&= 15\int \sin^2(u)\left(1-\sin^2(u)\right)\,\cos(u)\,du \\
&= 15\int w^2\,\left(1-w^2\right)\,dw \\
&= 15\int \left(w^2-w^4\right)\,dw \\
&= 15\left(\frac{w^3}{3}-\frac{w^5}{5}\right)+C \\
&= 5\,\sin^3(u)-3\,\sin^5(u)+C \\
&= 5\,\sin^3\left(\frac{x}{3}\right)-3\,\sin^5\left(\frac{x}{3}\right)+C.
\end{aligned}$$

8. Making the changes of variable $u=2x/3, du=(2/3)\,dx$ first and $w=\cos(u), dw=-\sin(u)\,du$ later, we

have

$$\begin{aligned}
\int 7\sin^3\left(\frac{2x}{3}\right)\sqrt{\cos\left(\frac{2x}{3}\right)}\,dx &= \frac{21}{2}\int \sin^3(u)\sqrt{\cos(u)}\,du \\
&= \frac{21}{2}\int \sin^2(u)\sqrt{\cos(u)}\,\sin(u)\,du \\
&= \frac{21}{2}\int \left(1-\cos^2(u)\right)\sqrt{\cos(u)}\,\sin(u)\,du \\
&= -\frac{21}{2}\int \left(1-w^2\right)\sqrt{w}\,dw \\
&= -\frac{21}{2}\int \left(w^{1/2}-w^{5/2}\right)dw \\
&= C-\frac{21}{2}\left(\frac{2}{3}w^{3/2}-\frac{2}{7}w^{7/2}\right) \\
&= C-7\cos^{3/2}\left(\frac{2x}{3}\right)+3\cos^{7/2}\left(\frac{2x}{3}\right).
\end{aligned}$$

9. We have

$$\int \frac{\cos^3(x)}{\sin(x)}\,dx = \int \frac{\cos^2(x)}{\sin(x)}\cos(x)\,dx = \int \frac{1-\sin^2(x)}{\sin(x)}\cos(x)\,dx = \int \left(\frac{1}{\sin(x)}-\sin(x)\right)\cos(x)\,dx.$$

Make the substitution $u=\sin(x)$, $du=\cos(x)\,dx$ to obtain

$$\int \frac{\cos^3(x)}{\sin(x)}\,dx = C-\frac{1}{2}\sin^2(x)+\ln\left(|\sin(x)|\right).$$

10. Write

$$\cos^3(2x)=\cos^2(2x)\cos(2x)=\left(1-\sin^2(2x)\right)\cos(2x).$$

The change of variable $u=\sin(2x)\,, du=2\cos(2x)\,dx$ converts the integral to $(5/2)\int\left(1-u^2\right)/\sqrt{u}\,du$, or $5\sqrt{u}-u^{5/2}+C$. After resubstituting $u=\sin(2x)$, we obtain $5\sqrt{\sin(2x)}-\sin^{5/2}(2x)+C$.

11. Answer: 4/35

12. Write

$$\cos^3(x)=\cos^2(x)\cos(x)=\left(1-\sin^2(x)\right)\cos(x).$$

The change of variable $u=\sin(x)\,, du=\cos(x)\,dx$ converts the integral to $\int_0^1\sqrt{u}\left(1-u^2\right)du$, or $\int_0^1\left(u^{1/2}-u^{5/2}\right)du$, or 8/21.

13. 12

Answer: 16/21

16. Answer: 3/8

17. To begin, we change $\cos^2(x)$ to $1-\sin^2(x)$, expand the integrand, and integrate term-by-term. This results in two integrals: one with $\sin^2(x)$ as the integrand and one with $\sin^4(x)$ as the integrand. The

integral of $\sin^2(x)$ is handled as in Exercise 1. We will omit the details of that calculation. For the integral of $\sin^4(x)$, we use the formula in Exercise 14. Here are some of the details:

$$\begin{aligned}
\int_0^{\pi/4} \cos^2(t)\,\sin^2(t)\,dt &= \int_0^{\pi/4} \left(1-\sin^2(t)\right)\,\sin^2(t)\,dt \\
&= \int_0^{\pi/4} \left(\sin^2(t)-\sin^4(t)\right)\,dt \\
&= \int_0^{\pi/4} \sin^2(t)\,dt - \int_0^{\pi/4} \sin^4(t)\,dt \\
&= \left(\frac{\pi}{8}-\frac{1}{4}\right) - \int_0^{\pi/4} \sin^4(t)\,dt \\
&= \left(\frac{\pi}{8}-\frac{1}{4}\right) - \left(-\frac{1}{4}\,\sin^3(t)\cos(t) - \frac{3}{8}\,\cos(t)\,\sin(t) + \frac{3}{8}\,t\Big|_0^{\pi/4}\right) \\
&= \left(\frac{\pi}{8}-\frac{1}{4}\right) - \left(-\frac{1}{4}\left(\frac{1}{\sqrt{2}}\right)^4 - \frac{3}{8}\left(\frac{1}{\sqrt{2}}\right)^2 + \frac{3}{8}\,\frac{\pi}{4}\right) \\
&= \frac{\pi}{32}.
\end{aligned}$$

18. The first step is to make the substitution $u = x/2$, $du = (1/2)\,dx$. Then we use the reduction formula of Exercise 15 with $n = 6$. Finally, we use the formula for the integral of $\cos(u)^4$ found in Exercise 15. Here are the details:

$$\begin{aligned}
\int_0^{\pi} \cos^6(x/2)\,dx &= 2\int_0^{\pi/2} \cos^6(u)\,du \\
&= 2\left(\frac{1}{6}\,\cos^5(u)\,\sin(u)\Big|_{u=0}^{u=\pi/2} + \frac{5}{6}\int_0^{\pi/2} \cos^4(u)\,du\right) \\
&= \frac{5}{3}\int_0^{\pi/2} \cos^4(u)\,du \\
&= \frac{5}{3}\left(\frac{1}{4}\,\cos^3(u)\,\sin(u) + \frac{3}{8}\,\cos(u)\,\sin(u) + \frac{3}{8}\,u\Big|_0^{\pi/2}\right) \\
&= \frac{5}{3}\,\frac{3}{8}\,\frac{\pi}{2} \\
&= \frac{5\,\pi}{16}.
\end{aligned}$$

19. To begin, we change $\cos^2(x)$ to $1-\sin^2(x)$, expand the integrand, and integrate term-by-term. This results in two integrals: one with $\sin^4(x)$ as the integrand and one with $\sin^6(x)$ as the integrand. We work with the higher power, using the reduction formula in Exercise 14 with $n = 6$ (and $u = x$). After combining terms, we use the formula for the integral of $\sin^4(x)$ developed in Exercise 14. (The alternative would be to continue to use the reduction formula.) In the evaluations at the limits of integration, we

use $\cos(\pi/6) = \sin(\pi/3) = \sqrt{3}/2$ and $\sin(\pi/6) = \cos(\pi/3) = 1/2$. Here are the details:

$$\begin{aligned}
\int_{\pi/6}^{\pi/3} 64\cos^2(x)\sin^4(x)\,dx &= 64\int_{\pi/6}^{\pi/3} \left(1-\sin^2(x)\right)\,\sin^4(x)\,dx \\
&= 64\left(\int_{\pi/6}^{\pi/3} \sin^4(x)\,dx - \int_{\pi/6}^{\pi/3} \sin^6(x)\,dx\right) \\
&= 64\left(\int_{\pi/6}^{\pi/3} \sin^4(x)\,dx - \left(-\frac{1}{6}\cos(x)\,\sin^5(x)\Big|_{x=\pi/6}^{x=\pi/3} + \frac{5}{6}\int_{\pi/6}^{\pi/3} \sin^4(x)\,dx\right)\right) \\
&= 64\left(\int_{\pi/6}^{\pi/3} \sin^4(x)\,dx + \frac{\sqrt{3}}{48} - \frac{5}{6}\int_{\pi/6}^{\pi/3} \sin^4(x)\,dx\right) \\
&= 64\left(\frac{1}{6}\int_{\pi/6}^{\pi/3} \sin^4(x)\,dx + \frac{\sqrt{3}}{48}\right) \\
&= \frac{32}{3}\int_{\pi/6}^{\pi/3} \sin^4(x)\,dx + \frac{4\sqrt{3}}{3} \\
&= \frac{32}{3}\left(\left(-\frac{1}{4}\sin^3(x)\cos(x) - \frac{3}{8}\cos(x)\,\sin(x) + \frac{3}{8}\,x\right)\Big|_{x=\pi/6}^{x=\pi/3}\right) + \frac{4\sqrt{3}}{3} \\
&= \frac{32}{3}\left(\frac{\pi}{16} - \frac{\sqrt{3}}{32}\right) + \frac{4\sqrt{3}}{3} \\
&= \frac{2\pi}{3} + \sqrt{3}.
\end{aligned}$$

20. Letting $u = \tan(x)$, $du = \sec^2(x)$, we have

$$\begin{aligned}
\int \tan^3(x)\,dx &= \int \tan(x)\,\tan^2(x)\,dx \\
&= \int \tan(x)\,\left(\sec^2(x) - 1\right)\,dx \\
&= \int \tan(x)\,\sec^2(x)\,dx - \int \tan(x)\,dx \\
&= \int u\,du - \int \tan(x)\,dx \\
&= \frac{1}{2}\,u^2 - \ln(\,|\sec(x)|\,) + C \\
&= \frac{1}{2}\,\tan^2(x) - \ln(\,|\sec(x)|\,) + C.
\end{aligned}$$

21. We have,

$$\begin{aligned}
\int \tan^2(x)\,\sec(x)\,dx &= \int \left(\sec^2(x) - 1\right)\,\sec(x)\,dx \\
&= \int \sec^3(x)\,dx - \int \sec(x)\,dx \\
&= \left(\frac{1}{2}\sec(x)\,\tan(x) + \frac{1}{2}\ln\left(\,|\sec(x) + \tan(x)\,|\,\right)\right) - \ln\left(\,|\sec(x) + \tan(x)\,|\,\right) + C \\
&= \frac{1}{2}\sec(x)\,\tan(x) - \frac{1}{2}\ln\left(\,|\sec(x) + \tan(x)\,|\,\right) + C.
\end{aligned}$$

22. Making the change of variable $u = \sec(x)$, $du = \sec(x)\,\tan(x)\,dx$, we have,

$$\begin{aligned}
\int \tan^3(x)\,\sec(x)\,dx &= \int \tan^2(x)\,\sec(x)\,\tan(x)\,dx \\
&= \int \left(\sec^2(x) - 1\right)\,\sec(x)\,\tan(x)\,dx \\
&= \int \left(u^2 - 1\right)\,du \\
&= \frac{1}{3}\,u^3 - u + C \\
&= \frac{1}{3}\,\sec^3(x) - \sec(x) + C.
\end{aligned}$$

Solutions: Trigonometric Substitution (Corresponds to Stewart 7.3)

1. Make the change of variable $x = \sin(\theta), dx = \cos(\theta)\, d\theta$. Then

$$\begin{aligned}
\int \frac{24\,x^2}{\sqrt{1-x^2}}\,dx &= 24\int \frac{\sin^2(\theta)}{\sqrt{1-\sin^2(\theta)}}\cos(\theta)\,d\theta \\
&= 24\int \frac{\sin^2(\theta)}{\cos(\theta)}\cos(\theta)\,d\theta \\
&= 24\int \sin^2(\theta)\,d\theta \\
&= 24\int \frac{1}{2}\left(1-\cos(2\,\theta)\right)\,d\theta \\
&= 12\left(\theta - \frac{1}{2}\sin(2\,\theta)\right) \\
&= 12\Big(\theta - \sin(\theta)\,\cos(\theta)\Big) \\
&= 12\left(\arcsin(x) - x\sqrt{1-x^2}\right).
\end{aligned}$$

Therefore,

$$\int_0^{\sqrt{3}/2} \frac{24\,x^2}{\sqrt{1-x^2}}\,dx = 12\left(\arcsin\left(\frac{\sqrt{3}}{2}\right) - \frac{\sqrt{3}}{2}\sqrt{1-\left(\frac{\sqrt{3}}{2}\right)^2}\right) = 12\cdot\frac{\pi}{3} - 12\cdot\frac{\sqrt{3}}{2}\cdot\frac{1}{2} = 4\,\pi - 3\,\sqrt{3}.$$

2. Make the change of variable $x = 5\,\sin(\theta), dx = 5\,\cos(\theta)\,d\theta$. Then

$$\begin{aligned}
\int 2\,\sqrt{25-x^2}\,dx &= 2\times 25\int\sqrt{1-\sin^2(\theta)}\,\cos(\theta)\,d\theta \\
&= 2\times 25\int\cos^2(\theta)\,d\theta \\
&= 2\times 25\int\frac{1}{2}\,(1+\cos(2\,\theta))\,d\theta \\
&= 25\,\left(\theta+\frac{1}{2}\sin(2\,\theta)\right) \\
&= 25\,\Big(\theta+\sin(\theta)\,\cos(\theta)\Big) \\
&= 25\,\left(\arcsin\left(\frac{x}{5}\right)+\frac{x}{5}\,\frac{\sqrt{25-x^2}}{5}\right) \\
&= 25\,\arcsin\left(\frac{x}{5}\right)+x\,\sqrt{25-x^2}.
\end{aligned}$$

Therefore,

$$\int_0^3 2\,\sqrt{25-x^2}\,dx = 25\,\arcsin\left(\frac{3}{5}\right)+12.$$

3. Make the change of variable $x = 2\,\sin(\theta), dx = 2\,\cos(\theta)\,d\theta$. Then

$$\begin{aligned}
\int\left(4-x^2\right)^{3/2}\,dx &= 2\times 4^{3/2}\int\left(1-\sin^2(\theta)\right)^{3/2}\,\cos(\theta)\,d\theta \\
&= 16\int\cos^3(\theta)\,\cos(\theta)\,d\theta \\
&= 16\,\left(\frac{1}{4}\,\cos^3(\theta)\,\sin(\theta)+\frac{3}{8}\,\cos(\theta)\,\sin(\theta)+\frac{3}{8}\,\theta\right)+C \\
&= 4\,\left(\frac{\sqrt{4-x^2}}{2}\right)^3\,\frac{x}{2}+6\,\frac{\sqrt{4-x^2}}{2}\,\frac{x}{2}+6\,\arcsin\left(\frac{x}{2}\right)+C \\
&= \frac{1}{4}x\,\left(4-x^2\right)^{3/2}+\frac{3}{2}x\,\left(4-x^2\right)^{1/2}+6\,\arcsin\left(\frac{x}{2}\right)+C.
\end{aligned}$$

4. Make the change of variable $x = \tan(\theta)/2, dx = \big(\sec^2(\theta)/2\big)\,d\theta$. Then

$$\begin{aligned}
\int\frac{2}{\sqrt{1+4x^2}}\,dx &= \int\frac{1}{\sqrt{1+\tan^2(\theta)}}\,\sec^2(\theta)\,d\theta \\
&= \int\frac{1}{\sec(\theta)}\,\sec^2(\theta)\,d\theta \\
&= \int\sec(\theta)\,d\theta \\
&= \ln\left(\,|\sec(\theta)+\tan(\theta)|\,\right)+C \\
&= \ln\left(\,|\sqrt{1+4x^2}+2x|\,\right)+C \\
&= \ln\left(\sqrt{1+4x^2}+2x\right)+C.
\end{aligned}$$

In the last line, the absolute value function has been omitted because it is superfluous: the expression $\sqrt{1+4x^2}+2x$ cannot assume a negative value for real x.

5. Make the change of variable $x = 3\,\tan(\theta), dx = 3\,\sec^2(\theta)\,d\theta$. Then, using the formula

$$\int \tan^3(\theta)\,d\theta = \frac{1}{2}\,\tan^2(x) - \ln(\,|\sec(x)|\,) + C$$

from the last set of exercises, we have

$$\begin{aligned}
\int \frac{2x^3}{9+x^2}\,dx &= \int \frac{2\times 3^3\,\tan^3(\theta)}{9+9\,\tan^2(\theta)}\,3\,\sec^2(\theta)\,d\theta \\
&= \int \frac{2\times 3\,\tan^3(\theta)}{1+\tan^2(\theta)}\,3\,\sec^2(\theta)\,d\theta \\
&= 18\int \frac{\tan^3(\theta)}{\sec^2(\theta)}\,\sec^2(\theta)\,d\theta \\
&= 18\int \tan^3(\theta)\,d\theta \\
&= 9\,\tan^2(\theta) - 18\,\ln(\,|\sec(\theta)|\,) + C \\
&= x^2 - 18\,\ln\left(\frac{\sqrt{9+x^2}}{3}\right) + C \\
&= x^2 - 18\,\ln\left(\sqrt{9+x^2}\right) + C \\
&= x^2 - 9\,\ln\left(9+x^2\right) + C.
\end{aligned}$$

6. Let $a = \sqrt{3}$. Make the change of variable $x = (a/2)\,\tan(\theta)$, $dx = (a/2)\,\sec^2(\theta)\,d\theta$. We have

$$\begin{aligned}
\int \frac{16\,x^2}{\sqrt{3+4\,x^2}}\,dx &= 16\left(\frac{a}{2}\right)^3 \int \frac{\tan^2(\theta)}{\sqrt{3+3\tan^2(\theta)}}\,\sec^2(\theta)\,d\theta \\
&= 16\left(\frac{a}{2}\right)^3 \frac{1}{a}\int \frac{\tan^2(\theta)}{\sqrt{1+\tan^2(\theta)}}\,\sec^2(\theta)\,d\theta \\
&= 16\left(\frac{a}{2}\right)^3 \frac{1}{a}\int \frac{\tan^2(\theta)}{\sec(\theta)}\,\sec^2(\theta)\,d\theta \\
&= 2\,a^2 \int \tan^2(\theta)\,\sec(\theta)\,d\theta \\
&= 6\int \left(\sec^2(\theta) - 1\right)\,\sec(\theta)\,d\theta \\
&= 6\int \sec^3(\theta)\,d\theta - 6\int \sec(\theta)\,d\theta \\
&= 3\,\sec(\theta)\,\tan(\theta) + 3\,\ln(\,|\,\sec(\theta)+\tan(\theta)\,|\,) - 6\,\ln(\,|\,\sec(\theta)+\tan(\theta)\,|\,) + C \\
&= 3\,\sec(\theta)\,\tan(\theta) - 3\,\ln(\,|\,\sec(\theta)+\tan(\theta)\,|\,) + C \\
&= 3\,\frac{\sqrt{a^2+4x^2}}{a}\,\frac{2x}{a} - 3\,\ln\left(\frac{\sqrt{a^2+4x^2}}{a} + \frac{2x}{a}\right) + C \\
&= 2\,x\,\sqrt{a^2+4x^2} - 3\,\ln\left(\sqrt{a^2+4x^2} + 2x\right) + C \\
&= 2\,x\,\sqrt{3+4x^2} - 3\,\ln\left(\sqrt{3+4x^2} + 2x\right) + C.
\end{aligned}$$

7. Make the change of variable $x = (3/2)\tan(\theta)$, $dx = (3/2)\sec^2(\theta)\,d\theta$. Then

$$\begin{aligned}
\int \frac{36}{(9+4\,x^2)^2}\,dx &= 36\int \frac{1}{\left(9+9\,\tan^2(\theta))^2\right)^2}\,\frac{3}{2}\,\sec^2(\theta)\,d\theta \\
&= \frac{36\times 3}{9^2\times 2}\int \frac{1}{\left(1+\tan^2(\theta)\right)^2}\,\sec^2(\theta)\,d\theta \\
&= \frac{2}{3}\int \frac{1}{\sec^4(\theta)}\,\sec^2(\theta)\,d\theta \\
&= \frac{2}{3}\int \frac{1}{\sec^2(\theta)}\,d\theta \\
&= \frac{2}{3}\int \cos^2(\theta)\,d\theta \\
&= \frac{2}{3}\int \frac{1}{2}\,(1+\cos(2\theta))\,d\theta \\
&= \frac{1}{3}\Big(\theta+\sin(\theta)\cos(\theta)\Big)+C \\
&= \frac{1}{3}\left(\arctan\left(\frac{2x}{3}\right)+\frac{2x}{\sqrt{9+4\,x^2}}\,\frac{3}{\sqrt{9+4\,x^2}}\right)+C \\
&= \frac{1}{3}\arctan\left(\frac{2x}{3}\right)+\frac{2x}{9+4\,x^2}+C.
\end{aligned}$$

The method of this problem can be used to derive the following general formula:

$$\int \frac{2}{(a^2+b^2\,x^2)^2}\,dx = \frac{1}{a^2}\,\frac{x}{(a^2+b^2x^2)}+\frac{1}{a^3b}\,\arctan\left(\frac{bx}{a}\right)+C.$$

8. Make the change of variable $x = 5\sec(\theta)$, $dx = 5\sec(\theta)\tan(\theta)\,d\theta$ and, to simplify the final answer, observe that the addition of a specific constant such as $-\ln(5)$ to an arbitrary constant results in an arbitrary constant:

$$\begin{aligned}
\int \frac{1}{\sqrt{x^2-25}}\,dx &= \int \frac{1}{\sqrt{25\sec^2(\theta)-25}}\,5\sec(\theta)\tan(\theta)\,d\theta \\
&= \frac{5}{\sqrt{25}}\int \frac{1}{\sqrt{\sec^2(\theta)-1}}\,\sec(\theta)\tan(\theta)\,d\theta \\
&= \int \frac{1}{\sqrt{\tan^2(\theta)}}\,\sec(\theta)\tan(\theta)\,d\theta \\
&= \int \sec(\theta)\,d\theta \\
&= \ln(\,|\sec(\theta)+\tan(\theta)|\,)+C \\
&= \ln\left(\left|\frac{x}{5}+\frac{\sqrt{x^2-25}}{5}\right|\right)+C \\
&= \ln\left(\left|\frac{x+\sqrt{x^2-25}}{5}\right|\right)+C \\
&= \ln\left(\left|x+\sqrt{x^2-25}\right|\right)+C-\ln(5) \\
&= \ln\left(\left|x+\sqrt{x^2-25}\right|\right)+C.
\end{aligned}$$

9. Make the change of variable $x = (2/3)\sec(\theta)$, $dx = (2/3)\sec(\theta)\tan(\theta)\,d\theta$ and use the integral formula of Exercise 21 of the preceding section:

$$\begin{aligned}
\int \sqrt{9x^2-4}\,dx &= \int \sqrt{4\sec^2(\theta)-4}\,\frac{2}{3}\sec(\theta)\tan(\theta)\,d\theta \\
&= \frac{4}{3}\int \sqrt{\sec^2(\theta)-1}\,\sec(\theta)\tan(\theta)\,d\theta \\
&= \frac{4}{3}\int \sqrt{\tan^2(\theta)}\,\sec(\theta)\tan(\theta)\,d\theta \\
&= \frac{4}{3}\int \sec(\theta)\tan^2(\theta)\,d\theta \\
&= \frac{4}{3}\cdot\frac{1}{2}\Big(\sec(\theta)\tan(\theta) - \ln(|\sec(\theta)+\tan(\theta)|)\Big) + C \\
&= \frac{2}{3}\left(\frac{3x}{2}\frac{\sqrt{9x^2-4}}{2} - \ln\left(\left|\frac{3x}{2}+\frac{\sqrt{9x^2-4}}{2}\right|\right)\right) + C \\
&= \frac{1}{2}x\sqrt{9x^2-4} - \frac{2}{3}\ln\left(\left|\frac{3x+\sqrt{9x^2-4}}{2}\right|\right) + C \\
&= \frac{1}{2}x\sqrt{9x^2-4} - \frac{2}{3}\ln\left(\left|3x+\sqrt{9x^2-4}\right|\right) + \frac{2}{3}\ln(2) + C \\
&= \frac{1}{2}x\sqrt{9x^2-4} - \frac{2}{3}\ln\left(\left|3x+\sqrt{9x^2-4}\right|\right) + C.
\end{aligned}$$

10. We complete the square before obtaining the expression $1 - \text{stuff}^2$ that can be reduced by the trigonometric substitution $\text{stuff} = \sin(\theta)$. The expression we will identify as "stuff", namely $(x-2)/2$, finalizes two lines above the last line of the following chain of equalities:

$$\begin{aligned}
4x - x^2 &= 0 + \left(-x^2 + 4x\right) \\
&= 0 - \left(x^2 - 4x\right) \\
&= 0 + \left(\frac{4}{2}\right)^2 - \left(x^2 - 4x + \left(\frac{4}{2}\right)^2\right) \\
&= 4 - \left(x^2 - 4x + 4\right) \\
&= 4 - (x-2)^2 \\
&= 4\left(1 - \frac{1}{4}(x-2)^2\right) \\
&= 4\left(1 - \frac{1}{4}(x-2)^2\right) \\
&= 4\left(1 - \left(\frac{x-2}{2}\right)^2\right) \\
&= 4\left(1 - \sin^2(\theta)\right) \\
&= 4\cos^2(\theta),
\end{aligned}$$

where $(x-2)/2 = \sin(\theta)$, or $x = 2\sin(\theta) + 2$, $dx = 2\cos(\theta)\,d\theta$. Thus,

$$\int \frac{1}{\sqrt{4x-x^2}}\,dx = \int \frac{1}{\sqrt{4\cos^2(\theta)}}\,2\cos(\theta)\,d\theta = \int 1\,d\theta = \theta + C = \arcsin\left(\frac{x-2}{2}\right) + C.$$

11. We complete the square before obtaining the expression $1+\text{stuff}^2$ that can be reduced by the trigonometric substitution $\text{stuff}=\tan(\theta)$. The expression we will identify as "stuff", namely $(x+4)/2$, finalizes two lines above the last line of the following chain of equalities:

$$\begin{aligned}
x^2+8\,x+20 &= \big(x^2+8\,x\big)+20\\
&= \left(x^2+8\,x+\left(\frac{8}{2}\right)^2\right)+20-\left(\frac{8}{2}\right)^2\\
&= \big(x^2+8\,x+16\big)+4\\
&= (x+4)^2+4\\
&= 4\left(\frac{1}{4}\,(x+4)^2+1\right)\\
&= 4\left(\left(\frac{x+4}{2}\right)^2+1\right)\\
&= 4\,\big(\tan^2(\theta)+1\big)\\
&= 4\,\sec^2(\theta),
\end{aligned}$$

where $(x+4)/2=\tan(\theta)$, or $x=2\,\tan(\theta)-4$, $dx=2\,\sec^2(\theta)\,d\theta$. We also note that the equation $(x+4)/2=\tan(\theta)$ leads to $\theta=\arctan\big((x+4)/2\big)$. Corresponding to $x=-4$ we have $\theta=\arctan\big((-4+4)/2\big)=\arctan(0)=0$ and corresponding to $x=-24$ we have $\theta=\arctan\big((-2+4)/2\big)=\arctan(1)=\pi/4$. Thus,

$$\int_{-4}^{-2}\frac{1}{x^2+8\,x+20}\,dx=\int_0^{\pi/4}\frac{1}{4\,\sec^2(\theta)}\,2\,\sec^2(\theta)\,d\theta=\frac{1}{2}\int_0^{\pi/4}1\,d\theta=\frac{1}{2}\,\theta\Big|_0^{\pi/4}=\frac{\pi}{8}.$$

12. We complete the square before obtaining the expression stuff^2-1 that can be reduced by the trigonometric substitution $\text{stuff}=\sec(\theta)$. The expression we will identify as "stuff", namely $5\,x+2$, finalizes two lines above the last line of the following chain of equalities:

$$\begin{aligned}
25\,x^2+20\,x+3 &= 25\left(x^2+\frac{4}{5}\,x\right)+3\\
&= 25\left(x^2+\frac{4}{5}\,x+\left(\frac{1}{2}\cdot\frac{4}{5}\right)^2\right)+3-25\left(\frac{1}{2}\cdot\frac{4}{5}\right)^2+3\\
&= 25\left(x^2+\frac{4}{5}\,x+\left(\frac{2}{5}\right)^2\right)+3-4\\
&= 25\left(x+\frac{2}{5}\right)^2-1\\
&= (5\,x+2)^2-1\\
&= \sec^2(\theta)-1\\
&= \tan^2(\theta),
\end{aligned}$$

where $5\,x + 2 = \sec(\theta)$, or $x = \frac{1}{5}\,\sec(\theta) - \frac{2}{5}$, $dx = \frac{1}{5}\,\sec(\theta)\,\tan(\theta)\,d\theta$. Thus,

$$\begin{aligned}
\int \frac{x^2}{(25\,x^2 + 20\,x + 3)^{3/2}}\,dx &= \int \left(\frac{1}{5}\,\sec(\theta) - \frac{2}{5}\right)^2 \frac{1}{\tan^3(\theta)}\,\frac{1}{5}\,\sec(\theta)\,\tan(\theta)\,d\theta \\
&= \frac{1}{125}\int (\sec(\theta) - 2)^2\,\frac{\sec(\theta)}{\tan^2(\theta)}\,d\theta \\
&= \frac{1}{125}\int \left(\sec^2(\theta) - 4\sec(\theta) + 4\right)\frac{\sec(\theta)}{\tan^2(\theta)}\,d\theta \\
&= \frac{1}{125}\int \frac{\sec^3(\theta) - 4\sec^2(\theta) + 4\sec(\theta)}{\tan^2(\theta)}\,d\theta.
\end{aligned}$$

No more was asked, but, for the curious, here is the evaluation of the integral:

$$\int \frac{x^2}{(25\,x^2 + 20\,x + 3)^{3/2}}\,dx = \frac{1}{125}\left(\ln\left(5\,x + 2 + \sqrt{25\,x^2 + 20\,x + 3}\right) - \frac{25\,x + 6}{\sqrt{25\,x^2 + 20\,x + 3}}\right).$$

This evaluation can be accomplished using methods of integration that are covered in Calculus II, but the task is somewhat tricky.

Solutions: Using Partial Fractions to Integrate Rational Functions (Corresponds to Stewart 7.4)

1. The partial fraction expansion is
$$\frac{x-11}{x^2-x-2} = \frac{4}{x+1} - \frac{3}{x-2},$$
which leads to the evaluation
$$\int \frac{x-11}{x^2-x-2}\,dx = 4\,\ln(\,|\,x+1\,|\,) - 3\,\ln(\,|\,x-2\,|\,).$$

2. The partial fraction expansion is
$$\frac{2}{x^3-x} = \frac{1}{x+1} - \frac{2}{x} + \frac{1}{x-1},$$
which leads to the evaluation
$$\int \frac{2}{x^3-x}\,dx = \ln(\,|\,x+1\,|\,) - 2\,\ln(\,|\,x\,|\,) + \ln(\,|\,x-1\,|\,).$$

3. The partial fraction expansion is
$$\frac{4\,x^2+3\,x+2}{x^3+x^2} = \frac{1}{x} + \frac{2}{x^2} + \frac{3}{x+1},$$
which leads to the evaluation
$$\int \frac{4\,x^2+3\,x+2}{x^3+x^2}\,dx = \ln(\,|\,x\,|\,) - \frac{2}{x} + 3\,\ln(\,|\,x+1\,|\,).$$

4. The partial fraction expansion is
$$\frac{6\,x^2+17\,x-8}{(x+4)(x^2+4)} = \frac{1}{x+4} + \frac{5\,x-3}{x^2+4},$$
which leads to the evaluation
$$\begin{aligned}\int \frac{6\,x^2+17\,x-8}{(x+4)(x^2+4)}\,dx &= \int \frac{1}{x+4}\,dx + \int \frac{5\,x-3}{x^2+4}\,dx \\ &= \ln(\,|\,x+4\,|\,) + \frac{5}{2}\int \frac{2\,x}{x^2+4}\,dx - 3\int \frac{1}{x^2+4}\,dx \\ &= \ln(\,|\,x+4\,|\,) + \frac{5}{2}\,\ln\left(x^2+4\right) - \frac{3}{2}\,\arctan\left(\frac{x}{2}\right) + C.\end{aligned}$$

5. The partial fraction expansion is

$$\frac{x^3+5\,x^2-x+30}{(x^2+4)\,(x^2+9)}=\frac{2-x}{x^2+4}+\frac{2\,x+3}{x^2+9},$$

which leads to the evaluation

$$\int\frac{x^3+5\,x^2-x+30}{(x^2+4)\,(x^2+9)}\,dx=-\frac{1}{2}\,\ln\left(x^2+4\right)+\arctan\left(\frac{x}{2}\right)+\ln\left(x^2+9\right)+\arctan\left(\frac{x}{3}\right)+C.$$

6. The partial fraction expansion is

$$\frac{2\,x^3+5\,x-1}{\left(x^2+1\right)^2}=\frac{2\,x}{x^2+1}+\frac{3\,x-1}{\left(x^2+1\right)^2},$$

which leads to the evaluation

$$\int\frac{2\,x^3+5\,x-1}{\left(x^2+1\right)^2}\,dx=\ln\left(x^2+1\right)-\frac{1}{2}\,\frac{x+3}{x^2+1}-\frac{1}{2}\,\arctan(x)+C.$$

7. The partial fraction expansion is

$$\frac{5\,x^4+6\,x^3+30\,x^2+45}{x^2\,\left(x^2+3\right)^2}=\frac{5}{x^2}+\frac{6\,x}{\left(x^2+3\right)^2},$$

which leads to the evaluation

$$\int\frac{5\,x^4+6\,x^3+30\,x^2+45}{x^2\,\left(x^2+3\right)^2}\,dx=-\frac{5}{x}-\frac{3}{x^2+3}+C.$$

8. Completing the square, we have

$$x^2+6\,x+18=\left(x^2+6\,x\right)+18=\left(x^2+6\,x+\left(\frac{6}{2}\right)^2\right)+18-\left(\frac{6}{2}\right)^2=\left(x^2+6\,x+9\right)+18-9=(x+3)^2+9,$$

which becomes u^2+a^2 by letting $a=3$ and making the change of variable $u=x+3$, $du=dx$. Thus,

$$\int\frac{6}{x^2+6\,x+18}\,dx=6\int\frac{1}{a^2+u^2}\,du=\frac{6}{a}\,\arctan\left(\frac{u}{a}\right)=2\,\arctan\left(\frac{x}{3}+1\right).$$

Solutions: Numerical Integration (Corresponds to Stewart's Section 7.7)

1. Let $f(x) = 6/x$. For $N = 2$, we have $\Delta x = (5-1)/2 = 2$, and the Trapezoidal, Midpoint, and Simpson's approximations, T_2, M_2, S_2, are

$$\begin{aligned}
T_2 &= \frac{\Delta x}{2}\,(f(1) + 2\,f(3) + f(5)) \\
&= \frac{56}{5}, \\
M_2 &= \Delta x\,(f(2) + f(4)) \\
&= 9, \\
S_2 &= \frac{\Delta x}{3}\,(f(1) + 4\,f(3) + f(5)) \\
&= \frac{152}{15}.
\end{aligned}$$

For $N = 4$, we have $\Delta x = (5-1)/4 = 1$, and the Trapezoidal, Midpoint, and Simpson's approximations, T_4, M_4, S_4, are

$$\begin{aligned}
T_4 &= \frac{\Delta x}{2}\,(f(1) + 2\,f(2) + 2\,f(3) + 2\,f(4) + f(5)) \\
&= \frac{101}{10}, \\
M_4 &= \Delta x\,(f(3/2) + f(5/2) + f(7/2) + f(9/2)) \\
&= \frac{993}{105}, \\
S_4 &= \frac{\Delta x}{3}\,(f(1) + 4\,f(2) + 2\,f(3) + 4\,f(4) + f(5)) \\
&= \frac{146}{15}.
\end{aligned}$$

For $N = 6$, we have $\Delta x = (5-1)/6 = 2/3$, and the Trapezoidal, Midpoint, and Simpson's approximations, T_6, M_6, S_6, are

$$\begin{aligned}
T_6 &= \frac{\Delta x}{2}\,(f(1)+2\,f(1+\Delta)+2\,f(1+2\,\Delta x)+2\,f(1+3\,\Delta x)+2\,f(1+4\,\Delta x)+2\,f(1+5\,\Delta x)+f(5)) \\
&= \frac{1}{3}\left(f(1)+2\,f\left(\frac{5}{3}\right)+2\,f\left(\frac{7}{3}\right)+2\,f\,(3)+2\,f\left(\frac{11}{3}\right)+2\,f\left(\frac{13}{3}\right)+f(5)\right) \\
&= \frac{148072}{15015}, \\
M_6 &= \Delta x\,(f(1+\Delta/2)+f(1+3\,\Delta/2)+f(1+5\,\Delta/2)+f(1+7\,\Delta/2)+f(1+9\,\Delta/2)+f(1+11\,\Delta/2)) \\
&= \frac{669}{70}, \\
S_6 &= \frac{\Delta x}{3}\,(f(1)+4\,f(1+\Delta x)+2\,f(1+2\,\Delta x)+4\,f(1+3\,\Delta x)+2\,f(1+4\,\Delta x)+4\,f(1+5\,\Delta x)+f(5)) \\
&= \frac{2}{9}\left(f(1)+4\,f\left(\frac{5}{3}\right)+2\,f\left(\frac{7}{3}\right)+4\,f\,(3)+2\,f\left(\frac{11}{3}\right)+4\,f\left(\frac{13}{3}\right)+f(5)\right) \\
&= \frac{435976}{45045}.
\end{aligned}$$

For the numerical evaluations of these 9 rational numbers, see the following table. For reference, the exact value of the given integral is $6\,\ln(5)=9.6566\ldots$.

N	2	4	6
Trapezoidal Rule	56/5, or 11.2	10.1	9.862
Midpoint Rule	9	9.448	9.557
Simpson's Rule	152/15, or 10.133	9.733	9.679

2. The values of L needed are $L(0)=0, L(20)=5.13, L(40)=L(20)+9.45=14.58, L(60)=L(40)+13.85=28.43, L(80)=L(60)+20.33=48.76, L(100)=100$. Because $\Delta x=(100-0)/5=20$, we have

$$\begin{aligned}
\gamma &= \frac{1}{5000}\int_0^{100}\big(x-L(x)\big)\,dx \\
&= \frac{1}{5000}\int_0^{100} x\,dx-\frac{1}{5000}\int_0^{100} L(x)\,dx \\
&= 1-\frac{1}{5000}\int_0^{100} L(x)\,dx \\
&\approx 1-\frac{1}{5000}\cdot\frac{20}{2}\big(0+2\times 5.13+2\times 14.58+2\times 28.43+2\times 48.76+100\big) \\
&= 0.4124.
\end{aligned}$$

The World Bank gives 0.411 for the 2010 Gini coefficient. Source: `https://en.wikipedia.org/wiki/List_of_countries_by_income_equality`

3. Because $\Delta x=10$, we have, using the Trapezoidal Rule,

$$\gamma\approx 1-\frac{1}{5000}\cdot\frac{10}{2}\,(0+2\times3+2\times7+2\times12+2\times18+2\times24+2\times32+2\times41+2\times54+2\times73+100)=0.372$$

and, using Simpson's Rule,

$$\gamma\approx 1-\frac{1}{5000}\cdot\frac{10}{3}\,(0+4\times3+2\times7+4\times12+2\times18+4\times24+2\times32+4\times41+2\times54+4\times73+100)=0.3773.$$

4. First,

$$\begin{aligned}\int_0^9 c(t)\,dt &\approx \frac{1.5}{3}\,(0+4\times 2.4+2\times 6.3+4\times 9.7+2\times 7.1+4\times 2.3+0)\\ &= 42.2\,\frac{\text{mg}\quad\text{s}}{\text{L}}\\ &= \frac{42.2}{60}\,\frac{\text{mg}\quad\text{min}}{\text{L}}\\ &= 0.7033\,\frac{\text{mg}\quad\text{min}}{\text{L}}.\end{aligned}$$

Therefore,

$$r=\frac{5}{0.7033}\,\frac{\text{mg}\quad\text{L}}{\text{mg}\quad\text{min}}=7.109\,\frac{\text{L}}{\text{min}}.$$

5. The cross-sectional area A is approximately given by

$$A\approx\frac{2}{3}\,(0.77+4\times 1.5+2\times 2.0+4\times 2.4+2\times 2.8+4\times 3.1+2\times 3.4+4\times 3.6+3.9)=42.3133\,\text{m}^2.$$

The volume V is therefore approximately given by $V\approx 6\,\text{m}\;\times\;42.3133\,\text{m}^2=253.88\,\text{m}^3$.

6. The distance s (in meters) that the athlete ran was

$$\begin{aligned}s &= \int_0^5 s'(t)\,dt\\ &\approx \frac{1/2}{3}\Big(0+4\times 5.26+2\times 6.67+4\times 7.41+2\times 8.33\\ &\qquad +4\times 8.33+2\times 9.52+4\times 9.52+2\times 10.64+4\times 10.64+10.87\Big)\\ &= 40.97.\end{aligned}$$

7. The nodes are $x_0=0$, $x_1=1$, $x_2=2$, $x_3=3$, $x_4=4$, $x_5=5$, , $x_6=6$. Thus, $N=6$ and $\Delta x=(6-0)/N=1$. The heights of the vertical line segments between the curves and over the nodes are $y_0=0$, $y_1=5$, $y_2=3$, $y_3=4$, $y_4=6, y_5=3$, $y_6=0$. The trapezoidal Rule approximation is

$$\begin{aligned}T_6 &= \frac{\Delta x}{2}\,(1\times y_0+2\times y_1+2\times y_2+2\times y_3+2\times y_4+2\times y_5+1\times y_6)\\ &= \frac{1}{2}\,(1\times 0+2\times 5+2\times 3+2\times 4+2\times 6+2\times 3+1\times 0)\\ &= 21.\end{aligned}$$

and the Simpson Rule approximation is

$$\begin{aligned}S_6 &= \frac{\Delta x}{3}\,(1\times y_0+4\times y_1+2\times y_2+4\times y_3+2\times y_4+4\times y_5+1\times y_6)\\ &= \frac{1}{3}\,(1\times 0+4\times 5+2\times 3+4\times 4+2\times 6+4\times 3+1\times 0)\\ &= 22.\end{aligned}$$

Solutions: Improper Integrals (Corresponds to Stewart 7.8)

1. The following calculation both shows that the given improper integral is convergent and evaluates it:

$$\int_2^4 (4-x)^{-0.9}\,dx = \lim_{\epsilon\to 0^+}\int_2^{4-\epsilon}(4-x)^{-0.9}\,dx = \lim_{\epsilon\to 0^+}\left(-\frac{(4-x)^{0.1}}{0.1}\right)\Bigg|_2^{4-\epsilon} = \lim_{\epsilon\to 0^+}\left(-\frac{(2)^{0.1}}{0.1}+\frac{(\epsilon)^{0.1}}{0.1}\right) = 10{\cdot}2^{0.1}.$$

2. The following calculation both shows that the given improper integral is convergent and evaluates it:

$$\int_1^2 (x-2)^{-1/5}\,dx = \lim_{\epsilon\to 0^+}\int_1^{2-\epsilon}(x-2)^{-1/5}\,dx = \lim_{\epsilon\to 0^+}\left(\frac{(x-2)^{4/5}}{4/5}\right)\Bigg|_1^{2-\epsilon} = \lim_{\epsilon\to 0^+}\left(\frac{5\times\epsilon^{4/5}}{4}-\frac{5\times 1^{4/5}}{4}\right) = -\frac{5}{4}.$$

3. The following calculation both shows that the given improper integral is convergent and evaluates it:

$$\int_{-1}^3 \frac{1}{\sqrt{x+1}}\,dx = \lim_{\epsilon\to 0^+}\int_{-1+\epsilon}^3 \frac{1}{\sqrt{x+1}}\,dx = \lim_{\epsilon\to 0^+} 2\sqrt{x+1}\ \Big|_{-1+\epsilon}^3 = \lim_{\epsilon\to 0^+} 4-\sqrt{\epsilon} = 4.$$

4. The following calculation both shows that the given improper integral is convergent and evaluates it:

$$\int_0^1 \frac{x}{(1-x^2)^{1/4}}\,dx = \lim_{\epsilon\to 0^+}\int_0^{1-\epsilon}\frac{x}{(1-x^2)^{1/4}}\,dx = \lim_{\epsilon\to 0^+}\left(-\frac{2}{3}\left(1-x^2\right)^{3/4}\right)\Bigg|_0^{1-\epsilon} = \lim_{\epsilon\to 0^+}\left(-\frac{2}{3}\left(2\epsilon-\epsilon^2\right)^{3/4}+\frac{2}{3}\right) = \frac{2}{3}.$$

5. The following calculation both shows that the given improper integral is convergent and evaluates it:

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{\epsilon\to 0^+}\int_0^{1-\epsilon}\frac{1}{\sqrt{1-x^2}}\,dx = \lim_{\epsilon\to 0^+}\arcsin(x)\ \Big|_0^{1-\epsilon} = \lim_{\epsilon\to 0^+}\arcsin(1-\epsilon) = \arcsin(1) = \frac{\pi}{2}.$$

6. Expand the integrand:

$$\begin{aligned}\int_0^3 x^{-1/2}(1+x)\,dx &= \int_0^3 x^{1/2}\,dx + \lim_{\epsilon\to 0^+}\int_\epsilon^3 x^{-1/2}\,dx\\ &= 2\sqrt{3} + \lim_{\epsilon\to 0^+} 2\,x^{1/2}\Big|_\epsilon^3\\ &= 2\sqrt{3}+2\sqrt{3}-2\sqrt{\epsilon}\\ &= 4\sqrt{3}.\end{aligned}$$

7. Make the change of variable $u = \ln(x)$, $du = (1/x)\,dx$ and observe that the limits of integration for u are from $u = \ln(1) = 0$ to $u = \ln(e) = 1$:

$$\begin{aligned}
\int_1^e \frac{1}{x \cdot \ln^{1/3}(x)}\,dx &= \int_0^1 u^{-1/3}\,du \\
&= \lim_{\epsilon \to 0^+} \int_\epsilon^1 u^{-1/3}\,du \\
&= \lim_{\epsilon \to 0^+} \frac{3}{2}\,u^{2/3}\Big|_\epsilon^1 \\
&= \frac{3}{2} - \frac{3}{2}\,\epsilon^{2/3} \\
&= \frac{3}{2}.
\end{aligned}$$

8. We use the integral formula $\int \ln(x)\ dx = x\ \ln(x) - x + C$, which was obtained in the section on integration by parts. In the third line of the calculation that follows, the indeterminate form $0 \times \infty$ arises. A change of variable is made to convert the indeterminate form to the form ∞/∞, which we evaluate by applying L'Hôpital's Rule:

$$\begin{aligned}
\int_0^1 \ln(x)\ dx &= \lim_{\epsilon \to 0^+} \int_\epsilon^1 \ln(x)\ dx \\
&= \lim_{\epsilon \to 0^+} (x\ \ln(x) - x)\ \Big|_\epsilon^1 \\
&= (1 \times \ln(1) - 1) - \lim_{\epsilon \to 0^+} (\epsilon \times \ln(\epsilon) - \epsilon) \\
&= -1 - \lim_{\epsilon \to 0^+} \epsilon \times \ln(\epsilon) \\
&= -1 - \lim_{\epsilon \to 0^+} \frac{\ln(\epsilon)}{1/\epsilon} \\
&= -1 + \lim_{\epsilon \to 0^+} \frac{\ln(1/\epsilon)}{1/\epsilon} \\
&= -1 + \lim_{t \to +\infty} \frac{\ln(t)}{t} \\
&= -1 + \lim_{t \to +\infty} \frac{\frac{d}{dt}\ln(t)}{\frac{d}{dt}t} \\
&= -1 + \lim_{t \to +\infty} \frac{1/t}{1} \\
&= -1.
\end{aligned}$$

9. The solution uses a basic integration formula involving the arcsecant as the antiderivative:

$$\begin{aligned}\int_1^{\sqrt{2}} \frac{1}{x\sqrt{x^2-1}}\,dx &= \lim_{\epsilon\to 0^+}\int_{1+\epsilon}^{\sqrt{2}} \frac{1}{x\sqrt{x^2-1}}\,dx \\ &= \lim_{\epsilon\to 0^+} \operatorname{arcsec}(x)\ \Big|_{1+\epsilon}^{\sqrt{2}} \\ &= \lim_{\epsilon\to 0^+}\left(\operatorname{arcsec}\left(\sqrt{2}\right) - \operatorname{arcsec}(1+\epsilon)\right) \\ &= \frac{\pi}{4} - 0 \\ &= \frac{\pi}{4}.\end{aligned}$$

10. The singularity is at $x = -1$ in the interior of the interval of integration, which means that the given integral must be decomposed into two improper integrals. Both are of types already treated in detail, so no further detail will be given here:

$$\int_{-2}^{4}(x+1)^{-2/3}\,dx = \int_{-2}^{-1}(x+1)^{-2/3}\,dx + \int_{-1}^{4}(x+1)^{-2/3}\,dx = 3 + 3\cdot 5^{1/3} = 3\left(5^{1/3}+1\right).$$

11. The integrand is very similar to the one in Exercise 7, and it is handled in exactly the same way. The current exercise does have, however, an additional complication: the simgularity is at $x = 1$, which is inside the interval of integration. So the given integral must be decomposed into two improper integrals, as was done in the preceding exercise. We have

$$\int_{1/e}^{e} \frac{1}{x\ \ln^{2/7}(x)}\,dx = \int_{1/e}^{1}\frac{1}{x\ln^{2/7}(x)}\,dx + \int_{1}^{e}\frac{1}{x\ln^{2/7}(x)}\,dx = \frac{7}{5}+\frac{7}{5} = \frac{14}{5}.$$

12. Issues at both limits of integration must be treated:

$$\begin{aligned}\int_{-2}^{2}\frac{1}{\sqrt{4-x^2}}\,dx &= \int_{-2}^{0}\frac{1}{\sqrt{4-x^2}}\,dx + \int_{0}^{2}\frac{1}{\sqrt{4-x^2}}\,dx \\ &= \lim_{\epsilon\to 0^+}\int_{-2+\epsilon}^{0}\frac{1}{\sqrt{4-x^2}}\,dx + \lim_{\epsilon\to 0^+}\int_{0}^{2-\epsilon}\frac{1}{\sqrt{4-x^2}}\,dx \\ &= \lim_{\epsilon\to 0^+}\arcsin\left(\frac{x}{2}\right)\Big|_{-2+\epsilon}^{0} + \lim_{\epsilon\to 0^+}\arcsin\left(\frac{x}{2}\right)\Big|_{0}^{2-\epsilon} \\ &= -\lim_{\epsilon\to 0^+}\arcsin\left(-1+\frac{\epsilon}{2}\right) + \lim_{\epsilon\to 0^+}\arcsin\left(1-\frac{\epsilon}{2}\right) \\ &= -\arcsin(-1) + \arcsin(1) \\ &= -\left(-\frac{\pi}{2}\right) + \frac{\pi}{2} \\ &= \pi.\end{aligned}$$

13. Integrating by parts twice yields the integral formula

$$\int \ln^2(x)\ dx = x\left(\ln^2(x) - 2\ \ln(x) + 2\right) + C.$$

In addition to this integral formula, we will need two limit formulas. One of them,

$$\lim_{\epsilon\to 0^+}\epsilon\ \ln(\epsilon) = 0$$

was derived in Exercise 7. The other is obtained in a similar way:

$$\begin{aligned}
\lim_{\epsilon\to0^+} \epsilon \ln^2(\epsilon) &= \lim_{\epsilon\to0^+} \frac{\ln^2(\epsilon)}{1/\epsilon} \\
&= \lim_{\epsilon\to0^+} \frac{\left(-\ln(1/\epsilon)\right)^2}{1/\epsilon} \\
&= \lim_{t\to+\infty} \frac{\left(\ln(t)\right)^2}{t} \\
&= \lim_{t\to+\infty} \frac{\frac{d}{dt}\left(\ln(t)\right)^2}{\frac{d}{dt}t} \\
&= 2 \lim_{t\to+\infty} \frac{\ln(t)}{t} \\
&= 2 \lim_{t\to+\infty} \frac{\frac{d}{dt}\ln(t)}{\frac{d}{dt}t} \\
&= 2 \lim_{t\to+\infty} \frac{1}{t} \\
&= 0.
\end{aligned}$$

Using these three equations, we have

$$\begin{aligned}
\int_0^1 \ln^2(x)\, dx &= \lim_{\epsilon\to0^+} \int_\epsilon^1 \ln^2(x)\, dx \\
&= \lim_{\epsilon\to0^+} \left(x\left(\ln^2(x) - 2\ln(x) + 2\right)\right)\Big|_\epsilon^1 \\
&= 2 - \lim_{\epsilon\to0^+} \epsilon\left(\ln^2(\epsilon) - 2\ln(\epsilon) + 2\right) \\
&= 2.
\end{aligned}$$

14. Integrating by parts, we have

$$\int x^{-1/2}\ln(x)\,dx = 2\,x^{1/2}\ln(x) - 2\int x^{-1/2}\frac{1}{x}\,dx = 2\,x^{1/2}\ln(x) - 4\,x^{1/2} + C = 2\,x^{1/2}\left(\ln(x) - 2\right).$$

Additionally, we will need one limit formula, which we obtain from a limit formula already derived

$$\lim_{\epsilon\to0^+} \epsilon^{1/2}\ln(\epsilon) = 2\lim_{\epsilon\to0^+} \epsilon^{1/2}\ln\left(\epsilon^{1/2}\right) = 2\lim_{u\to0^+} u\ln(u) = 0.$$

Therefore

$$\begin{aligned}
\int_0^4 x^{-1/2}\ln(x)\,dx &= \lim_{\epsilon\to0^+}\int_\epsilon^4 x^{-1/2}\ln(x)\,dx \\
&= \lim_{\epsilon\to0^+} 2\,x^{1/2}\left(\ln(x) - 2\right)\Big|_\epsilon^4 \\
&= 2\,4^{1/2}\left(\ln(4) - 2\right) - \lim_{\epsilon\to0^+} 2\,\epsilon^{1/2}\left(\ln(\epsilon) - 2\right) \\
&= 4\left(2\ln(2) - 2\right) - 2\lim_{\epsilon\to0^+}\epsilon^{1/2}\ln(\epsilon) \\
&= 8\left(\ln(2) - 1\right).
\end{aligned}$$

15. Issues at both limits of integration must be treated. The method is to expand the logarithm. Doing so leads to two similar integrations for which we use the formula

$$\int \ln(1 + a\,x)\,dx = \left(x + \frac{1}{a}\right)\,\ln(1 + a\,x) - x + C$$

with $a = -1$ and $a = 1$. The cited formula may be obtained by integrating by parts with $u = \ln(1+a\,x)$ and $dv = dx$. For $a = -1$ it yields

$$\int \ln(1 - x)\,dx = (x - 1)\,\ln(1 - x) - x + C$$

an for $a = 1$ it yields

$$\int \ln(1 + x)\,dx = (x + 1)\,\ln(1 + x) - x + C.$$

The following evaluation makes use of these integral formulas, as well as the limit formula $\lim_{\epsilon\to 0^+} \epsilon\,\ln(\epsilon) = 0$, which was derived in Exercise 8:

$$\begin{aligned}
\int_{-1}^{1} \ln\left(1 - x^2\right)\,dx &= \int_{-1}^{1} \ln\big((1-x)(1+x)\big)\,dx \\
&= \int_{-1}^{1} \ln(1-x)\,dx + \int_{-1}^{1} \ln(1+x)\,dx \\
&= \lim_{\epsilon\to 0^+} \int_{-1}^{1-\epsilon} \ln(1-x)\,dx + \lim_{\epsilon\to 0^+} \int_{-1+\epsilon}^{1} \ln(1+x)\,dx \\
&= \lim_{\epsilon\to 0^+} \big((x-1)\,\ln(1-x) - x\big)\,\Big|_{-1}^{1-\epsilon} + \lim_{\epsilon\to 0^+} \big((x+1)\,\ln(1+x) - x\big)\,\Big|_{-1+\epsilon}^{1} \\
&= \lim_{\epsilon\to 0^+} (-\epsilon\,\ln(\epsilon) - 1 + \epsilon) - (-2\,\ln(2) + 1) + 2\,\ln(2) - 1 - \lim_{\epsilon\to 0^+} (\epsilon\,\ln(\epsilon) + 1 - \epsilon) \\
&= 4\,(\ln(2) - 1)\,.
\end{aligned}$$

16. We have

$$\int_4^{\infty} x^{-3/2}\,dx = \lim_{N\to\infty} \int_4^{N} x^{-3/2}\,dx = \lim_{N\to\infty} \quad -2\,x^{-1/2}\Big|_4^N = 1 - 2\,\lim_{N\to\infty} \frac{1}{N^{1/2}} = 1.$$

17. We have

$$\int_0^{\infty} \frac{1}{1+x^2}\,dx = \lim_{N\to\infty} \int_0^{N} \frac{1}{1+x^2}\,dx = \lim_{N\to\infty} \quad \arctan(x)\Big|_0^N = \lim_{N\to\infty} \arctan(N) = \frac{\pi}{2}.$$

18. We have

$$\int_0^{\infty} \frac{x}{\left(1+x^2\right)^2}\,dx = \lim_{N\to\infty} \int_0^{N} \frac{x}{\left(1+x^2\right)^2}\,dx = -\frac{1}{2}\,\lim_{N\to\infty} \quad \left(1+x^2\right)^{-1}\Big|_0^N = \frac{1}{2}\left(1 - \lim_{N\to\infty} \frac{1}{1+N^2}\right) = \frac{1}{2}.$$

19. We have

$$\int_2^{\infty} e^{-x/2}\,dx = \lim_{N\to\infty} \int_2^{N} e^{-x/2}\,dx = -2\,\lim_{N\to\infty} \quad e^{-x/2}\Big|_2^N = 2\,\left(e^{-1} - \lim_{N\to\infty} e^{-N/2}\right) = \frac{2}{e}.$$

20. A partial fraction decomposition, the details of which are omitted, is key. We have

$$\begin{aligned}
\int_2^\infty \frac{1}{x\,(x-1)}\,dx &= \lim_{N\to\infty}\int_2^N \frac{1}{x\,(x-1)}\,dx \\
&= \lim_{N\to\infty}\int_2^N \left(\frac{1}{x-1}-\frac{1}{x}\right)dx \\
&= \lim_{N\to\infty}\left(\ln(\,|x-1|\,)-\ln(\,|x|\,)\right)\Big|_2^N \\
&= \lim_{N\to\infty}\ln\left(\frac{|x-1|}{|x|}\right)\Big|_2^N \\
&= \lim_{N\to\infty}\ln\left(\frac{N-1}{N}\right)-\ln\left(\frac{1}{2}\right) \\
&= \lim_{N\to\infty}\ln\left(1-\frac{1}{N}\right)+\ln(2) \\
&= \ln(1)+\ln(2) \\
&= \ln(2)\,.
\end{aligned}$$

21. Begin with the substitution $u = e^x$, $du = e^x\,dx$. The limits of integration for u are $u = e^0 = 1$ and $u = e^\infty = \infty$. It follows that

$$\begin{aligned}
\int_0^\infty \frac{e^x}{e^{2x}+1}\,dx &= \int_1^\infty \frac{1}{u^2+1}\,du \\
&= \lim_{N\to\infty}\int_1^N \frac{1}{u^2+1}\,du \\
&= \lim_{N\to\infty}\arctan(u)\Big|_1^N \\
&= \lim_{N\to\infty}\arctan(N)-\arctan(1) \\
&= \frac{\pi}{2}-\frac{\pi}{4} \\
&= \frac{\pi}{4}.
\end{aligned}$$

22. Begin with the substitution $u = e^x$, $du = e^x\,dx$. The limits of integration for u are $u = e^0 = 1$ and $u = e^\infty = \infty$. It follows that

$$\begin{aligned}
\int_0^\infty \frac{e^x}{(e^x+1)^3}\,dx &= \int_1^\infty \frac{1}{(u+1)^3}\,du \\
&= \lim_{N\to\infty}\int_1^N \frac{1}{(u+1)^3}\,du \\
&= -\frac{1}{2}\lim_{N\to\infty}(u+1)^{-2}\Big|_1^N \\
&= \frac{1}{2}\left(\frac{1}{4}-\lim_{N\to\infty}\frac{1}{(N+1)^2}\right) \\
&= \frac{1}{2}\cdot\frac{1}{4}-0 \\
&= \frac{1}{8}.
\end{aligned}$$

23. A partial fraction decomposition, the details of which are omitted, is key. We have

$$\begin{aligned}
\int_0^\infty \left(\frac{x+1}{x^2+1}\right)^2 dx &= \lim_{N\to\infty} \int_0^N \left(\frac{x+1}{x^2+1}\right)^2 dx \\
&= \lim_{N\to\infty} \int_0^N \left(\frac{1}{x^2+1} + \frac{2x}{(x^2+1)^2}\right) dx \\
&= \lim_{N\to\infty} \left(\arctan(x) - \frac{1}{x^2+1}\right)\Bigg|_0^N \\
&= \lim_{N\to\infty} \left(\arctan(N) - \frac{1}{N^2+1}\right) - \left(\arctan(0) - \frac{1}{0^2+1}\right) \\
&= \frac{\pi}{2} + 1.
\end{aligned}$$

24. Begin with the substitution $u = 3x^2$, $du = 6x\,dx$. The limits of integration for u are $u = 3$ and $u = \infty$. It follows that

$$\begin{aligned}
\int_1^\infty x\,e^{-3x^2}\,dx &= \frac{1}{6}\int_3^\infty e^{-u}\,du \\
&= \frac{1}{6}\lim_{N\to\infty}\int_3^N e^{-u}\,du \\
&= -\frac{1}{6}\lim_{N\to\infty} e^{-u}\ \Big|_3^N \\
&= \frac{1}{6}\left(e^{-3} - \lim_{N\to\infty} e^{-N}\right) \\
&= \frac{1}{6e^3}.
\end{aligned}$$

25. Begin with the substitution $u = \ln(x)$, $du = (1/x)\,dx$. The limits of integration for u are $u = 1$ and $u = \infty$. It follows that

$$\begin{aligned}
\int_e^\infty \frac{1}{x\ln^2(x)}\,dx &= \int_1^\infty \frac{1}{u^2}\,du \\
&= \lim_{N\to\infty}\int_1^N \frac{1}{u^2}\,du \\
&= -\lim_{N\to\infty} \frac{1}{u}\Big|_1^N \\
&= 1 - \lim_{N\to\infty}\frac{1}{N} \\
&= 1.
\end{aligned}$$

26. We have

$$\int_0^\infty \left(\frac{2}{3}\right)^x dx = \lim_{N\to\infty}\int_0^N \left(\frac{2}{3}\right)^x dx = \frac{1}{\ln(2/3)}\lim_{N\to\infty}\left(\frac{2}{3}\right)^x\Bigg|_0^N = -\frac{1}{\ln(2/3)} = \left(\ln\left(\frac{3}{2}\right)\right)^{-1}.$$

27. We have

$$\int_{-\infty}^{-2} x^{-3}\,dx = \lim_{N\to\infty}\int_{-N}^{-2} x^{-3}\,dx = -\frac{1}{2}\lim_{N\to\infty} x^{-2}\ \Big|_{-N}^{-2} = -\frac{1}{8}.$$

28. We have

$$\int_{-\infty}^{-2} \frac{1}{(1+x)^{4/3}}\,dx = \lim_{N\to\infty}\int_{-N}^{-2} \frac{1}{(1+x)^{4/3}}\,dx = -3\lim_{N\to\infty}(1+x)^{-1/3}\Big|_{-N}^{-2} = 3.$$

29. We have

$$\int_{-\infty}^{1} \frac{x}{(1+x^2)^2}\,dx = \lim_{N\to\infty}\int_{-N}^{1} \frac{x}{(1+x^2)^2}\,dx = -\frac{1}{2}\lim_{N\to\infty}\frac{1}{1+x^2}\Big|_{-N}^{1} = -\frac{1}{4}.$$

30. We integrate by parts to obtain

$$\int x\exp(x/2)\,dx = 2\,(x-2)\,\exp(x/2) + C.$$

We use this antiderivative formula to evaluate the given improper integral. To obtain the last line of the calculation, we apply L'Hôpital's Rule to show that $\lim_{N\to\infty}(-N-2)\,\exp(-N/2) = 0$:

$$\int_{-\infty}^{0} x\exp(x/2)\,dx = \lim_{N\to\infty}\int_{-N}^{0} x\exp(x/2)\,dx = \lim_{N\to\infty} 2\,(x-2)\,\exp(x/2)\Big|_{-N}^{0} = -4.$$

31. We have

$$\int_{0}^{\infty} \frac{1}{4+x^2}\,dx = \lim_{N\to\infty}\int_{0}^{N} \frac{1}{4+x^2}\,dx = \lim_{N\to\infty}\frac{1}{2}\arctan\left(\frac{x}{2}\right)\Big|_{0}^{N} = \frac{\pi}{4}.$$

A similar calculation provides the same value for the integration over the interval $(-\infty, 0]$. Thus,

$$\int_{-\infty}^{\infty} \frac{1}{4+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{4+x^2}\,dx + \int_{0}^{\infty} \frac{1}{4+x^2}\,dx = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}.$$

Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^2+2x+10}\,dx.$

We will need the basic integral formula

$$\int \frac{1}{u^2+a^2}\,du = \frac{1}{a}\arctan\left(\frac{u}{a}\right) + C.$$

With $u = x+1$ and $a = 3$, it follows that

$$\int \frac{1}{x^2+2x+10}\,dx = \int \frac{1}{(x+1)^2+3^2}\,dx = \frac{1}{3}\arctan\left(\frac{x+1}{3}\right) + C.$$

Thus,

$$\begin{aligned}
\int_{0}^{\infty} \frac{1}{x^2+2x+10}\,dx &= \lim_{N\to\infty}\int_{0}^{N} \frac{1}{x^2+2x+10}\,dx \\
&= \lim_{N\to\infty}\frac{1}{3}\arctan\left(\frac{x+1}{3}\right)\Big|_{0}^{N} \\
&= \lim_{N\to\infty}\frac{1}{3}\arctan\left(\frac{N+1}{3}\right) - \arctan\left(\frac{1}{3}\right) \\
&= \frac{\pi}{6} - \arctan\left(\frac{1}{3}\right).
\end{aligned}$$

Similarly,

$$\begin{aligned}\int_{-\infty}^{0} \frac{1}{x^2+2x+10}\,dx &= \lim_{N\to\infty}\int_{-N}^{0} \frac{1}{x^2+2x+10}\,dx \\ &= \lim_{N\to\infty} \frac{1}{3} \arctan\left(\frac{x+1}{3}\right)\Big|_{-N}^{0} \\ &= \arctan\left(\frac{1}{3}\right) - \lim_{N\to\infty}\frac{1}{3}\arctan\left(\frac{-N+1}{3}\right) \\ &= \arctan\left(\frac{1}{3}\right) - \left(\frac{-\pi}{6}\right).\end{aligned}$$

It follows that

$$\int_{-\infty}^{\infty} \frac{1}{x^2+2x+10}\,dx = \frac{\pi}{6} - \arctan\left(\frac{1}{3}\right) + \arctan\left(\frac{1}{3}\right) - \left(\frac{-\pi}{6}\right) = \frac{\pi}{3}.$$

32. The integrand is a rational fraction that is already in partial fraction form. We expand the numerator and split the interval of integration—the result is a sum of four improper integrals:

$$\int_{-\infty}^{\infty} \frac{x+2}{(x^2+1)^2}\,dx = \int_{-\infty}^{0} \frac{x}{(x^2+1)^2}\,dx + \int_{-\infty}^{0} \frac{2}{(x^2+1)^2}\,dx + \int_{0}^{\infty} \frac{x}{(x^2+1)^2}\,dx + \int_{0}^{\infty} \frac{2}{(x^2+1)^2}\,dx.$$

We will refer to the four improper integrals on the right side of this equation as, from left to right, I_1, I_2, I_3, and I_4. We have

$$\begin{aligned} I_1 &= \lim_{N\to\infty}\int_{-N}^{0} \frac{x}{(x^2+1)^2}\,dx \\ &= -\frac{1}{2}\lim_{N\to\infty} \frac{1}{x^2+1}\Big|_{-N}^{0} \\ &= -\frac{1}{2}\lim_{N\to\infty}\left(1 - \frac{1}{N^2+1}\right) \\ &= -\frac{1}{2}.\end{aligned}$$

From this calculation we deduce that I_3 is also convergent and equal to $-I_1$, or $1/2$. Thus, $I_1 + I_2 = 0$. Next, using the general formula that concludes the solution of Exercise 7 from the set on Trigonometric Substitution, we have

$$\begin{aligned} I_2 &= \lim_{N\to\infty}\int_{-N}^{0} \frac{2}{(x^2+1)^2}\,dx \\ &= \lim_{N\to\infty}\left(\frac{x}{1+x^2} + \arctan(x)\right)\Big|_{-N}^{0} \\ &= 0 - \lim_{N\to\infty}\left(\frac{-N}{1+N^2} + \arctan(-N)\right) \\ &= 0 - \left(0 + \left(-\frac{\pi}{2}\right)\right) \\ &= \frac{\pi}{2}.\end{aligned}$$

From this calculation we deduce that I_4 is also convergent and equal to I_2, or $\pi/2$. Thus,

$$\int_{-\infty}^{\infty} \frac{x+2}{(x^2+1)^2}\,dx = I_1 + I_2 + I_3 + I_4 = -\frac{1}{2} + \frac{\pi}{2} + \frac{1}{2} + \frac{\pi}{2} = \pi.$$

Solutions: Arc Length (Corresponds to Stewart 8.1)

1. Let $f(x) = x^2/8 - \ln(x)$, we have

$$\begin{aligned} 1 + \left(\frac{d}{dx} f(x)\right)^2 &= 1 + \left(\frac{x}{4} - \frac{1}{x}\right)^2 \\ &= 1 + \left(\left(\frac{x}{4}\right)^2 - 2\,\frac{x}{4}\,\frac{1}{x} + \left(\frac{1}{x}\right)^2\right) \\ &= 1 + \left(\left(\frac{x}{4}\right)^2 - \frac{1}{2} + \left(\frac{1}{x}\right)^2\right) \\ &= \quad \left(\left(\frac{x}{4}\right)^2 + \frac{1}{2} + \left(\frac{1}{x}\right)^2\right) \\ &= \left(\frac{x}{4} + \frac{1}{x}\right)^2. \end{aligned}$$

The requested arc length L is therefore given by

$$L = \int_1^2 \sqrt{1 + \left(\frac{d}{dx} f(x)\right)^2}\, dx = \int_1^2 \sqrt{\left(\frac{x}{4} + \frac{1}{x}\right)^2}\, dx = \int_1^2 \left(\frac{x}{4} + \frac{1}{x}\right)\, dx = \frac{3}{8} + \ln(2).$$

2. i) We have

$$\left(\frac{d}{dx} x^3\right)\left(\frac{d}{dx} x^p\right) = \left(3x^2\right)\left(p\,x^{p-1}\right) = 3\,p\,x^{p+1},$$

which is a constant only if $p + 1 = 0$, or $p = -1$.

ii) Let $f(x) = a\,x^3 + b\,x^{-1}$. Then

$$\left(\frac{d}{dx} f(x)\right)^2 = \left(\left(3\,a\,x^2\right)^2 + 2\,\left(3\,a\,x^2\right)\,\left(-b\,x^{-2}\right) + \left(-b\,x^{-2}\right)^2\right).$$

The cross term $2\,\left(3\,a\,x^2\right)\,\left(-b\,x^{-2}\right)$ simplifies to $-6\,a\,b$, and for this to equal -1/2 we must have $b = \dfrac{1}{12\,a}$.

iii) Let $a = 1$ and $b = 1/12$. Then $f(x) = x^3 + 1/(12\,x)$. Then

$$\begin{aligned} 1 + \left(\frac{d}{dx} f(x)\right)^2 &= 1 + \left(3x^2 - \frac{1}{12\,x^2}\right)^2 \\ &= 1 + \left(3x^2\right)^2 - \frac{1}{2} + \left(\frac{1}{12\,x^2}\right)^2 \\ &= \left(3x^2\right)^2 + \frac{1}{2} + \left(\frac{1}{12\,x^2}\right)^2 \\ &= \left(3x^2 + \frac{1}{12\,x^2}\right)^2. \end{aligned}$$

The requested arc length is

$$\int_1^2 \sqrt{1 + \left(\frac{d}{dx} f(x)\right)^2}\, dx = \int_1^2 \sqrt{\left(3x^2 + \frac{1}{12\,x^2}\right)^2}\, dx = \int_1^2 \left(3x^2 + \frac{1}{12\,x^2}\right) dx = \frac{169}{24}.$$

3. The arc length is $\int_{1/2}^{1} \sqrt{1 + 4x^2}\, dx$. With $n = 4$, we have $\Delta x = 1/8$ and nodes $1/2$, $5/8$, $3/4$, $7/8$, 1. Letting $g(x) = \sqrt{1 + 4x^2}$, the requested Simpson's Rule approximation is

$$\frac{\Delta x}{3}\left(g\left(\frac{1}{2}\right) + 4\,g\left(\frac{5}{8}\right) + 2\,g\left(\frac{3}{4}\right) + 4\,g\left(\frac{7}{8}\right) + g\,(1)\right), \quad \text{or} \quad 0.9050506.$$

The exact length is 0.9050460693, to 10 decimal places.

4. We have two options: we can either write the equation of the curve as $y = f(x)$ or as $x = g(y)$. The former involves solving a cubic, whereas the latter involves solving a quadratic. Being sensible, we choose the latter. We can either use the Quadratic Formula, or we can complete the square by adding 1 to both side of the equation. Either way, we find $x = g(y) = -1 + \sqrt{y^3 + y + 7}$. (The negative sign in the $\pm$ of the solution for y results in a branch for which $y \le -1$, and we want a branch that includes the values $-1.5 \le y \le 1$). The requested length L is given by

$$L = \int_{-1.5}^{1} \sqrt{1 + \frac{\left(3\,y^2 + 1\right)^2}{4\,\left(y^3 + y + 7\right)}}\, dy.$$

An accurate numerical integration, not required, yields the value 3.099775857.

Life is filled with choices. Three types of choices, in fact: good choices and bad choices. Had we solved the equation $x^2 + 2\,x = y^3 + y + 6$ for y instead of x, we would have obtained, with a little bit of work, the details of which are omitted, the following formula for $y = f(x)$:

$$f(x) = \frac{1}{6}\left(108\,x^2 + 216\,x - 648 + 12\sqrt{81\,x^4 + 324\,x^3 - 648\,x^2 - 1944\,x + 2928}\right)^{1/3} - 2\left(108\,x^2 + 216\,x - 648 + 12\sqrt{81\,x^4 + 324\,x^3 - 648\,x^2 - 1944\,x + 2928}\right)^{-1/3}$$

We will omit the expression for $\sqrt{1 + f'(x)^2}$, which is rather intimidating: the expression for $f(x)$ is already sufficiently convincing that it presents a more difficult route to the arc length. Nevertheless. as far as the numerical value of the arc length, it is five of one, five-sixths of half a dozen of the other, as they say. A screen capture from a Maple session shows the same answer when the length is expressed as an integral with respect to x:

```
> Int( 'sqrt(1 + D(f)(x)^2 )', x = 0.4577379737..2)
                    = int( sqrt(1 + D(f)(x)^2 ), x = 0.4577379737..2);
```

$$\int_{0.4577379737}^{2} \sqrt{1 + \mathrm{D}(f)(x)^2}\, dx = 3.099775857$$

5. We have

$$\begin{aligned} 1 + \left(\frac{d}{dx}\,\frac{a}{2}\,(\exp(x/a) + \exp(-x/a))\right)^2 &= 1 + \left(\frac{1}{2}\,(\exp(x/a) - \exp(-x/a))\right)^2 \\ &= 1 + \left(\frac{1}{4}\,\exp(2x/a) - \frac{1}{2} + \frac{1}{4}\,\exp(-x/a)\right) \\ &= \frac{1}{4}\,\exp(2x/a) + \frac{1}{2} + \frac{1}{4}\,\exp(-x/a) \\ &= \frac{1}{4}\,(\exp(2x/a) + 2 + \exp(-2x/a)) \\ &= \frac{1}{4}\,(\exp(x/a) + \exp(-x/a))^2\,. \end{aligned}$$

The requested arc length L is given by

$$\begin{aligned} L &= \int_{-a}^{a} \sqrt{1 + \left(\frac{d}{dx}\,\frac{a}{2}\,(\exp(x/a) + \exp(-x/a))\right)^2}\, dx \\ &= \int_{-a}^{a} \sqrt{\frac{1}{4}\,(\exp(x/a) + \exp(-x/a))^2}\, dx \\ &= \int_{-a}^{a} \frac{1}{2}\,(\exp(x/a) + \exp(-x/a))\, dx \\ &= a\left(e - \frac{1}{e}\right) \end{aligned}$$

Solutions: Area of a Surface of Revolution (Corresponds to Stewart 8.2)

1. Making the change of variable $u = 1 + 9\,x^4$, $du = 36\,x^3\,dx$, we find that the surface area is given by

$$\int_0^1 2\,\pi\,f(x)\,\sqrt{1 + f'(x)^2}\,dx = \int_0^1 2\,\pi\,x^3\sqrt{1 + 9\,x^4}\,dx = \frac{\pi}{18}\int_1^{10}\sqrt{u}\,du = \frac{\pi}{27}\left(10\sqrt{10} - 1\right).$$

2. We write $x = g(y) = y^3$. The calculation is identical to that of the preceding exercise except for the limits of integration (and using y for the variable of integration, although the symbol used for the variable of integration has no effect on the numerical value of a definite integral):

$$\int_1^2 2\,\pi\,f(y)\,\sqrt{1 + g'(y)^2}\,dy = \int_1^2 2\,\pi\,y^3\sqrt{1 + 9\,y^4}\,dy = \frac{\pi}{18}\int_{10}^{145}\sqrt{u}\,du = \frac{\pi}{27}\left(145\sqrt{145} - 10\sqrt{10}\right).$$

3. Let $f(x) = \exp(-x)$ and $F(u) = \pi\,u\,\sqrt{1 + u^2} + \pi\,\ln\left(u + \sqrt{1 + u^2}\right) + C$. Making the change of variable $u = \exp(-x)$, $du = -\exp(-x)\,dx$, we find that the surface area is given by

$$\int_0^{\ln(2)} 2\,\pi\,f(x)\,\sqrt{1 + f'(x)^2}\,dx = \int_0^{\ln(2)} 2\,\pi\,\exp(-x)\sqrt{1 + \exp(-x)^2}\,dx = 2\,\pi\int_{1/2}^1 \sqrt{1 + u^2}\,du = F(1) - F\left(\frac{1}{2}\right) = 3.9438245$$

4. For $f(x) = \sqrt{25 - x^2}$ we have

$$\sqrt{1 + f'(x)^2} = \sqrt{1 + \frac{x^2}{25 - x^2}} = \sqrt{\frac{25}{25 - x^2}} = \frac{5}{\sqrt{25 - x^2}}.$$

The surface area S is given by

$$S = \int_{-3}^4 2\,\pi\,f(x)\,\sqrt{1 + f'(x)^2}\,dx = 2\,\pi\int_{-3}^4 \sqrt{25 - x^2}\,\frac{5}{\sqrt{25 - x^2}}\,dx = 10\,\pi\int_{-3}^4 1\,dx = 70\,\pi.$$

5. Solving $y = f(x) = x^2$ for x, we have $x = g(y) = \sqrt{y}$. The surface area S is given by

$$\begin{aligned} S &= \int_0^4 2\,\pi\, g(y)\, \sqrt{1 + g'(y)^2}\, dy \\ &= 2\,\pi \int_0^4 \sqrt{y}\, \sqrt{1 + \frac{1}{4\,y}}\, dy \\ &= 2\,\pi \int_0^4 \sqrt{y + \frac{1}{4}}\, dy \\ &= 2\,\pi \int_0^4 \frac{1}{2}\, \sqrt{4\,y + 1}\, dy \\ &= \frac{1}{4}\,\pi \int_1^{17} u^{1/2}\, du \qquad (u = 4\,y + 1,\ du = 4\,dy) \\ &= \frac{\pi}{6}\, \left(17\,\sqrt{17} - 1\right). \end{aligned}$$

6. Let $f(x) = x^2$. The surface area S is given by

$$\begin{aligned} S &= \int_0^2 2\,\pi\, x\, \sqrt{1 + f'(x)^2}\, dx \\ &= 2\,\pi \int_0^2 x\, \sqrt{1 + 4\,x^2}\, dx \\ &= \frac{2\,\pi}{8} \int_1^{17} u^{1/2}\, du \qquad (u = 1 + 4\,x^2,\ du = 8\,x\,dx) \\ &= \frac{\pi}{6}\, \left(17\,\sqrt{17} - 1\right). \end{aligned}$$

7. We calculate

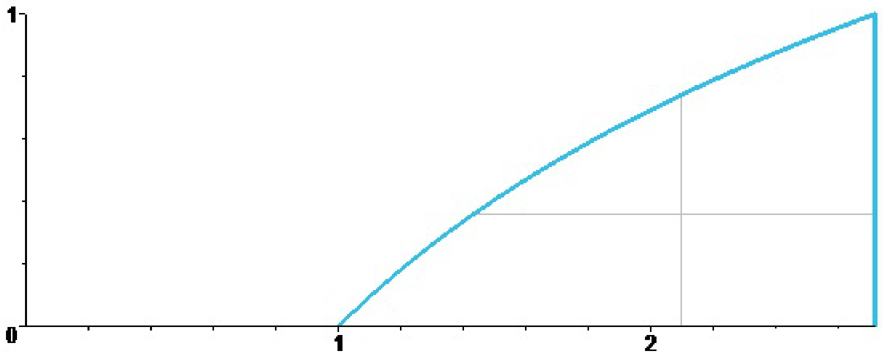
$$1 + f'(x)^2 = 1 + \left(2\,x - \frac{1}{8\,x}\right)^2 = 1 + \left((2\,x)^2 - \frac{1}{2}\left(\frac{1}{8\,x}\right)^2\right) = (2\,x)^2 + \frac{1}{2}\left(\frac{1}{8\,x}\right)^2 = \left(2\,x + \frac{1}{8\,x}\right)^2.$$

The arc length is

$$\begin{aligned} \int_1^2 2\,\pi\, f(x)\, \sqrt{1 + f'(x)^2}\, dx &= 2\,\pi \int_1^2 \left(x^2 - \frac{1}{8}\ln(x)\right) \left(2\,x + \frac{1}{8\,x}\right) dx \\ &= 2\,\pi \int_1^2 \left(2\,x^3 + \frac{1}{8}\,x - \frac{1}{4}\,x\,\ln(x) - \frac{1}{64}\,\frac{1}{x}\,\ln(x)\right) dx \\ &= \pi\, \left(\frac{63}{4} - \frac{1}{64}\ln^2(2) - \ln(2)\right) \\ &\approx 47.278914. \end{aligned}$$

Solutions: Center of Mass (Corresponds to Stewart 8.3)

1. The area of the region is 1. The center of mass is $\left(\left(e^2+1\right)/4, (e-2)/2\right)$



2. The area A of the region is

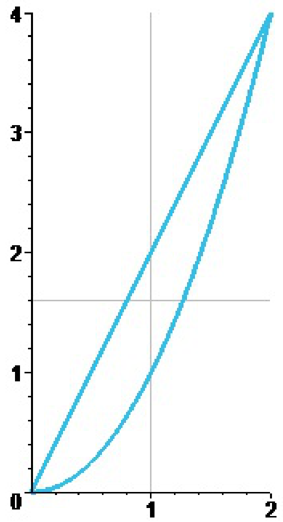
$$A = \int_0^2 \left(2\,x - x^2\right)\,dx = \frac{4}{3}.$$

Therefore,

$$\begin{aligned}
\bar{x} &= \frac{1}{A}\int_0^2 x\,\left(2\,x - x^2\right)\,dx \\
&= \frac{1}{A}\left(\frac{2}{3}\,x^3 - \frac{1}{4}\,x^4\right)\Big|_0^2 \\
&= \frac{1}{A}\,\frac{4}{3} \\
&= 1
\end{aligned}$$

and

$$\begin{aligned}\bar{y} &= \frac{1}{2A}\int_0^2 \left((2\,x)^2 - \left(x^2\right)^2\right)\,dx\\ &= \frac{1}{2A}\left(\frac{4}{3}\,x^3 - \frac{1}{5}\,x^5\right)\Big|_0^2\\ &= \frac{1}{2A}\,\frac{64}{15}\\ &= \frac{8}{5}.\end{aligned}$$



3. The area A of the region is calculated as follows:

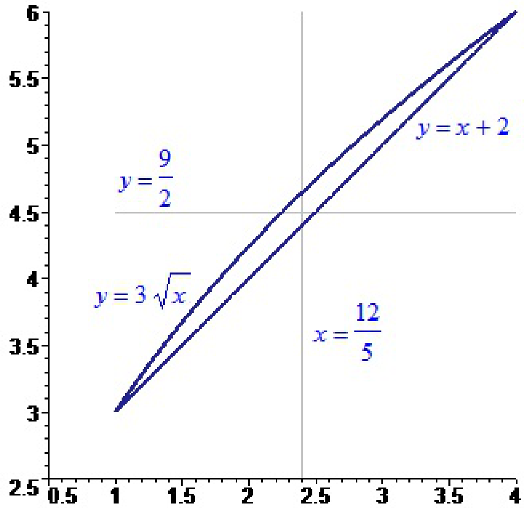
$$A = \int_1^4 \left(3\,\sqrt{x} - (x+2)\right)\,dx = \left(2\,x^{3/2} - \frac{1}{2}\,x^2 - 2\,x\right)\Big|_1^4 = \frac{1}{2}.$$

The x-center of mass $\bar{x}$ is calculated as follows:

$$\begin{aligned}\bar{x} &= \frac{1}{A}\cdot\int_1^4 x\,\left(3\,\sqrt{x} - (x+2)\right)\,dx\\ &= \frac{1}{A}\cdot\int_1^4 \left(3\,x^{3/2} - x^2 - 2\,x\right)\,dx\\ &= \frac{1}{A}\cdot\left(\frac{6}{5}\,x^{5/2} - \frac{1}{3}\,x^3 - x^2\right)\Big|_1^4\\ &= \frac{1}{A}\left(\frac{192}{5} - \frac{64}{3} - 16 - \frac{6}{5} + \frac{1}{3} + 1\right)\\ &= \frac{1}{1/2}\cdot\frac{6}{5}\\ &= \frac{12}{5}.\end{aligned}$$

The y-center of mass $\bar{y}$ is calculated as follows:

$$\begin{aligned}
\bar{y} &= \frac{1}{2A} \cdot \int_1^4 \left((3\sqrt{x})^2 - (x+2)^2\right)\, dx \\
&= \frac{1}{2A} \cdot \int_1^4 \left(5x - x^2 - 4\right)\, dx \\
&= \frac{1}{2A} \cdot \left(\frac{5}{2}x^2 - \frac{1}{3}x^3 - 4x\right)\Bigg|_1^4 \\
&= \frac{1}{2A}\left(40 - \frac{64}{3} - 16 - \frac{5}{2} + \frac{1}{3} + 4\right) \\
&= \frac{1}{2(1/2)} \cdot \frac{9}{2} \\
&= \frac{9}{2}.
\end{aligned}$$



4. The area A of the region is $2h + \pi/2$, the sum of the areas of a $2 \times h$ rectangle and a half-disk of radius 1. The equation of the semicircle is $(x-1)^2 + (y-h)^2 = 1$, $h \le y$, or $y = f(x) = h + \sqrt{1-(x-1)^2}$. We calculate

$$\begin{aligned}
\int_0^2 f(x)^2\, dx &= \int_0^2 \left(h^2 + 2h\sqrt{1-(x-1)^2} + \left(1-(x-1)^2\right)\right)\, dx \\
&= 2h^2 + 2h\frac{\pi}{2} + \int_0^2 \left(1-(x-1)^2\right)\, dx \\
&= 2h^2 + h\pi + \frac{4}{3}.
\end{aligned}$$

When this quantity is divided by $2A$, the y-center of mass, namely h, results. Therefore

$$\frac{2h^2 + h\pi + 4/3}{4h + \pi} = h \qquad \text{or} \qquad 2h^2 + h\pi + \frac{4}{3} = 4h^2 + h\pi.$$

As a result, $6\,h^2 = 4$, or $h = \dfrac{\sqrt{6}}{3}$.

Solutions: Separable Differential Equations (Corresponds to Stewart 9.3)

1. We have $\displaystyle\int \frac{1}{y}\,dy = \int \frac{1}{1+x}\,dx$, or $\ln(|y|) = \ln(|1+x|)+C$. It follows that $|y| = A\,|1+x|$ for $A = \exp(C)$. Substituting $x = 3$ and $y = 8$ results in $A = 2$. Thus, $|y| = 2\,|1+x|$. For values of x such as the cited values 3 and 5, the absolute values on the right side of the solution are not needed. That is, $|y| = 2\,(1+x)$ for $x > 1/2$. The solution $y = -2\,(1+x)$ is not consistent with the given initial value. Therefore $y(x) = 2\,(1+x)$ and $y(5) = 12$.

2. After separating variables and integrating, we obtain $y(x)^3 + 2\,y(x) = x^2 + x + C$. Substituting $x = 2$ and $y(2) = 1$ results in $C = -3$. Substituting $y(x) = 0$ in the equation $y(x)^3 + 2\,y(x) = x^2 + x - 3$ results in a quadratic equation in x. the positive root of which is $\left(\sqrt{13} - 1\right)/2$.

3. Answer: $y\,(x) = \sqrt[3]{3\,x + 3\,x^2 + 2}$

4. Answer: $y\,(x) = \left(2\,x^{3/2} + 1\right)^2$

5. Answer: $y\,(x) = \sqrt{5\,\exp(2\,x) - 1}$

6. Let $y(t)$ be the mass of salt in the tank (measured in kg) at time t (measured in min). The differential equation is
$$\frac{dy}{dt} = 0 - 20 \cdot \frac{y}{1000} \qquad \text{(rate = rate in - rate out)}$$
The solution of the initial value problem is $y(t) = 100\,\exp(-t/50)$. let T be the time when $y(T) = 10$. Solve the equation $10 = 100\,\exp(-T/50)$ to obtain $T = 50\,\ln(10)$.

7. Let $y(t)$ be the mass of salt in the mixing tank at time t. The rate equation for the change of the mass of salt in the mixing tank, namely overall rate = rate in - rate out, is, in units of kg per min,
$$\frac{dy}{dt} = \frac{2}{10}\cdot 10 - \frac{y}{200}\cdot 10 = \frac{1}{20}\,(40 - y)\,.$$
This differential equation leads to the equation $20\displaystyle\int \frac{1}{40-y}\,dy = \int 1\,dt$, or $-20\,\ln(\,|\,40-y\,|\,) = t + C$. Into this last equation, substitute $t = 0$ and $y = 5$, which is the initial condition $y(0) = 5$, to obtain $C = -20\,\ln(35)$. We therefore have $-20\,\ln(\,|\,40-y\,|\,) = t - 20\,\ln(35)$, or $\ln(\,|\,40-y\,|\,) = \ln(35) - t/20$, or, on exponentiating, $|\,40-y\,| = \exp\left(\ln(35) - t/20\right)$, or $|\,40-y\,| = \exp\big(\ln(35)\big)\,\exp(-t/20)$, or $|\,40-y\,| = 35\,\exp(-t/20)$. Are the absolute values needed? Initially y is equal to 5, which is less than 40, but y is increasing. If y were to grow larger than 40, the concentration in the mixing

tank would exceed 40 kg per 200 L, or 1 kg per 5 L. Such a concentration would exceed both the initial concentration in the mixing tank and the concentration of the inflow. Clearly this could happen only if a brine sorcerer cast a spell on our mixing tank. Absent such magic, we may safely drop the absolute values. Thus, $40 - y = 35\ \exp(-t/20)$, or $y = 40 - 35\ \exp(-t/20)$. Set $t = 60$ min (1 hour) to obtain $y(60) = 40 - 35\ \exp(-3)$, or $y(60) = 38.2575$. The concentration is 38.2575 kg per 200 L, or 0.1913 kg/L.

8. The rate equation, overall rate = rate in - rate out, is

$$\frac{dy}{dt} = \frac{1}{5} \cdot 10 - \frac{y}{2000} \cdot 10 = \frac{1}{200}\,(400 - y)\,.$$

where $y(t)$ is the mass in kg of the salt in the mixing tank's solution at time t minutes. To solve this separable ode, we set

$$\int \frac{1}{400 - y}\, dy = \int \frac{1}{200}\, dt.$$

After integrating we obtain $-\ln(\,|\,400 - y\,|\,) = C + t/200$, or $\ln(\,|\,400 - y\,|\,) = C - t/200$ (where we replaced the constant $-C$ with plain old C), or $|\,400 - y\,| = C\ \exp(-t/200)$ (where we replaced the constant $\exp(C)$ with plain old C). Because $y(0) = 100$, we solve $|\,400 - 100\,| = C\ \exp(-0/200)$ for C, the last manifestation of C, and find $C = 300$. Thus, $|\,400 - y\,| = 300\ \exp(-t/200)$. As long as $y(t)$ remains less than 400, we may dispense with the absolute values. Common sense tells us that this will be the case. Were $y(t)$ greater than 400 , the concentration of salt would be greater than 400 kg per 2000 liters, or 0.2 kg per liter. How could this possibly happen when the initial concentation is 100 kg per 2000 liters, or 0.05 kg per liter, and the inflow has concentration 1 kg per 5 liters, or 0.2 kg per liter? Thus $400 - y = 300\ \exp(-t/200)$, or $y = 400 - 300\ \exp(-t/200)$. Because the volume of water in the tank is always 2000 liters, the concentration doubles in strength when the salt content doubles. We solve $y(t) = 200$ to find $t = 200\ \ln(3/2)$, or about 81 minutes. Long term, the amount y_∞ of salt is about

$$y_\infty = \lim_{t\to\infty} y(t) = \lim_{t\to\infty} \left(400 - 300\ \exp\left(-\frac{t}{200}\right)\right) = 400.$$

Notice that the long term concentration, 400 kg per 2000 liters is the same concentration of the brine solution flowing into the mixing tank.

9. The Law of Radioactive Decay, $D(m)(t) = -\lambda\, m(t)$, gives us

$$\int \frac{1}{m}\, dm = -\lambda \int 1\, dt$$

or $\ln(\,|\,m(t)\,|\,) = -\lambda\, t + C$, or $m(t) = \exp(-\lambda\, t + C)$, or $m(t) = \exp(C)\ \exp(-\lambda\, t)$, or $m(t) = C\ \exp(-\lambda\, t)$. Two remarks: 1) the absolute values were dropped because $m(t)$, being a mass, is nonnegative, and 2) the constant $\exp(C)$, being an arbitrary (positive) constant rather than a specific constant was replaced with C, which is a simpler choice for representing an arbitrary constant. The initial value condition $m(0) = m_0$ allows us to solve for C: $C = m_0$. Thus, $m(t) = m_0\ \exp(-\lambda\, t)$. C

10. We have

$$m(t + \tau) = m_0\ \exp\bigl(-\lambda\,(t + \tau)\bigr) = m_0\ \exp(-\lambda\, t))\ \exp(-\lambda\,\tau) = m(t)\ \exp(-\ln(2)) = \frac{1}{2}\, m(t).$$

11. We have

$$m(t) = m_0\ \exp(-\lambda\, t) = m_0\ \exp(-\frac{t}{\tau}\ \ln(2)) = m_0\ \exp\left(\ln\left(2^{(-t/\tau)}\right)\right) = \frac{m_0}{2^{t/\tau}}.$$

12. First note that the answer is *not* "We lose two grams every hundred years so after 100 more years the isotope will have decayed from 6 grams to 4 grams." Instead, we use the formula $m(t) = m_0 \exp(-\lambda t) = 8 \exp(-\lambda t)$ for the amount of radioactive material at time t. Since $m(100) = 6$, we have

$$6 = m(100) = 8\,e^{-100\,\lambda} = 8\,\left(e^{-\lambda}\right)^{100}.$$

From this equation we conclude that

$$e^{-\lambda} = \left(\frac{6}{8}\right)^{1/100} = \left(\frac{3}{4}\right)^{1/100}.$$

Thus, the formula for the amount of isotope present at time t is

$$m(t) = 8\,e^{-\lambda t} = 8\,\left(e^{-\lambda}\right)^{t} = 8\,\left(\frac{3}{4}\right)^{t/100}.$$

There will be four grams of material present when

$$4 = m(t) = 8 \cdot \left(\frac{3}{4}\right)^{t/100}$$

or

$$\frac{1}{2} = \left(\frac{3}{4}\right)^{t/100}.$$

We solve for t by taking the natural logarithm of both sides:

$$\ln(1/2) = \ln\left(\left(\frac{3}{4}\right)^{t/100}\right) = \frac{t}{100} \cdot \ln(3/4),$$

or

$$t = 100 \cdot \frac{\ln(1/2)}{\ln(3/4)} \approx 240.942.$$

Thus, at $t = 240.942$, which is to say after 140.942 more years, there will be 4 grams of the isotope remaining.

13. We will use the form of the Law of Radioactive Decay obtained in Exercise 11: it is more convenient when a half-life is given as the parameter describing decay. We have

$$\frac{m(3700)}{m_0} = \frac{1}{2^{3700/5730}} \approx 0.63917.$$

14. We have

$$\frac{m(1994 - 1200)}{m_0} = \frac{1}{2^{794/5730}} \approx 0.90842.$$

15. We have

$$0.0879 \cdot m_0 = m(T) = \frac{m_0}{2^{T/5730}} \qquad \text{or} \qquad T = -\frac{5730\,\ln(0.0879)}{\ln(2)} \approx 20\,100 \text{ years.}$$

16. The differential equation
$$v \cdot \frac{dv}{dy} = -\frac{g\,R^2}{(R+y)^2}.$$
is separable. Following the general procedure, we obtain
$$\int v\ dv = \int \left(-\frac{gR^2}{(R+y)^2}\right)\,dy$$
or
$$\frac{1}{2}v^2 = \frac{g\,R^2}{(R+y)} + C.$$
When $y = 0$ we have $v = v_0$. Therefore
$$\frac{1}{2}v_0^2 = \frac{g\,R^2}{(R+0)} + C,$$
or $C = \frac{1}{2}\,v_0^2 - g\,R$. It follows that
$$\frac{1}{2}\,v^2 = \frac{g\,R^2}{(R+y)} + \left(\frac{1}{2}v_0^2 - g\,R\right),$$
or
$$v^2 = (2\,g\,R)\,\frac{R}{(R+y)} + \left(v_0^2 - 2\,g\,R\right).$$
Assuming $v_0 < \sqrt{2gR}$, we set $v = 0$ and solve for y. We find that the maximum height is $v_0^2\,R/\left(2gR - v_0^2\right)$.

Note: If $v_0 \geq \sqrt{2gR}$, then the second summand of the formula for v^2 is nonnegative, and
$$v^2 \geq 2\,g\,R\,\frac{R}{(R+y)}.$$
It is *not* possible for y to remain bounded. For if there was a constant K with $y \leq K$, then we would have $R + y \leq R + K$, and therefore $1/(R+y) \geq 1/(R+K)$ and $R/(R+y) \geq R/(R+K)$. It would follow that
$$v^2 \geq 2\,g\,R\,\frac{R}{(R+K)},$$
or
$$v \geq \sqrt{2\,g\,R\,\frac{R}{(R+K)}}.$$
Let ρ denote the positive constant on the right side of this inequality, so that $v \geq \rho$. A consequence of this ineqality is that the value of y at time t exceeds $\rho\,t$. By letting t tend to infinity in the inequality $y(t) \geq \rho\,t$, we see that $y \to \infty$ as $t \to \infty$. This contradicts the hypothesis that all values of y satisfy $y \leq K$. However you dice it, y must become unbounded if $v_0 \geq \sqrt{2gR}$. In other words, the projectile escapes Earth's gravity and does not fall back to Earth if $v_0 \geq \sqrt{2gR}$. For this reason we say that $\sqrt{2gR}$ is Earth's *escape velocity.* Now imagine an object BH so massive that the acceleration g_{BH} at the surface of the object is so great that $\sqrt{2g_{BH}R}$ exceeds the speed of light. What happens to a photon trying to escape the object's gravity?

Solutions: Models of Population Growth (Corresponds to Stewart 9.4)

1. The solution to the differential equation $D(P)(t) = k\,P(t)$ is $P(t) = P_0 \exp(k\,t)$ where $P_0 = P(0)$. It follows that
$$P(t+T) = P_0\, e^{k\,(t+T)} = P_0\, e^{k\,t} e^{k\,T} = P(t)\, e^{k\,T} = P(t)\, e^{\ln(2)} = 2\,P(t).$$

2. We have
$$P(t) = P_0\, e^{k\,t} = P_0 \,\exp\left(\frac{t}{T}\,\ln(2)\right) = P_0\,\exp\left(\ln\left(2^{t/T}\right)\right) = P_0\, 2^{t/T}.$$

3. Let us measure time t in hours with $t = 0$ corresponding to 10:00 AM, time $t = 1/3$ corresponding to 10:20 AM, and time $t = 2.5$ corresponding to 12:30 PM. Because $P(1/3) = 12000 = 2 \times 6000 = 2 \times P(0)$, we see that $T = 1/3 - 0 = 1/3$ is the doubling time for the population. From Exercise 2 we have $P(t) = 6000 \cdot 2^{3\,t}$, and $P(2.5) = 6000 \cdot 2^{7.5} \approx 1\,086\,116$. That's a lot of colonists. We're going to need more broth!

4. In integrated form, after separating variables to their respective sides of the equality sign, the differential equation $D(P)(t) = k\,P(t)\,\big(1 - P(t)\big)$ becomes
$$\int \frac{1}{P\,(1-P)}\,dP = \int k\,dt,$$
or, by finding the partial fraction form of the left side integrand,
$$\int \left(\frac{1}{P} + \frac{1}{1-P}\right)\,dP = k\,t + C.$$
This equation leads to $\ln(P) - \ln(\,|\,1-P\,|\,) = k\,t + C$. By using the identity $\ln(u) - \ln(v) = \ln(u/v)$ and discarding the absolute values, which are superfluous because $1 - P > 0$, we have $\ln(P/(1-P)) = k\,t + C$. Applying the natural exponential function results in $P/(1-P) = \exp(C)\exp(k\,t)$, which may be written more consicely as $P/(1-P) = C\,\exp(k\,t)$. (As usual, there is no need to write an arbitrary constant such as $\exp(C)$ in such a specific form.) A small algebra trick helps us solve for P: equate the reciprocals of each side of the preceding equation to obtain $(1-P)/P = C\,\exp(-k\,t)$ (writing the arbitrary constant $1/C$ as C). Then $1/P - 1 = C\,\exp(-k\,t)$. This is a good time to solve for C. Setting $P = P_0$ and $t - 0$, we find $C = 1/P_0 - 1$. Thus, $1/P - 1 = (1/P_0 - 1)\,\exp(-k\,t)$, or $1/P = 1 + (1/P_0 - 1)\,\exp(-k\,t)$, or or
$$P(t) = \frac{1}{1 + \left(\frac{1}{P_0} - 1\right)\,\exp(-k\,t)}.$$

Stewart gives the logistic differential equation as
$$\frac{dP}{dt} = k\,P\,\left(1 - \frac{P}{M}\right)$$

with solution

$$P(t) = \frac{M}{1 + A\,\exp(-k\,t)} \qquad \text{where} \qquad A = \frac{M - P_0}{P_0}.$$

The equation we have used to model the spread of an infection is the logistic differential equation with carrying capacity M equal to 1. If M is set equal to 1 in Stewart's solution of the logistic equation, then it matches the solution we have derived for the spread of an infection.

5. In the formula derived in the preceding problem, there are two unspecified constants: the initial population P_0 and the proportionality constant k. Let $t = 0$ correspond to 9 AM Monday morning. Then $P_0 = P(0) = 1/100$ and

$$P(t) = \frac{1}{1 + (100 - 1)\,\exp(-k\,t)}.$$

Let us use days as the unit of time. Then $t = 1$ corresponds to 9 AM Tuesday morning and $P(1) = 3/100$, or

$$\frac{3}{100} = \frac{1}{1 + 99\,\exp(-k)}.$$

We obtain $k = -\ln(97/297)$. Thus

$$P(t) = \frac{1}{1 + 99\,\exp(t\,\ln(97/297))},$$

or

$$P(t) = \frac{1}{1 + 99\,\left(\frac{97}{297}\right)^t}.$$

The equation $P(t) = 1/2$ has solution $t = 4.106\ldots$. According to our model, about 4.1 days after 9 AM Monday, that is to say, by about 11:30 Friday morning, the fiftieth child will be infected.

6. We may write the differential equation as

$$\frac{dP}{dt} = a\,P\left(1 - \frac{b}{a}\,P\right) = a\,P\left(1 - \frac{P}{a/b}\right) = a\,P\left(1 - \frac{P}{M}\right)$$

where $M = a/b$ is the carrying capacity.

Solutions: Linear Differential Equations (Corresponds to Stewart 9.5)

1. The given differential equation is already in the standard form for a linear differential equation, with $p(x) = 2$ and $q(x) = 1$. The integrating factor is $\exp\left(\int p(x)\,dx\right)$, or $\exp(2x)$. After mutiplying each side of the differential equation by the integrating factor, we have
$$\exp(2x)\ \frac{dy}{dx} + 2\ \exp(2x)\ y = \exp(2x)\,,$$
or
$$\frac{d}{dx}\left(\exp(2x)\ y\right) = \exp(2x)\,,$$
or
$$\exp(2x)\ y = \int \exp(2x)\ dx,$$
or
$$\exp(2x)\ y = \frac{1}{2}\ \exp(2x) + C.$$
Now is an auspicious time for using the initial condition $y(0) = 2$ to determine that $C = 3/2$. It follows that
$$y = \frac{1}{2} + \frac{3}{2}\ \exp(-2\,x)\,.$$

2. In standard form, the given linear differential equation is
$$\frac{dy}{dx} + \left(-\frac{1}{2}\right)\ y = 1$$
with $p(x) = 1/2$ and $q(x) = 1$. The integrating factor is $\exp\left(\int p(x)\,dx\right)$, or $\exp(-x/2)$. After mutiplying each side of the differential equation by the integrating factor, we have
$$\exp(-x/2)\ \frac{dy}{dx} + \left(-\frac{1}{2}\ \exp(-x/2)\right)\ y = \exp(-x/2)\,,$$
or
$$\frac{d}{dx}\left(\exp(-x/2)\ y\right) = \exp(-x/2)\,,$$
or
$$\exp(-x/2)\ y = \int \exp(-x/2)\ dx,$$

or

$$\exp(-x/2)\; y = -2\, \exp(-x/2) + C.$$

Now is a timely occasion for using the initial condition $y(0) = 3$ to determine that $C = 5$. It follows that

$$y = \exp(x/2)\; \big(5 - 2\, \exp(-x/2)\big),$$

or

$$y = 5\, \exp(x/2) - 2.$$

3. We rewrite the differential equation in the form

$$\frac{dy}{dx} + \frac{2x}{1+x^2}\, y = 15x^2$$

to see that $p(x) = 2x/\left(1+x^2\right)$ and $q(x) = 15\,x^2$. The integrating factor is $\exp\left(\int 2x/\left(1+x^2\right) dx\right)$, or $\exp\left(\ln\left(1+x^2\right)\right)$, or $1+x^2$. Multiplying the rewritten differential equation by the integrating factor leaves us with

$$\left(1+x^2\right) \frac{dy}{dx} + 2\,x\,y = 15\,\left(x^2 + x^4\right),$$

or

$$\frac{d}{dx}\left(\left(1+x^2\right)\, y\right) = 15\,\left(x^2 + x^4\right).$$

Integrating each side gives us

$$\left(1+x^2\right)\, y = 5\,x^3 + 3\,x^5 + C.$$

This is a splendid moment for calculating the constant of integration: $\left(1+1^2\right)\, 6 = 5\,1^3 + 3\,1^5 + C$, or $C = 4$. Substituting this value for C and solving for y, we obtain

$$y = \frac{5\,x^3 + 3\,x^5 + 4}{1+x^2}.$$

4. Writing the differential equation in the standard linear form $\dfrac{dy}{dx} + \dfrac{1}{x}\, y = 1$ directs us to the integrating factor $\exp\left(\int (1/x)\, dx\right)$, or $\exp(\ln(x))$, or x. Multiplying the rewritten differential equation by the integrating factor leaves us with

$$x\, \frac{dy}{dx} + y = x, \qquad \text{or} \qquad x\, \frac{d}{dx}(x\,y) = x.$$

Integrating each side results in $x\,y = x^2/2 + C$. Now is a dandy time to determine the constant of integration. Set $x = 4$ and $y = 5$ in $x\,y = x^2/2 + C$ to obtain $20 = 8 + C$, or $C = 12$. It follows that $x\,y = x^2/2 + 12$, or $y = x/2 + 12/x$.

5. Answer: $y = 2x^{3/2} + 7/x^2$ In Standard form, the given linear equation is $\frac{dy}{dx} + \left(\frac{2}{x}\right)\, y = 7\sqrt{x}$, so the integrating factor is $\exp\left(\int (2/x)\, dx\right)$, or $\exp(2\, \ln(x))$, or $\exp\left(\ln\left(x^2\right)\right)$, or x^2. Multiplying the differential equation by the integrating factor results in $x^2\, \frac{dy}{dx} + 2\,x\,y = 7\,x^{5/2}$, or $\dfrac{d}{dx}\left(x^2\, y\right) = 7\,x^{5/2}$. Integrating results in $x^2\, y = 2\,x^{7/2} + C$. Using the initial value to solve for C at this juncture would be hunky-dory. Setting $x = 1$ and $y = 9$, we have $1^2 \times 9 = 2 \times 1^{7/2} + C$, or $C = 7$. Therefore, $x^2\, y = 2\,x^{7/2} + 7$, or $y = 2\,x^{3/2} + 7/x^2$.

6. The given differential equation is in standard linear form as presented. The integrating factor is $\exp\left(\int x\right)$, or $\exp\left(x^2/2\right)$. We multiply the differential equation by this factor, obtaining $\exp\left(x^2/2\right)\,\dfrac{dy}{dx} + x\,\exp\left(x^2/2\right)\,y = x\,\exp\left(x^2/2\right)$, or $\dfrac{d}{dx}\left(\exp\left(x^2/2\right)\,y\right) = x\,\exp\left(x^2/2\right)$. Integration gives us $\exp\left(x^2/2\right)\,y = \exp\left(x^2/2\right) + C$. Solving for C now is an admirable next step. We substitute $x = 0$ and $y = 4$ to obtain $4 = 1 + C$, or $C = 3$. We now replace C with this value in our implicit solution and solve explicitly for y: $y = 1 + 3\,\exp\left(-x^2/2\right)$.

7. Matching the differential equation $\dfrac{dy}{dx} - 2\,x\,y = 4\,x$ with the standard form $\dfrac{dy}{dx} + p(x)\,y = q(x)$ of the linear differential equation, we set $p(x) = -2\,x$ to obtain the integrating factor $\exp\left(\int(-2\,x)\right)$, which evaluates to $\exp\left(-x^2\right)$. Multiplying the equation by this factor gives us $\exp\left(-x^2\right)\dfrac{dy}{dx} + (-2\,x)\,\exp\left(-x^2\right)\,y = 4\,x\,\exp\left(-x^2\right)$, or $\dfrac{d}{dx}\left(\exp\left(-x^2\right)\,y\right) = 4\,x\,\exp\left(-x^2\right)$. Integrating yields $\exp\left(-x^2\right)\,y = C - 2\,\exp\left(-x^2\right)$. Now is an admirable time for determining C. Set $x = 0$ and $y = 1$ in the equation just obtained to get $1 = C - 2$, or $C = 3$. It follows that $\exp\left(-x^2\right)\,y = 3 - 2\,\exp\left(-x^2\right)$, or $y = 3\,\exp\left(x^2\right) - 2$.

8. The given differential equation is equivalent to the differential equation of the preceding exercise. The answer is therefore $y = 3\,\exp\left(x^2\right) - 2$. However, in the form presented in the current exercise, it takes some unwinding to see that the equation is linear. On the other hand, in the present form it is obvious that it is separable. If we use that technique we obtain $\displaystyle\int \frac{1}{2+y}\,dy = \int 2\,x\,dx$, or $\ln(\,|\,2+y\,|\,) = x^2 + C$. The initial condition $y(0) = 1$ results in $C = \ln(3)$. It follows that $\ln(\,|\,2+y\,|\,) = x^2 + \ln(3)$, or $|\,2+y\,| = \exp\left(x^2 + \ln(3)\right)$, or $|\,2+y\,| = \exp\left(x^2\right)\exp\left(\ln(3)\right)$, or $|\,2+y\,| = 3\,\exp\left(x^2\right)$. Notice that y cannot be -2 because our equation tells us that $|\,2+y\,|$ must be positive. Therefore, an arc of a solution curve must either lie under the horizontal line $y = -2$ or above the horizontal line $y = -2$. An arc that lies under $y = -2$ is disconnected from an arc that lies above. Therefore, the initial condition $y(0) = 1$ can only determine an arc of a solution curve that lies above $y = -2$. For such an arc, $2 + y$ is positive and we my discard the troublesome absolute values. Thus, $2 + y = 3\,\exp\left(x^2\right)$, or $y = 3\,\exp\left(x^2\right) - 2$.

9. Writing the linear differential equation in the standard form $D(y)(x) + p(x)\,y(x) = q(x)$, we have $p(x) = -3$. The integrating factor is $\exp\left(\int p(x)\,dx\right)$, or $\exp\left(-3\,x\right)$. After multiplying each side by the integrating factor, we obtain $e^{-3x}\,\dfrac{dy}{dx} + (-3)e^{-3x}\,y = 1$, or $\frac{d}{dx}\left(e^{-3x}\,y\right) = 1$, or $e^{-3x}\,y = x + C$. Now is an estimable time for determining the constant of integration. Set $x = 2$ and $y = 0$, the values of the initial condition, in the equation just obtained. The result is $0 = 2 + C$, or $C = -2$. Thus, $e^{-3x}\,y = x - 2$, or $y = (x-2)\,e^{3x}$.

10. Writing the linear differential equation in the standard form $D(y)(x) + p(x)\,y(x) = q(x)$, we have $\dfrac{dy}{dx} + \left(-\cot(x)\right)y = \csc(x)$. Thus, $p(x) = -\cot(x)$, and the integrating factor is $\exp\left(\int p(x)\,dx\right)$, or $\exp\left(-\int \cot(x)\,dx\right)$, or $\exp\left(\ln(\csc(x))\right)$, or $\csc(x)$. After multiplying each side by the integrating factor, we obtain $\csc(x)\,\dfrac{dy}{dx} + \left(-\csc(x)\cot(x)\right)y = \csc^2(x)$, or $\frac{d}{dx}\left(\csc(x)\,y\right) = \csc^2(x)$, or $\csc(x)\,y = -\cot(x) + C$. Now is a meritorious time for determining the constant of integration. Set $x = \pi/2$ and $y = 4$, the values of the initial condition, in the equation just obtained. The result is $4 = 0 + C$, or $C = 4$. Thus, $\csc(x)\,y = 4 - \cot(x)$, or, on dividing by $\csc(x)$, which is equivalent to multiplying by the reciprocal $\sin(x)$, we obtain $y = 4\,\sin(x) - \cos(x)$.

11. The integrating factor is

$$\exp\left(\int \frac{1}{x\,\ln(x)}\,dx\right), \qquad \text{or} \qquad \exp\left(\ln(\ln(x))\right), \qquad \text{or} \qquad \ln(x).$$

Multiply each side of the differential equation by this integrating factor to obtain

$$\ln(x)\,\frac{dy}{dx} + \frac{1}{x}\,y = \ln(x),$$

or

$$\frac{d}{dx}\,(\ln(x)\,y) = \ln(x),$$

or

$$\ln(x)\,y = \int \ln(x)\,dx = x\,\ln(x) - x + C.$$

This is a propitious time for determining the value of C. Setting $y = 2\,e$ and $x = e$ results in the equation $2\,e = e - e + C$, or $C = 2\,e$. It follows that $\ln(x)\,y = \int \ln(x)\,dx = x\,\ln(x) - x + 2\,e$, or

$$y = x + \frac{2\,e - x}{\ln(x)}.$$

12. Let $y(t)$, the weight of the solute in the mixing tank t minutes after the valves have been opened. We are given that $y(0) = 80$. The differential equation for $y(t)$ is obtained by translating into mathematics the following observation: "The overall rate of change of the weight of the solute in the mixing tank equals the rate of the weight of the solute going in minus the rate of the weight of the solute going out." The rate of the volume of the fluid entering is 24 gal/min. Because the concentration of this solution is 1 lb/gal, we see that the rate of the weight of the solute going in is $(24\,\text{gal/min}) \times (1\,\text{lb/gal})$, or 24 lb/min. The rate of the volume of the fluid exiting is 16 gal/min. Because the concentration of this solution is $y/V(t)\,\text{lb/gal}$, where $V(t)$ is the volume of the solution in the tank at time t, we see that the rate of the weight of the solute going out is $(16\text{gal/min}) \times (y/V(t)\,\text{lb/gal})$, or $\big(16/V(t)\big)\,y\,\text{lb/min}$. We are given that $V(0) = 120\,\text{gal}$. The information we have been given also tells us that the net increase of the fluid in the tank is $(24 - 16)\,\text{gal}$ per minute. Therefore, $V(t) = 120 + 8\,t$. Thus, the differential equation for $y(t)$ is

$$\frac{dy}{dt} = 24 - \frac{16}{120 + 8\,t}\,y.$$

This linear differential equation has standard form

$$\frac{dy}{dt} + \frac{16}{120 + 8\,t}\,y = 24.$$

The integrating factor is $\exp\big(\int 16/(120 + 8\,t)\,dt\big)$, or $\exp\left(2\,\ln(120 + 8\,t)\right)$, or $(120 + 8\,t)^2$. Thus,

$$(120 + 8\,t)^2\,\frac{dy}{dt} + 16\,(120 + 8\,t)\,y = 24\,(120 + 8\,t)^2,$$

or

$$\frac{d}{dt}\,\big((120 + 8\,t)^2\,y\big) = 24\,(120 + 8\,t)^2.$$

Integrating each side, we obtain

$$(120 + 8\,t)^2\,y = (120 + 8\,t)^3 + C.$$

We can make the numbers smaller by factoring 8 out of each parenthesis and dividing both sides of the equation by 8^2:

$$(15 + t)^2\,y = 8\,(15 + t)^3 + C.$$

Now is the opportune time to solve for C using the initial condition $y(0) = 80$. We find $C = -9000$. Thus,

$$(15 + t)^2\,y = 8\,(15 + t)^3 - 9000,$$

or

$$y = 8\,(15+t) - \frac{9000}{(15+t)^2}.$$

Consequently, $y(15) = 8\,(15+15) - 9000/(15+15)^2$, or $y(15) = 230$.

Solutions: Sequences (Corresponds to Stewart 11.1)

1. $a_0 = 3^0/(2\times 0+1) = 1$, $a_1 = 3^1/(2\times 1+1) = 1$, $a_2 = 3^2/(2\times 2+1) = 9/5$, $a_3 = 3^3/(2\times 3+1) = 27/7$, $a_4 = 3^4/(2\times 4+1) = 9$.

2. Because $f'(x_n) = 3\,x^2$, we have

$$x_{n+1} = x_n - \frac{x_n^3 - 4}{3\,x_n^2} = \frac{2\left(x_n^3+2\right)}{3\,x_n^2}.$$

It follows that $x_1 = 1, x_2 = 2, x_3 = 2\left(2^3+2\right)/\left(3\times 2^2\right) = 20/12 = 5/3, x_4 = 2\left((5/3)^3+2\right)/\left(3\times(5/3))^2\right) = 358/225, x_5 = 2\left((358/225)^3+2\right)/\left(3\times(358/225))^2\right) = 34331981/21627675 \approx 1.58741$. To five decimal places, $4^{1/3} = 1.58740$.

3. Divide every term in sight by the highest power of n, namely n^2:

$$\lim_{n\to\infty}\frac{3+5\,n^2}{5+3\,n^2} = \lim_{n\to\infty}\frac{3/n^2+5}{5/n^2+3} = \frac{0+5}{0+3} = \frac{5}{3}.$$

4. Divide every term in the fraction by the highest power of n, namely n^6:

$$\lim_{n\to\infty}\sqrt{\frac{n^5+9\,n^6}{1+4\,n^6}} = \lim_{n\to\infty}\sqrt{\frac{1/n+9}{1/n^6+4}} = \sqrt{\frac{0+9}{0+4}} = \frac{3}{2}.$$

5. The given sequence decays exponentially:

$$\lim_{n\to\infty} 3^n\pi^{-n} = \lim_{n\to\infty}\frac{1}{(\pi/3)^n} = \lim_{n\to\infty}\frac{1}{\exp\left(\ln\left((\pi/3)^n\right)\right)} = \lim_{n\to\infty}\frac{1}{\exp\left(n\,\ln\left(\pi/3\right)\right)}.$$

Because $\pi > 3$, we have $\pi/3 > 1$ and $k = \ln(\pi/3) > 0$. It follows that $\lim_{n\to\infty}\exp\left(n\,\ln\left(\pi/3\right)\right) = \lim_{n\to\infty}\exp\left(k\,n\right) = \infty$. Consequently, the limit of the given sequence is 0.

6. We have

$$\lim_{n\to\infty}\left(\exp(1/\sqrt{n}) + \exp(-1/\sqrt{n})\right) = \exp\left(\lim_{n\to\infty} 1/\sqrt{n}\right) + \exp\left(\lim_{n\to\infty}(-1\sqrt{n})\right) = \exp(0) + \exp(0) = 2.$$

7. We deal with a ratio of factorials by expanding the factorial of the larger number until the factorial of the smaller number appears. In this exercise, we have $(2\,n+1)! = (2\,n+1)\cdot 2n\cdot(2\,n-1)!$. Thus,

$$\lim_{n\to\infty}\frac{(2\,n-1)!\,n^2}{(2\,n+1)!} = \lim_{n\to\infty}\frac{(2\,n-1)!\,n^2}{(2\,n+1)\cdot 2n\cdot(2\,n-1)!} = \frac{1}{2}\lim_{n\to\infty}\frac{n}{(2\,n+1)} = \frac{1}{2}\lim_{n\to\infty}\frac{1}{(2+1/n)} = \frac{1}{4}.$$

8. We have
$$0 \le \frac{|\sin(n)|}{n} \le \frac{1}{n}.$$
By the Pinching Theorem, we conclude that $|\sin(n)|/n \to 0$. It follows that $\sin(n)/n \to 0$.

9. Letting x be a continuous variable tending to infinity and making the change of variable $u = 1/x$, we have
$$\lim_{n\to\infty} n\,\sin(1/n) = \lim_{x\to\infty} x\,\sin(1/x) = \lim_{x\to\infty} \frac{\sin(1/x)}{1/x} = \lim_{u\to 0^+} \frac{\sin(u)}{u} = 1.$$

10. We again divide each term in the fraction by the highest power of n that occurs, namely n^1:
$$\lim_{n\to\infty} \exp\left(\frac{2\,n}{n+3}\right) = \lim_{n\to\infty} \exp\left(\frac{2}{1+3/n}\right) = \exp\left(\frac{2}{1+0}\right) = e^2.$$

11. The standard technique of dividing by a highest power works here. In this instance, we will divide by by the highest power of n in the numerator, namely n^2. We obtain
$$\lim_{n\to\infty} \frac{n^2}{\sqrt{n^5+1}} = \lim_{n\to\infty} \frac{n^2/n^2}{\sqrt{n^5/n^4+1/n^4}} = \lim_{n\to\infty} \frac{1}{\sqrt{n+1/n^4}} = \frac{1}{\sqrt{\infty+0}} = 0.$$
We could equally well have divided every term by what is effectively the highest power in the denominator, namely $n^{5/2}$, although that power doesn't really appear:
$$\lim_{n\to\infty} \frac{n^2}{\sqrt{n^5+1}} = \lim_{n\to\infty} \frac{n^2/n^{5/2}}{\sqrt{n^5/n^5+1/n^5}} = \lim_{n\to\infty} \frac{n^{-1/2}}{\sqrt{1+1/n^5}} = \frac{0}{\sqrt{1+0}} = 0.$$
Finally, a rough, intuitive calculation would lead us to the same limit. The 1 added to n^5 becomes negligible as n tends to infinity. For $n = 1000$, the difference between n^5+1 and n^5 is the differnece between 1,000,000,000,000,001 and 1,000,000,000,000,000. Some difference! It's just a rounding error. It is obvious that $\lim_{n\to\infty} n^2/\sqrt{n^5} = \lim_{n\to\infty} n^2/n^{5/2} = \lim_{n\to\infty} 1/n^{1/2} = 0$.

12. We are to evaluate $\displaystyle\lim_{n\to\infty} \frac{\ln(n)}{n^p}$ for $p = 1/2$. We will show that
$$\lim_{n\to\infty} \frac{\ln(n)}{n^p} = 0$$
for every positive constant p. Applying L'Hôpital's Rule, we have
$$\lim_{n\to\infty} \frac{\ln(n)}{n^p} = \lim_{n\to\infty} \frac{\frac{d}{dn}\ln(n)}{\frac{d}{dn}n^p} = \lim_{n\to\infty} \frac{1/n}{p\,n^{p-1}} = \frac{1}{p} \lim_{n\to\infty} \frac{1}{n^p} = 0.$$

13. This limit, for which a formal calculation leads to the indeterminate form ∞/∞, can be calculated using L'Hôpital's Rule:
$$\lim_{n\to\infty} \frac{\ln(n)}{\ln(3\,n)} = \lim_{n\to\infty} \frac{\frac{d}{dn}\ln(n)}{\frac{d}{dn}\ln(3\,n)} = \lim_{n\to\infty} \frac{\frac{d}{dn}\ln(n)}{\frac{d}{dn}\big(\ln(3)+\ln(n)\big)} = \lim_{n\to\infty} \frac{1/n}{1/n} = 1.$$
An algebraic approach is more elementary and equally effective:
$$\lim_{n\to\infty} \frac{\ln(n)}{\ln(3\,n)} = \lim_{n\to\infty} \frac{\ln(n)}{\ln(3)+\ln(n)} = \lim_{n\to\infty} \frac{\ln(n)/\ln(n)}{\ln(3)/\ln(n)+\ln(n)/\ln(n)} = \lim_{n\to\infty} \frac{1}{\ln(3)/\ln(n)+1} = \frac{1}{0+1} = 1.$$

14. L'Hôpital's Rule is an effective way to deal with the indeterminate form ∞/∞:

$$\lim_{n\to\infty} \frac{\ln(n)}{\ln(n^3)} = \lim_{n\to\infty} \frac{\frac{d}{dn}\ln(n)}{\frac{d}{dn}\ln(n^3)} = \lim_{n\to\infty} \frac{1/n}{3n^2/n^3} = \lim_{n\to\infty} \frac{1}{3} = \frac{1}{3}.$$

An even easier method is to simplify the denominator:

$$\lim_{n\to\infty} \frac{\ln(n)}{\ln(n^3)} = \lim_{n\to\infty} \frac{\ln(n)}{3\ln(n)} = \lim_{n\to\infty} \frac{1}{3} = \frac{1}{3}.$$

15. In this case we can resolve the indeterminate form $\infty - \infty$ by appealing to an algebraic property of logarithms to convert the indeterminate form to the more convenient indeterminate form ∞/∞:

$$\lim_{n\to\infty} \big(\ln(n+10) - \ln(n+1)\big) = \lim_{n\to\infty} \ln\left(\frac{n+10}{n+1}\right) = \lim_{n\to\infty} \ln\left(\frac{n/n+10/n}{n/n+1/n}\right) = \ln\left(\lim_{n\to\infty} \frac{n/n+10/n}{n/n+1/n}\right) = \ln\left(\frac{1+0}{1+0}\right) = 0.$$

16. The only significance of the number 1.1 is that it is greater than 1. We will write the solution with b denoting 1.1. The calculation shows that the same result holds for any $b > 1$. We write $n^3 \cdot (1.1)^{-n}$ as n^3/b^n. The formal limit is ∞/∞, which is an indeterminate form. However, it is an indeperminate form to which we may apply L'Hôpital's Rule without any preliminary algebraic manipulation (such as is necessary for the indeterminate forms $0\cdot\infty, 0^0, \infty^0, 1^\infty$, and $\infty - \infty$). In preparation for the use of L'Hôpital's Rule, we recall that $\frac{d}{dx} b^x = \ln(b)\cdot b^x$. Through repeated use of L'Hôpital's Rule, it follows that

$$\begin{aligned}
\lim_{n\to\infty} \frac{n^3}{b^n} &= \lim_{n\to\infty} \frac{\frac{d}{dn}n^3}{\frac{d}{dn}b^n} \\
&= \frac{3}{\ln(b)} \lim_{n\to\infty} \frac{n^2}{b^n} \\
&= \frac{3}{\ln(b)} \lim_{n\to\infty} \frac{\frac{d}{dn}n^2}{\frac{d}{dn}b^n} \\
&= \frac{3\cdot 2}{\ln(b)^2} \lim_{n\to\infty} \frac{n}{b^n} \\
&= \frac{3\cdot 2}{\ln(b)^2} \lim_{n\to\infty} \frac{\frac{d}{dn}n}{\frac{d}{dn}b^n} \\
&= \frac{3!}{\ln(b)^3} \lim_{n\to\infty} \frac{1}{b^n} \\
&= 0.
\end{aligned}$$

It is worth noting that if the exponent 3 is replaced with an arbitrary positive integer k and if we apply L'Hôpital's Rule k times, then we obtain

$$\lim_{n\to\infty} \frac{n^k}{b^n} = \frac{k!}{\ln(b)^k} \lim_{n\to\infty} \frac{1}{b^n} = 0.$$

The phrase "exponential growth" has become a cliché in English. Nobody speaks of "geometric growth". But the geometric growth b^n is exponential growth in disguise: $b^n = \exp(n\ln(b))$. What this exercise shows is that

$$\lim_{n\to\infty} \frac{\text{polynomial growth in } n}{\text{exponential growth in } n} = 0.$$

17. The solution to Exercise 16 shows that $\lim_{n\to\infty} \frac{n^5}{2^n} = 0$ and $\lim_{n\to\infty} \frac{n^2}{2^n} = 0$. On dividing by 2^n every term in the given fraction whose limit we are to determine, we have

$$\lim_{n\to\infty} \frac{2^n + n^5}{2^n + n^2} = \lim_{n\to\infty} \frac{2^n/2^n + n^5/2^n}{2^n/2^n + n^2/2^n} = \lim_{n\to\infty} \frac{1 + n^5/2^n}{1 + n^2/2^n} = \frac{1+0}{1+0} = 1.$$

18. The fastest growing term in the given fraction is 11^n. Divide every term in the fraction by it:

$$\lim_{n\to\infty} \frac{3\times 7^n + 6\times 11^n}{5^n + 2\times 11^n} = \lim_{n\to\infty} \frac{3\times 7^n/11^n + 6\times 11^n/11^n}{5^n/11^n + 2\times 11^n/11^n} = \lim_{n\to\infty} \frac{3\times (7/11)^n + 6}{(5/11)^n + 2} = \frac{0+6}{0+2} = 3.$$

19. Answer: $1/e$ (*Not* 1!) This exercise involves the indeterminate form 1^∞. Resist all temptation, no matter how great it might be, to think that 1^∞ is 1. It might be, but often it is not. Generally, indeterminate forms involving an exponent are handled by a preliminary application of the natural logarithm. Let

$$y_n = \ln(a_n) = n\,\ln\left(1 - \frac{1}{n}\right) = \frac{\ln(1-1/n)}{1/n}.$$

We now have the indeterminate form $0/0$, which is L'Hôpital Rule ready. To simplify the differentiation, let us make the change of variable $x = 1/n$. Then

$$\begin{aligned} \lim_{n\to\infty} y_n &= \lim_{x\to 0^+} \frac{\ln(1-x)}{x} \\ &= \lim_{x\to 0^+} \frac{\frac{d}{dx}\ln(1-x)}{\frac{d}{dx}x} \\ &= -\lim_{x\to 0^+} \frac{1}{1-x} \\ &= -1. \end{aligned}$$

It follows that

$$\begin{aligned} \lim_{n\to\infty} a_n &= \lim_{n\to\infty} \exp(\ln(a_n)) \\ &= \lim_{n\to\infty} \exp(y_n) \\ &= \exp\left(\lim_{n\to\infty} y_n\right) \\ &= \exp(-1). \end{aligned}$$

The same calculation shows that $\lim_{n\to\infty}(1 + a/n)^n = \exp(a)$ for any constant a.

20. The limit of the sequence, calculated with no refinement, is the indeterminate form ∞^0. To evaluate an indeterminate form involving a power, namely one of the three forms 0^0, 1^∞, and ∞^0, first apply the natural logarithm. The result of this application will be, in all three cases, $\pm 0\cdot\infty$, an indeterminate form with no power. To handle the indeterminate form $\pm 0\cdot\infty$, write the product as a quotient in which the denominator is the reciprocal of one of the two factors. The resulting quotient will be one of the indeterminate forms $0/0$ or ∞/∞. Both of these indeterminate forms can be treated with L'Hôpital's Rule. Thus, if $y_n = \ln(a_n)$, then

$$y_n = \ln\left(\left(1+n^2\right)^{1/2n}\right) = \frac{1}{2n}\,\ln\left(1+n^2\right) = \frac{\ln\left(1+n^2\right)}{2n}$$

and

$$\begin{aligned}
\lim_{n\to\infty} y_n &= \lim_{n\to\infty} \frac{\frac{d}{dn}\ln\left(1+n^2\right)}{\frac{d}{dn}2n} \\
&= \lim_{n\to\infty} \frac{2n}{2\left(1+n^2\right)} \\
&= \lim_{n\to\infty} \frac{n/n}{\left(1/n+n^2/n\right)} \\
&= \lim_{n\to\infty} \frac{1}{\left(1/n+n\right)} \\
&= \frac{1}{0+\infty} \\
&= 0.
\end{aligned}$$

It follows that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \exp\left(\ln\left(a_n\right)\right) = \lim_{n\to\infty} \exp\left(y_n\right) = \exp\left(\lim_{n\to\infty} y_n\right) = \exp(0) = 1.$$

21. We have $L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{2\,a_n} = \sqrt{2\,\lim_{n\to\infty} a_n} = \sqrt{2\,L}$, so $L^2 = 2\,L$, or $L = 2$.

22. We have $L = \lim_{n\to\infty} a_{n+1} = 1 + 1/\lim_{n\to\infty}\left(a_n+1\right) = 1 + 1/\left(L+1\right)$, so $L\left(L+1\right) = \left(L+1\right)+1$, or $L^2 = 2$. The terms of the sequence are all greater than or equal to 1, so L must be greater than or equal to 1. We therefore discard the negative root of $L^2 = 2$ and conclude that $L = \sqrt{2}$.

Solutions: Series (Corresponds to Stewart 11.2)

1. (i) We have

$$s_1 = \ln\left(\frac{2}{4}\right),$$

$$s_2 = \ln\left(\frac{2}{4}\right) + \ln\left(\frac{3}{5}\right) = \ln\left(\frac{2\times 3}{4\times 5}\right),$$

$$s_3 = s_2 + \ln\left(\frac{4}{6}\right) = \ln\left(\frac{2\times 3\times 4}{4\times 5\times 6}\right) = \ln\left(\frac{2\times 3}{5\times 6}\right),$$

$$s_4 = s_3 + \ln\left(\frac{5}{7}\right) = \ln\left(\frac{2\times 3\times 5}{5\times 6\times 7}\right) = \ln\left(\frac{2\times 3}{6\times 7}\right),$$

$$s_5 = s_4 + \ln\left(\frac{6}{8}\right) = \ln\left(\frac{2\times 3\times 6}{6\times 7\times 8}\right) = \ln\left(\frac{2\times 3}{7\times 8}\right).$$

(ii) The integers M and N are unequal because the given series begins with $n = 2$ instead of $n = 1$. The integer N in s_N is the number of summands in the partial sum s_N. Thus, when $N = 1$ there is only one summand. The partial sum begins with $n = 2$ and ends with $n = 2$. In other words, $M = 2$ for $N = 1$, which is to say that $M = N + 1$. This equation holds generally:

$$s_N = \sum_{n=2}^{M} \ln\left(\frac{n}{n+2}\right) = \sum_{n=2}^{N+1} \ln\left(\frac{n}{n+2}\right).$$

Remark: The point of part (ii) is to emphasize that the index N of a partial sum s_N refers to the number of terms a_n in the sum. It does not refer to the largest index of the summands a_n that constitute the sum. For a series $\sum_{n=2}^{\infty} a_n$ such as the one in this exercise, the N^{th} partial sum s_N is given by $S_N = a_2 + a_3 + \cdots + a_N + a_{N+1}$. For a series of the form $\sum_{n=0}^{\infty} a_n$, the N^{th} partial sum s_N is given by $S_N = a_0 + a_1 + \cdots + a_{N-2} + a_{N-1}$.

(iii) From the pattern in part (i), we have

$$s_N = \ln\left(\frac{2\times 3}{(N+2)(N+3)}\right).$$

(iv) We calculate

$$\begin{aligned}\lim_{N\to\infty} s_N &= \lim_{N\to\infty} \ln\left(\frac{2\times 3}{(N+2)(N+3)}\right)\\ &= \lim_{N\to\infty}\big(\ln(6)-\ln\big((N+2)(N+3)\big)\big)\\ &= \ln(6)-\lim_{N\to\infty}\ln\big((N+2)(N+3)\big)\\ &= \ln(6)-\infty\\ &= -\infty.\end{aligned}$$

The series diverges.

2. Starting at the beginning,

$$a_1 = s_1 = s_N\Big|_{N=1} = \frac{2\,N+1}{N+2}\Big|_{N=1} = \frac{3}{3} = 1,$$

$$a_2 = (a_1+a_2)-a_1 = s_2 - s_1 = \frac{2\,N+1}{N+2}\Big|_{N=2} - \frac{2\,N+1}{N+2}\Big|_{N=1} = \frac{5}{4}-\frac{3}{3} = \frac{1}{4},$$

$$a_3 = (a_1+a_2+a_3)-(a_1+a_2) = s_3 - s_2 = \frac{2\,N+1}{N+2}\Big|_{N=3} - \frac{2\,N+1}{N+2}\Big|_{N=2} = \frac{7}{5}-\frac{5}{4} = \frac{3}{20}.$$

In general

$$\begin{aligned}a_n &= (a_1+a_2+\cdots+a_{n-1}+a_n)-(a_1+a_2+\cdots+a_{n-1})\\ &= s_n - s_{n-1}\\ &= \frac{2\,N+1}{N+2}\Big|_{N=n} - \frac{2\,N+1}{N+2}\Big|_{N=n-1}\\ &= \frac{2\,n+1}{n+2}-\frac{2\,n-1}{n+1}\\ &= \frac{(n+1)(2\,n+1)-(n+2)(2\,n-1)}{(n+1)(n+2)}\\ &= \frac{3}{(n+1)(n+2)}.\end{aligned}$$

Thus, the series we are dealing with is

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}\frac{3}{(n+1)(n+2)} = 1+\frac{1}{4}+\frac{3}{20}+\frac{1}{10}+\cdots+\frac{3}{(n+1)(n+2)}+\cdots$$

It converges to 2:

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} s_N = \lim_{N\to\infty}\frac{2\,N+1}{N+2} = \lim_{N\to\infty}\frac{2\,N/N+1/N}{N/N+2/N} = \lim_{N\to\infty}\frac{2+1/N}{1+2/N} = 2.$$

3. We have

$$5-2.5+1.25-0.625+0.3125-0.15625+\cdots = 5\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n = \frac{5}{1-(-1/2)} = 10/3.$$

4. We have

$$\sum_{n=2}^{\infty}\frac{(-2)^{n-1}}{3^n} = (-2)^{-1}\sum_{n=2}^{\infty}\frac{(-2)^n}{3^n} = -\frac{1}{2}\sum_{n=2}^{\infty}\left(-\frac{2}{3}\right)^n = -\frac{1}{2}\left(\frac{(-2/3)^2}{1-(-2/3)}\right) = -\frac{2}{15}.$$

5. We have

$$\sum_{n=1}^{\infty}\frac{3^{n+1}}{4^{n-1}} = \frac{3}{4^{-1}}\sum_{n=1}^{\infty}\frac{3^n}{4^n} = 12\sum_{n=1}^{\infty}\left(\frac{3}{4}\right)^n = 12\left(\frac{(3/4)^1}{1-3/4}\right) = 36.$$

6. We have

$$\sum_{n=1}^{\infty}\frac{2^{2n+1}}{5^n} = 2\sum_{n=1}^{\infty}\frac{4^n}{5^n} = 2\sum_{n=1}^{\infty}\left(\frac{4}{5}\right)^n = 2\left(\frac{(4/5)^1}{1-4/5}\right) = 8.$$

7. Partial fractions lead to the identity

$$\frac{2}{n^2-1} = \frac{1}{n-1} - \frac{1}{n+1}.$$

Thus,

$$\begin{aligned} s_N &= \sum_{n=2}^{N+1}\frac{2}{n^2-1} \\ &= \sum_{n=2}^{N+1}\left(\frac{1}{n-1}-\frac{1}{n+1}\right) \\ &= \left(1-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{4}-\frac{1}{6}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)+\cdots\left(\frac{1}{N-1}-\frac{1}{N+1}\right)+\left(\frac{1}{N}-\frac{1}{N+2}\right) \\ &= 1+\frac{1}{2}-\frac{1}{N+1}-\frac{1}{N+2}. \end{aligned}$$

Thus,

$$\sum_{n=2}^{\infty}\frac{2}{n^2-1} = \lim_{N\to\infty} s_N = \lim_{N\to\infty}\left(1+\frac{1}{2}-\frac{1}{N+1}-\frac{1}{N+2}\right) = \frac{3}{2}.$$

8. The first four series can be shown to be divergent. However, that conclusion can be attained by the Test for Divergence only for the series in parts (c) and (d). Clearly $\lim_{n\to\infty} 1/\sqrt{n} = 0$, and, by L'Hôpital's Rule, $\lim_{n\to\infty} \ln(n)/n = 0$. Therefore, the Test for Divergence cannot be applied to the series in parts (a) and (b). Because $\lim_{n\to\infty} n^{1/n} = 1$, the Test for Divergence does apply to the series in (c), and shows that it is divergent. Because $\lim_{n\to\infty} \arctan(n) = \pi/2$, the Test for Divergence shows that the series in part (d) is divergent. Finally, for any constant r, we have $\lim_{n\to\infty}(1 + r/n)^n = \exp(r)$. Therefore, the numerator of $\left(\dfrac{1+2/n}{7}\right)^n$ tends to $\exp(2)$ while the denominator tends to infinity. The ratio tends to 0, and the Test for Divergence cannot be applied. The series is, in fact, convergent. The Test for Divergence can never be successful when applied to a convergent series.

9. (i) "may" Convergence can occur, as for example with a geometric series with ratio r between -1 and 1. But divergence can occur, as for example with the harmonic series.
(ii) "must" The Test for Divergence is logically equivalent to the statement filled in by "must."
(iii) "may" The Test for Divergence establishes the divergence of $\sum(-1)^n$, but it does not establish the divergence of the harmonic series.
(iv) "may" The partial sums of $\sum(1/2)^n$ and of $\sum(-1)^n$ are bounded. The first of these series converges

but the second does not.
(v) "may" The series $\sum(-1)^n$ and $\sum 1/n$ both diverge. The terms of the first of these two series diverge, but the terms of the second series converge.
(vi) "must" This is by definition: an infinite series is convergent if and only if its partial sums are convergent.

Solutions: The Integral Test (Corresponds to Stewart 11.3)

1. The first three terms of $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^3+32}$ are 0.30303, 0.1, and 0.1525. These terms are increasing, not decreasing, so the Integral Test cannot be applied. However, let us be persistent. Let $f(x) = x^2/\left(x^3+32\right)$. Notice that $f(n)$ is the n^{th} term of S_1. After some simplification we find that

$$f'(x) = -\frac{x\left(x^3-64\right)}{\left(x^3+32\right)^2}.$$

We see that $f'(x) \leq 0$ for $x \geq 4$. This tells us that $f(x)$ decreases for $x \geq 4$, and therefore the terms of S_1 are decreasing beginning with $M = 4$. The Integral Test can be applied to every tail S_M of S_1 with $M \geq 4$.

2. As noted in the preceding exercise, the Integral Test leads to a conclusion for this series by applying it to a tail. The Integral Test shows that $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^3+32}$ diverges:

$$\begin{aligned}
\int_4^{\infty} \frac{x^2}{x^3+32}\,dx &= \lim_{N\to\infty} \int_4^{N} \frac{x^2}{x^3+32}\,dx \\
&= \lim_{N\to\infty} \frac{1}{3}\ln\left(x^3+32\right)\Big|_4^N \\
&= \frac{1}{3}\lim_{N\to\infty}\left(\ln\left(N^3+32\right) - \frac{1}{3}\ln(96)\right) \\
&= \infty.
\end{aligned}$$

3. Let $f(x) = x^2/\left(nx^3+32\right)^2$. Then $f'(x) = -4\,x\,\left(x^3-16\right)/\left(x^3+32\right)^3 \leq 0$ for $x \geq 3$. The Integral Test

may be applied to the tail that begins with the third term. It shows that $\sum_{n=1}^{\infty} \frac{n^2}{(n^3+32)^2}$ converges:

$$\begin{aligned}
\int_3^{\infty} \frac{x^2}{(x^3+32)^2}\,dx &= \lim_{N\to\infty} \int_3^{N} \frac{x^2}{(x^3+32)^2}\,dx \\
&= -\lim_{N\to\infty} \left. \frac{1}{3\,(x^3+32)} \right|_3^N \\
&= -\lim_{N\to\infty} \left(\frac{1}{3\,(N^3+32)} - \frac{1}{3\times 59} \right) \\
&= \frac{1}{177} \\
&< \infty.
\end{aligned}$$

4. For r in the interval (0,1), the factor r^n is decreasing. However, the other factor, n, of $a_n = n\,r^n$ is increasing. It is not so easy to see which of the two opposing factors wins the battle. Following the method used in preceding exercises, set $f(x) = x\,r^x$ and differentiate: $f'(x) = (1 + x\,\ln(r))\,r^x$, which is negative for $1 + x\,\ln(r) < 0$. We deduce that the terms of the series decrease for $1 + n\,\ln(r) < 0$, or $n\,\ln(r) < -1$, or $n > -1/\ln(r)$. Why has the direction of the inequality changed? It is because $\ln(r) < 0$ thanks to r being less than 1. Let M denote the smallest integer greater than $-1/\ln(r)$. If r is slightly less than 1, this ineger might be fairly large. For example, if $r = 0.99$, then $M = 100$. The first 101 terms of the sequence $\{n\,(0.99)^n\}_{n=1}^{\infty}$ are 0.99, 1.96, 2.91, 3.84, 4.75, ..., 36.59949949, 36.60323412,36.60323413, 36.59957. As our calculation predicted, the decrease only begins once the summation index reaches $M = 100$. Integrating by parts, we find

$$\int x\,r^x\,dx = \frac{1}{\ln(r)^2}\,(x\,\ln(r) - 1)\,r^x.$$

L'Hôpital's Rule shows that $\lim_{x\to\infty} x\,r^x = 0$ for $0 < r < 1$. The improper integral is convergent and, therefore, so is the series.

5. We calculate $\int_{10}^{\infty} (1/x^2)\,dx = 0.1$ and $\int_{11}^{\infty} (1/x^2)\,dx = 0.909090909$ to nine decimal places. Therefore $\ell = 1.549767731 + 0.909090909 = 1.640676822$ and $m = 1.5497677318 + 0.1 = 1.6497677318$. The midpoint approximation to S is $(1.64067682 + 1.6497677318)/2$, or 1.645222276.

6. We calculate $\int_6^{\infty}(1/x^3)\,dx = 1/72$ and $\int_6^{\infty}(1/x^3)\,dx = 1/50$. The value of S, whatever it may be, lies in the interval $[1.185662037 + 1/72, 1.185662037 + 1/50]$, or $[1.199550926, 1.205662037]$. The midpoint is 1.202606482. If this value is used to estimate the exact value of S, then the maximum error is the distance of the midpoint to an endpoint of the interval, or half the width of the interval, or 0.0030555555. The (unknown) exact value of S is 1.202056903 to nine decimal places.

Solutions: Series with Positive Terms (Corresponds to Stewart 11.4 and 11.3)

1. Converges. The convergent series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ may be used for comparison.

2. Converges. The convergent p-series $\sum_{n=1}^{\infty} \frac{5}{n^{5/4}}$ may be used for comparison.

3. Converges. The convergent p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ may be used for comparison.

4. Converges. The convergent p-series $\sum_{n=1}^{\infty} \frac{3}{n^4}$ may be used for comparison.

5. Converges. The convergent geometric series $\sum_{n=1}^{\infty} 2\left(\frac{1}{e}\right)^n$ may be used for comparison.

6. Converges. The convergent p-series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ may be used for comparison.

7. Converges. The convergent p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ may be used for comparison.

8. Converges. The convergent geometric series $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$ may be used for comparison.

9. Converges. The convergent geometric series $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$ may be used for comparison.

10. Converges. The convergent p-series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ may be used for comparison.

11. Converges. The convergent p-series $\sum_{n=1}^{\infty} \frac{1}{n^{1.01}}$ may be used for comparison.

12. Converges. The convergent geometric series $\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n$ may be used for comparison.

13. Diverges. The divergent p-series $\sum_{n=1}^{\infty}\frac{1}{2n}$ may be used for comparison.

14. Diverges. The divergent p-series $\sum_{n=2}^{\infty}\frac{1}{n}$ may be used for comparison.

15. Diverges. The divergent p-series $\sum_{n=2}^{\infty}\frac{1}{3n}$ may be used for comparison.

16. Diverges. The divergent p-series $\sum_{n=2}^{\infty}\frac{2}{n}$ may be used for comparison.

17. Diverges. The divergent p-series $\sum_{n=2}^{\infty}\frac{1}{n}$ may be used for comparison.

18. Diverges. The series $\sum_{n=1}^{\infty}\frac{1}{2}$, which diverges by the Divergence Test, may be used for comparison:

$$\sum_{n=1}^{\infty}\frac{3^n+1}{3^n+2^n} > \sum_{n=1}^{\infty}\frac{3^n}{3^n+3^n} = \sum_{n=1}^{\infty}\frac{1}{2}.$$

This line of investigation also shows divergence by means of the Divergence Test.

19. Diverges. The divergent p-series $\sum_{n=1}^{\infty}\frac{1}{2\sqrt{n}}$ may be used for comparison. In the following comparison, observe that, at each step with an inequality, a smaller right side is obtained by either using a numerator that is smaller than the one on the left, or a denominator that is larger than the one on the left:

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{3^n+n}{\sqrt{n}3^n+1} &> \sum_{n=1}^{\infty}\frac{3^n}{\sqrt{n}3^n+1}\\ &> \sum_{n=1}^{\infty}\frac{3^n}{\sqrt{n}3^n+\sqrt{n}3^n}\\ &= \sum_{n=1}^{\infty}\frac{1}{2\sqrt{n}}.\end{aligned}$$

20. Diverges. The divergent series $\sum_{n=1}^{\infty}\frac{1}{2}$ may be used for comparison. In the following comparison, observe that, at each step with an inequality, a smaller right side is obtained by either using a numerator that

is smaller than the one on the left, or a denominator that is larger than the one on the left:

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{2^n+1}{2^n+n^2} &> \sum_{n=1}^{\infty}\frac{2^n}{2^n+n^2}\\ &> \sum_{n=1}^{\infty}\frac{2^n}{2^n+2^n}\\ &= \sum_{n=1}^{\infty}\frac{1}{2}.\end{aligned}$$

This line of investigation also shows divergence by means of the Divergence Test.

21. Diverges. The divergent harmonic series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\frac{1}{n}$ can be used for comparison: $\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{3}{2}$.

22. Converges. The convergent p-series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ can be used for comparison: $\lim_{n\to\infty}\frac{a_n}{b_n} = 1$.

23. Converges. The convergent p-series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\frac{1}{n^2}$ can be used for comparison: $\lim_{n\to\infty}\frac{a_n}{b_n} = 1$.

24. Converges. The convergent p-series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\frac{1}{n^2}$ can be used for comparison: $\lim_{n\to\infty}\frac{a_n}{b_n} = 1$.

25. Converges. The convergent p-series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ can be used for comparison: $\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{1}{2}$.

26. Converges. The convergent geometric series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n$ can be used for comparison: $\lim_{n\to\infty}\frac{a_n}{b_n} = 3$.

27. Converges. The convergent geometric series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n$ can be used for comparison: $\lim_{n\to\infty}\frac{a_n}{b_n} = 1$.

28. Converges. The convergent p-series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\frac{1}{n^2}$ can be used for comparison: $\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\arctan(n) = \frac{\pi}{2}$.

29. Diverges. The (divergent) harmonic series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\frac{1}{n}$ can be used for comparison: $\lim_{n\to\infty}\frac{a_n}{b_n} = 1$.

30. Diverges. Observe that the denominator of a_n is comparable to $n^{5/4}$ and so a_n is comparable to $b_n = \frac{n^{1/3}}{n^{5/4}} = \frac{1}{n^{11/12}}$. The divergent p-series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\frac{1}{n^{11/12}}$ can be used for comparison: $\lim_{n\to\infty}\frac{a_n}{b_n} = 1$.

31. Converges. The convergent p-series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\frac{1}{n^2}$ can be used for comparison: $\lim_{n\to\infty}\frac{a_n}{b_n} = 1$.

32. Converges. The convergent p-series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\frac{1}{n^2}$ can be used for comparison: $\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\left(1+\frac{\ln(n)}{n}\right) = 1+0$.

33. Diverges. The (divergent) harmonic series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ can be used for comparison: $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{1+\ln(n)/n} = \frac{1}{1+0}$.

34. Converges. The convergent geometric series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$ can be used for comparison: $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{1+n^3/2^n}{1+n^2/3^n}\right) = \frac{1+0}{1+0}$.

35. Converges. The convergent geometric series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \left(\frac{7}{10}\right)^n$ can be used for comparison: $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(1 + \frac{n^{10}}{7^n}\right) = 1$.

36. Converges. The convergent geometric series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$ can be used for comparison: $\lim_{n\to\infty} \frac{a_n}{b_n} = \exp\left(\frac{1}{6}\right)$. Remark: The limit of a_n/b_n might initially seem to be 1 because $\frac{a_n}{b_n} = \left(\frac{1+5/(2\,n)}{1+7/(3\,n)}\right)^n$, which has a base tending to 1. After closer inspection, we see that the limit is the indeterminate form 1^∞ and must be calculated more carefully (using L'Hôpital's Rule, for example).

37. Diverges. We calculate

$$\begin{aligned}
\int_1^\infty \frac{x}{x^2+1}\,dx &= \lim_{N\to\infty} \int_1^N \frac{x}{x^2+1}\,dx \\
&= \frac{1}{2} \lim_{N\to\infty} \ln\left(x^2+1\right)\Big|_1^N \\
&= \frac{1}{2} \lim_{N\to\infty} \ln\left(N^2+1\right) - \frac{\ln(2)}{2} \\
&= \infty.
\end{aligned}$$

38. Converges. Using $\lim_{N\to\infty} \arctan(N) = \pi/2$, we calculate

$$\int_1^\infty \frac{1}{x^2+4}\,dx = \lim_{N\to\infty} \int_1^N \frac{1}{x^2+4}\,dx = \lim_{N\to\infty} \left(\frac{1}{2}\arctan(N) - \frac{1}{2}\arctan(1)\right) = \left(\frac{1}{2}\frac{\pi}{2} - \frac{1}{2}\frac{\pi}{4}\right) = \frac{\pi}{8}.$$

39. Converges.

$$\begin{aligned}
\int_1^\infty \frac{1}{x(x+2)}\,dx &= \lim_{N\to\infty}\int_1^N \frac{1}{x(x+2)}\,dx \\
&= \frac{1}{2}\lim_{N\to\infty}\int_1^N \left(\frac{1}{x}-\frac{1}{x+2}\right)dx \\
&= \frac{1}{2}\lim_{N\to\infty}\left(\ln(|x|)-\ln(|x+2|)\right)\Big|_1^N \\
&= \frac{1}{2}\lim_{N\to\infty}\left(\ln(N)-\ln(N+2)\right)+\frac{\ln(3)}{2} \\
&= \frac{1}{2}\lim_{N\to\infty}\ln\left(\frac{N}{N+2}\right)+\frac{\ln(3)}{2} \\
&= \frac{1}{2}\lim_{N\to\infty}\ln\left(\frac{1}{1+2/N}\right)+\frac{\ln(3)}{2} \\
&= 0+\frac{\ln(3)}{2}.
\end{aligned}$$

40. Converges.

$$\begin{aligned}
\int_1^\infty x^2\exp\left(-x^3\right)dx &= \lim_{N\to\infty}\int_1^N x^2\exp\left(-x^3\right)dx \\
&= -\frac{1}{3}\lim_{N\to\infty}\exp\left(-x^3\right)\Big|_1^N \\
&= \frac{1}{2}\lim_{N\to\infty}\exp\left(-N^3\right)+\exp(-1) \\
&= \exp(-1).
\end{aligned}$$

Remark: The hypotheses of the Integral Test are satisfied. If $f(x)=x^2\exp\left(-x^3\right)$, then $f'(x)=x\left(2-3x^3\right)\exp\left(-x^3\right)$. For $x\geq 1$, the first and last factors of $f'(x)$ are positive and the middle factor is negative. Therefore $f'(x)<0$ for $x\geq 1$, and $f(x)$ is decreasing, as is required for the Integral Test to be applicable.

41. Diverges.

$$\begin{aligned}
\int_2^\infty \frac{2x^2}{x^3+4}\,dx &= \lim_{N\to\infty}\int_2^N \frac{2x^2}{x^3+4}\,dx \\
&= \frac{2}{3}\lim_{N\to\infty}\ln\left(x^3+4\right)\Big|_2^N \\
&= \frac{2}{3}\lim_{N\to\infty}\left(\ln\left(N^3+4\right)-\ln(12)\right) \\
&= \infty.
\end{aligned}$$

Remark: The hypotheses of the Integral Test are satisfied. If $f(x)=2x^2/\left(x^3+4\right)$, then $f'(x)=-2x\left(x^3-8\right)\left(x^3+4\right)^{-2}$. For $x\geq 2$, the factors of $f'(x)$ involving x are all nonnegative. Therefore $f'(x)\leq 0$ for $x\geq 2$, and $f(x)$ is decreasing, as is required for the Integral Test to be applicable.

42. Diverges.

$$\begin{aligned}
\int_2^\infty \frac{1}{x\ln(x)}\,dx &= \lim_{N\to\infty}\int_2^N \frac{1}{x\ln(x)}\,dx \\
&= \lim_{N\to\infty}\ln\big(\ln(x)\big)\Big|_1^N \\
&= \infty.
\end{aligned}$$

Remark: For this exercise, the Integral Test is the best test we have available.

43. Converges.

$$\begin{aligned}\int_2^\infty \frac{1}{x\ln^2(x)}\,dx &= \lim_{N\to\infty}\int_2^N \frac{1}{x\ln^2(x)}\,dx \\ &= -\lim_{N\to\infty}\left(\frac{1}{\ln(x)}\right)\Bigg|_1^N \\ &= 1.\end{aligned}$$

Remark: For this exercise, the Integral Test is the best test we have available.

44. Converges. By making the substitution $u = \exp(x)$, we obtain the integral formula

$$\int \frac{\exp(x)}{(1+\exp(x))^2}\,dx = -\frac{1}{1+\exp(x)} + C.$$

From this we deduce that the improper integral $\displaystyle\int_1^\infty \frac{\exp(x)}{(1+\exp(x))^2}\,dx$ is convergent.

Remark: The hypotheses of the Integral Test are satisfied. If $f(x) = \exp(x)/(1+\exp(x))^2$, then $f'(x) = -\exp(x)\,(-1+\exp(x))/(1+\exp(x))^3$. For $x \geq 1$, all factors of $f'(x)$ involving x are nonnegative. Therefore $f'(x) \leq 0$ for $x \geq 1$, and $f(x)$ is decreasing, as is required for the Integral Test to be applicable.

45. Converges. Integrating by parts, we find

$$\int x\,10^{-x}\,dx = -\frac{(1+x\,\ln(10))\,10^{-x}}{\ln^2(10)} + C.$$

Using this formula we see that the improper integral $\displaystyle\int_1^\infty \frac{x}{10^x}\,dx$ is convergent. Remark: The hypotheses of the Integral Test are satisfied. If $f(x) = x/10^x$, then $f'(x) = (1 - x\,\ln(10))\,/10^x$. For $x \geq 1$, the numerator of $f'(x)$ is nnegative. Therefore $f'(x) \leq 0$ for $x \geq 1$, and $f(x)$ is decreasing, as is required for the Integral Test to be applicable.

46. Divergent. The integral formula

$$\int \frac{\ln(x)}{x}\,dx = \frac{1}{2}\ln^2(x) + C,$$

which we obtain by making the substitution $u = \ln(x)$, shows that the improper integral $\displaystyle\int_1^\infty \frac{\ln(x)}{x}\,dx$ is divergent.

47. Converges. We integrate by parts to find $\displaystyle\int_2^\infty x\,\exp(-x/2)\,dx = \frac{8}{e}$.

Remark: The hypotheses of the Integral Test are satisfied. If $f(x) = x\,\exp(-x/2)$, then $f'(x) = (1-x/2)\,\exp(-x/2)$. For $x \geq 2$, the first factor of $f'(x)$ is nonpositive. Therefore $f'(x) \leq 0$ for $x \geq 2$, and $f(x)$ is decreasing, as is required for the Integral Test to be applicable.

48. Converges. Use the integral formula $\displaystyle\int \frac{1}{x\sqrt{x^2-1}}\,dx = \operatorname{arcsec}(x) + C$ to obtain $\displaystyle\int_2^\infty \frac{1}{x\sqrt{x^2-1}}\,dx = \frac{\pi}{6}$.

Solutions: Alternating Series

In each of Exercises 1–6 a convergent alternating series $S = \sum_{n=1}^{\infty} (-1)^{n+1}\, a_n$ is given. It is easy to see that, because the terms a_n decrease to 0, the Alternating Series Test can be applied to each of these series, establishing convergence. Find a value of N such that the partial sum $S_N = \sum_{n=1}^{N} (-1)^{n+1}\, a_n$ approximates the given infinite series to within 0.01. That is, find N so that $|S - S_N| \leq 0.01$ holds. The point of these exercises

1. $N = 9999$
   In this exercise and the others in this group, the solution is obtained by finding the *smallest* index ν for which $a_{nu} \leq 0.01$. For $N = \nu - 1$ we then have

   $$|S - S_N| \leq a_{N+1} = a_\nu \leq 0.01.$$

   We therefore must find the least ν such that $1/\sqrt{\nu} \leq 0.01 = 1/100$, or $sqrt\nu \geq 100$, or $\nu = 100^2 = 10000$. Thus, we can be certain that $N = 10000 - 1 = 9999$ does the job. We can be sure that every larger index results in the required accuracy, but, if that accuracy suffices, why would we add up more terms than are necessary for that accuracy? In the other direction, what can we say about partial sums S_K with $K < 9999$? Nothing at all with certainty! There may well be indices smaller than 9999 that result in the desired accuracy, but theory does not allow us to state for certain that there are such smaller indices. In summary, $N = 9999$ provides the required accuracy and is the smallest index that theory ensures us does the job. It turns out that there *are* indices much smaller than 9999 that do the job: $S = 0.6048986434$ and $S_{2500} = 0.5948996442$, so $|S - S_{2500}| = 0.0099989992 < 0.01$. Thus, we see that 2500 does the job, and 2500 is much smaller than 9999. The problem is, we can find 2500 only by performing a tremendous amount of numerical computation (including the calculation of S to more accuracy than the exercise required): we have no theoretical basis for knowing that S_{2500} is sufficiently accurate.

2. $N = 4$

3. $N = 200$

4. $N = 4$

5. $N = 4$

6. $N = 29$

7. $M = 10$

8. $M = 4$

9. $M = 7$

10. $M = 7$

11. $1 - 1/2 + 1/6 - 1/24 + 1/120 - 1/720 \approx 0.63194$
12. $1/10 - 3/200 + 1/600 \approx 0.08667$

Solutions: The Ratio Test and the Root Test

1. $\sum_{n=1}^{\infty} \frac{n}{e^n}$ converges:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n+1}{n} \frac{e^n}{e^{n+1}} = \frac{1}{e} \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = \frac{1}{e} < 1.$$

2. $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2 \frac{2^n}{2^{n+1}} = \frac{1}{2} \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^2 = \frac{1}{2} < 1.$$

3. $\sum_{n=1}^{\infty} \frac{2^n}{n^3}$ diverges:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^3 \frac{2^{n+1}}{2^n} = 2 \lim_{n\to\infty} \left(\frac{1}{1+1/n}\right)^3 = 2 > 1.$$

4. $\sum_{n=1}^{\infty} \frac{10^n}{n!}$ converges:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n!}{(n+1)!} \frac{10^{n+1}}{10^n} = 10 \lim_{n\to\infty} \frac{n!}{(n+1)\cdot n!} = 10 \lim_{n\to\infty} \frac{1}{(n+1)} = 0 < 1.$$

5. $\sum_{n=1}^{\infty} \frac{n^{100}}{n!}$ converges:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n!}{(n+1)!} \left(\frac{n+1}{n}\right)^{100} = \lim_{n\to\infty} \frac{n!}{(n+1)\cdot n!} \left(1 + \frac{1}{n}\right)^{100} = \lim_{n\to\infty} \frac{1}{(n+1)} \left(1 + \frac{1}{n}\right)^{100} = 0 \times 1^{100} = 0 < 1.$$

6. $\sum_{n=1}^{\infty} \frac{n!}{n^5 \cdot 7^n}$ diverges:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!}{n!} \left(\frac{n}{n+1}\right)^5 \frac{7^n}{7^{n+1}} = \frac{1}{7} \lim_{n\to\infty} \frac{(n+1)\cdot n!}{n!} \left(\frac{1}{1+1/n}\right)^5 = \frac{1}{7} \frac{1^5}{(1+0)^5} \lim_{n\to\infty} (n+1) = \infty > 1.$$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n + n}{2^n + n^3}$ diverges:

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{3^{n+1} + n + 1}{3^n + n} \times \frac{2^n + n^3}{2^{n+1} + (n+1)^3} \\
&= \lim_{n\to\infty} \frac{3 + (n+1)/3^n}{1 + n/3^n} \times \frac{1 + n^3/2^n}{2 + (n+1)^3/2^n} \\
&= \frac{3+0}{1+0} \times \frac{1+0}{2+0} \\
&= \frac{3}{2} \\
&> 1.
\end{aligned}$$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{\sqrt{n!}}$ converges:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\sqrt{n!}}{\sqrt{(n+1)!}} \, \frac{2^{n+1}}{2^n} = 2 \lim_{n\to\infty} \sqrt{\frac{n!}{(n+1)\cdot n!}} = 2 \lim_{n\to\infty} \frac{1}{\sqrt{n+1}} = 0 < 1.$$

9. Let p be any exponent and let c be any constant. Then, using L'Hôpital's Rule, we have

$$\lim_{n\to\infty} \frac{c \ln^p(n+1)}{c \ln^p(n)} = \lim_{n\to\infty} \left(\frac{\ln(n+1)}{\ln(n)} \right)^p = \lim_{n\to\infty} \left(\frac{1/(n+1)}{1/n} \right)^p = \lim_{n\to\infty} \left(\frac{n}{n+1} \right)^p = \lim_{n\to\infty} \left(\frac{1}{1+1/n} \right)^p = 1.$$

In a similar way, if q is any exponent, we have

$$\lim_{n\to\infty} \frac{c\,(n+1)^q}{c\,n^q} = \lim_{n\to\infty} \left(\frac{n+1}{n} \right)^q = \lim_{n\to\infty} \left(1 + \frac{1}{n} \right)^q = 1.$$

Thus, if $a_n = \dfrac{\ln^p(n)}{n^q}$, then

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\ln^p(n+1)}{\ln^p(n)} \times \frac{n^q}{(n+1)^q} = 1 \times 1 = 1.$$

Using $p = q = 1$ for this example, we see that the Ratio Test is inconclusive. Using the Alternating Series Test, we see that the given series is convergent. Its series of absolute values, $\displaystyle\sum_{n=2}^{\infty} \frac{\ln(n)}{n}$, is seen to be divergent by using the Comparison Test with the harmonic series for comparison. Hence, the given series is **conditionally convergent**.

10. First, simplify the given series, writing it as $\displaystyle\frac{1}{2} \sum_{n=2}^{\infty} (-1)^n \frac{1}{\ln(n)}$. Use the calculation of Exercise 9 with $p = -1$ and $q = 0$ to see that the Ratio Test is inconclusive. Using the Alternating Series Test, we see that the given series is convergent. Its series of absolute values, $\displaystyle\frac{1}{2} \sum_{n=2}^{\infty} \frac{1}{\ln(n)}$, is seen to be divergent by using the Comparison Test with the harmonic series for comparison. Hence, the given series is **conditionally convergent**.

11. Use the calculation of Exercise 9 with $p = 1$ and $q = 2$ to see that the Ratio Test is inconclusive. L'Hôpital's Rule shows that, for every $q > 0$,

$$\lim_{n\to\infty} \frac{\ln^p(n)}{n^q} = 0.$$

By taking $p = 1$ and $q = 1/2$, we obtain

$$\lim_{n\to\infty} \frac{\ln(n)}{\sqrt{n}} = 0.$$

It follows that there exists and M with

$$\frac{\ln(n)}{\sqrt{n}} < 1$$

for all n with $n \geq M$. That is, $\sqrt{n} > \ln(n)$ for $n \geq M$. Therefore

$$b_n = \frac{1}{n^{3/2}} = \frac{\sqrt{n}}{n^2} > \frac{\ln(n)}{n^2}$$

for all n with $n \geq M$. Using the convergent p-series $\displaystyle\sum_{n=M}^{\infty} \frac{1}{n^{3/2}}$ for comparison, we deduce that $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n^2}$ is convergent. Therefore, the given series is **absolutely convergent**.

12. The series $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n^3+1}$ is seen to be convergent by using the Limit Comparison Test with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ for the comparison series. Therefore, the given series is **absolutely convergent**.

13. The given series is **divergent** by the Divergence Test.

14. Using the Alternating Series Test, we see that the given series is convergent. Its series of absolute values, $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+4}$, is seen to be divergent by using the Limit Comparison Test with the comparison series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$. Hence, the given series is **conditionally convergent**.

15. The series of absolute values, $\displaystyle\sum_{n=1}^{\infty} \frac{e^n + \ln(n)}{e^n \cdot n^2}$, is seen to be conergent by using the Comparison Test with the comparison series $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2}$:

$$\frac{e^n + \ln(n)}{e^n \cdot n^2} < \frac{e^n + e^n}{e^n \cdot n^2} = \frac{2\,e^n}{e^n \cdot n^2} = \frac{2}{n^2}.$$

Hence, the given series is **absolutely convergent**.

16. Because

$$\lim_{n\to\infty} \frac{n!}{n! + 2^n} = \lim_{n\to\infty} \frac{1}{1 + 2^n/n!} = \frac{1}{1+0} = 1,$$

the given series is **divergent** by the Divergence Test.

17. Because $\lim_{n\to\infty} \arctan(n) = \pi/2$, the given series is **divergent** by the Divergence Test.

18. The given series is convergent by the Alternating Series Test. The series of absolute values, $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$, is divergent by the Limit Comparison Test with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ for the comparison series. Hence, the given series is **conditionally convergent**.

19. Because
$$\lim_{n\to\infty} \left(n^{-n/2}\right)^{1/n} = \lim_{n\to\infty} \frac{1}{n^{1/2}} = 0 < 1,$$
the given series $\sum_{n=1}^{\infty} n^{-n/2}$ **converges** by the Root Test.

20. Because
$$\lim_{n\to\infty} \left(\frac{2^{3n}}{3^{2n}}\right)^{1/n} = \lim_{n\to\infty} \frac{2^3}{3^2} = \frac{8}{9} < 1,$$
the given series $\displaystyle\sum_{n=1}^{\infty} \frac{2^{3n}}{3^{2n}}$ **converges** by the Root Test.

21. Because
$$\lim_{n\to\infty} \left(\frac{n}{2^n}\right)^{1/n} = \lim_{n\to\infty} \frac{n^{1/n}}{2} = \frac{1}{2} \lim_{n\to\infty} n^{1/n} = \frac{1}{2} < 1,$$
the given series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$ **converges** by the Root Test.

22. Because
$$\lim_{n\to\infty} \left(\frac{10^n}{n^{10}}\right)^{1/n} = \lim_{n\to\infty} \frac{10}{n^{10/n}} = \frac{10}{\left(\lim_{n\to\infty} n^{1/n}\right)^{10}} = 10 > 1,$$
the given series $\displaystyle\sum_{n=1}^{\infty} \frac{10^n}{n^{10}}$ **diverges** by the Root Test.

23. Because
$$\lim_{n\to\infty} \left(\frac{n^7}{\ln^n(n)}\right)^{1/n} = \frac{\left(\lim_{n\to\infty} n^{1/n}\right)^7}{\lim_{n\to\infty} \ln(n)} = \frac{1^7}{\infty} = 0 < 1,$$
the given series $\displaystyle\sum_{n=2}^{\infty} \frac{n^7}{\ln^n(n)}$ **converges** by the Root Test.

24. Using L'Hôpital's Rule, we have
$$\begin{aligned}
\lim_{n\to\infty} \left(\left(\frac{\ln(n+4)}{\ln\left(n^2+4\right)}\right)^n\right)^{1/n} &= \lim_{n\to\infty} \frac{\ln(n+4)}{\ln\left(n^2+4\right)} \\
&= \lim_{n\to\infty} \frac{1/(n+4)}{2n/\left(n^2+4\right)} \\
&= \lim_{n\to\infty} \frac{n^2+4}{2n(n+4)} \\
&= \lim_{n\to\infty} \frac{1+4/n^2}{2+8/n} \\
&= \frac{1}{2} \\
&< 1.
\end{aligned}$$

Therefore, the given series $\sum_{n=1}^{\infty}\left(\frac{\ln(n+4)}{\ln(n^2+4)}\right)^n$ converges.

25. Because

$$\lim_{n\to\infty}\left(\left(\frac{37}{n}\right)^n\right)^{1/n} = \lim_{n\to\infty}\frac{37}{n} = \frac{37}{\infty} = 0 < 1,$$

the given series $\sum_{n=1}^{\infty}\left(\frac{37}{n}\right)^n$ **converges** by the Root Test.

26. **Absolutely convergent** by the Rato Test: if $a_n = (2n)!/(3n)!$, then $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = 0 < 1$.

27. **Absolutely convergent** by the Rato Test: if $a_n = (n!)^2/(2n)!$, then $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \frac{1}{4} < 1$.

28. **Divergent** by the Divergence Test:

$$\lim_{n\to\infty}\frac{n+3^n}{n^3+2^n} = \lim_{n\to\infty}\frac{n/2^n+(3/2)^n}{n^3/2^n+1} = \frac{0+\infty}{0+1} = \infty \neq 0.$$

29. **Divergent** by the Root Test:

$$\lim_{n\to\infty}\left(\left(n^{1/n}+1/2\right)^n\right)^{1/n} = \lim_{n\to\infty}\left(n^{1/n}+1/2\right) = \frac{3}{2} > 1.$$

30. **Absolutely convergent** by the Root Test: $\lim_{n\to\infty}\left(\frac{2^n}{1+\ln^n(n)}\right)^{1/n} = \frac{2}{\infty} = 0 < 1$.

31. **Absolutely convergent** by the Root Test: $\lim_{n\to\infty}\left(\frac{\exp(n)}{\ln^n(n)}\right)^{1/n} = \lim_{n\to\infty}\frac{e}{\ln(n)} = 0 < 1$.

32. **Conditionally convergent**. The given series is convergent by the Alternating Series Test. The series of absolute values diverges by the Limit Comparison Test using the divergent p-series with $p = 1/2$ for comparison.

33. **Divergent** by the Divergence Test: $\lim_{n\to\infty}\frac{n!}{3^n} = \infty \neq 0$.

34. **Conditionally convergent**

35. **Absolutely convergent**. Use the Integral Test

$$\int_2^{\infty}\frac{1}{x\ \ln^3(x)}\,dx = \frac{1}{2\ \ln^2(2)}.$$

36. **Conditionally convergent**

37. **Absolutely convergent**. Use the limit Comparison Test to compare the series of absolute values with the convergent p-series with $p = e > 1$.

38. **Conditionally convergent** The given series has the behavior of the alternating harmonic series because

$$\lim_{n\to\infty}\frac{\ln(1+1/n)}{1/n} = 1.$$

39. **Divergent** by the Divergence Test: $\lim_{n\to\infty}(1+1/n)^n = e$.

40. **Conditionally convergent**

Solutions: Power Series, Intervals of Convergence (Corresponds to Stewart 11.8)

1. $(-3, 3)$
2. $(-1, 1)$
3. $[-1, 1)$
4. $(-1/10, 1/10]$
5. $[-1/4, 1/4]$
6. $(-5/2, 3/2)$
7. $(-\infty, \infty)$
8. $(-1, 1]$
9. $[-1/2, 3/2)$
10. $[-7/3, -5/3]$
11. $(-2, 1)$
12. $[-7, -5)$
13. $(\pi - 1, \pi + 1]$
14. $[-5, -3]$
15. $(-1, -1/3)$
16. $(-3, -2)$
17. For $u = \big(x - (-3)\big)^2$, we have

$$\sum_{n=0}^{\infty} \frac{4^n}{n+1} (x+3)^{2n} = \sum_{n=0}^{\infty} \frac{4^n}{n+1} u^n.$$

For this power series in u, we have $c = 0$ and $a_n = 4^n/(n+1)$. It follows that

$$\begin{aligned} R &= \lim_{n\to\infty} \frac{a_n}{a_{n+1}} \\ &= \lim_{n\to\infty} \frac{4^n}{n+1} \frac{n+2}{4^{n+1}} \\ &= \lim_{n\to\infty} \frac{4^n}{4^{n+1}} \frac{n+2}{n+1} \\ &= \lim_{n\to\infty} \frac{1}{4} \frac{1+2/n}{1+1/n} \\ &= \frac{1}{4}. \end{aligned}$$

For the left endpoint, set $u = c - R = 0 - 1/4 = -1/4$ in the power series:

$$\sum_{n=0}^{\infty} \frac{4^n}{n+1} u^n \bigg|_{u=-1/4} = \sum_{n=0}^{\infty} \frac{4^n}{n+1} \frac{-(1)^n}{4^n} = \sum_{n=0}^{\infty} -(1)^n \frac{1}{n+1},$$

which is convergent by the Alternating Series Test. For the right endpoint, set $u = c+R = 0+1/4 = -1/4$ in the power series:

$$\sum_{n=0}^{\infty} \frac{4^n}{n+1} u^n \bigg|_{u=1/4} = \sum_{n=0}^{\infty} \frac{4^n}{n+1} \frac{1}{4^n} = \sum_{n=0}^{\infty} \frac{1}{n+1},$$

which is a tail of a divergent p-series. It follows that $[-1/4, 1/4)$ is the interval of convergence for the powers series in u. Thus, the given power series in x converges for $-1/4 \le \big(x - (-3)\big)^2 < 1/4$, which is the same interval as $0 \le \big(x - (-3)\big)^2 < 1/4$, or $|x - (-3)| < 1/2$, or $-1/2 < x - (-3) < 1/2$, or $-3 - 1/2 < x < -3 + 1/2$, or $-7/2 < x < -5/2$. In interval notation, the answer is $(-7/2, -5/2)$.

18. $[-1/2, 5/2)$

Solutions: Series Related to $1/(1 \pm x)$ (Corresponds to Stewart 11.8)

1. With $u = x/2$ we have

$$\begin{aligned}
\frac{1}{2+x} &= \frac{1}{2} \times \frac{1}{1+u} \\
&= \frac{1}{2}\left(1 - u + u^2 - u^3 + u^4 - u^5 + \cdots\right) \\
&= \frac{1}{2} - \frac{1}{4}\,x + \frac{1}{8}\,x^2 - \frac{1}{16}\,x^3 + \frac{1}{32}\,x^4 \pm \cdots
\end{aligned}$$

2. With $u = 2x$ we have

$$\begin{aligned}
\frac{1}{1-2x} &= \frac{1}{1-u} \\
&= 1 + u + u^2 + u^3 + u^4 + u^5 + \cdots \\
&= 1 + 2x + 4x^2 + 8x^3 + 16x^4 + 32x^5 + \cdots
\end{aligned}$$

3. With $u = -3x/4$ we have

$$\begin{aligned}
\frac{x}{4+3x} &= \frac{x}{4}\,\frac{1}{1-u} \\
&= \frac{x}{4}\left(1 + u + u^2 + u^3 + u^4 + u^5 + \cdots\right) \\
&= \frac{x}{4}\left(1 - \frac{3}{4}x + \frac{9}{16}x^2 - \frac{27}{64}x^3 + \frac{81}{256}x^4 - \cdots\right) \\
&= \frac{1}{4}x - \frac{3}{16}x^2 + \frac{9}{64}x^3 - \frac{27}{256}x^4 + \frac{81}{1024}x^5 - \cdots
\end{aligned}$$

4. With $u = -x^2/4$ we have

$$\begin{aligned}
\frac{1}{4+x^2} &= \frac{1}{4}\,\frac{1}{1-u} \\
&= \frac{1}{4}\left(1 + u + u^2 + u^3 + u^4 + u^5 + \cdots\right) \\
&= \frac{1}{4}\left(1 - \frac{1}{4}x^2 + \frac{1}{16}x^4 - \cdots\right) \\
&= \frac{1}{4} - \frac{1}{16}x^2 + \frac{1}{64}x^4 - \cdots
\end{aligned}$$

5. With $u = x^2$ we have

$$\frac{1}{1-x^2} = \frac{1}{1-u} = 1 + u + u^2 + u^3 + u^4 + \cdots = 1 + x^2 + x^4 + \cdots$$

Therefore,

$$\frac{1+x^2}{1-x^2} = \left(1+x^2\right)\left(1 + x^2 + x^4 + \cdots\right) = 1 + 2x^2 + 2x^4 + \cdots$$

6. With $u = 3\,x^2/2$ we have

$$\frac{x}{2+3\,x^2} = \frac{x}{2} \cdot \frac{1}{1+u} = \frac{x}{2}\left(1 - u + u^2 - u^3 + u^4 + \cdots\right) = \frac{1}{2}\,x - \frac{3}{4}\,x^3 + \frac{9}{8}\,x^5 - \frac{27}{16}\,x^7 \pm \cdots$$

7. We have

$$\begin{aligned}
\frac{1}{(2+x)^2} &= \frac{1}{2+x} \cdot \frac{1}{2+x} \\
&= \left(\frac{1}{2} - \frac{1}{4}\,x + \frac{1}{8}\,x^2 - \frac{1}{16}\,x^3 + \frac{1}{32}\,x^4 \pm \cdots\right)\left(\frac{1}{2} - \frac{1}{4}\,x + \frac{1}{8}\,x^2 - \frac{1}{16}\,x^3 + \frac{1}{32}\,x^4 \pm \cdots\right) \\
&= \frac{1}{2}\cdot\frac{1}{2} + \left(-\frac{1}{2}\cdot\frac{1}{4} - \frac{1}{4}\cdot\frac{1}{2}\right)x + \left(\frac{1}{2}\cdot\frac{1}{8} + \frac{1}{4}\cdot\frac{1}{4} + \frac{1}{8}\cdot\frac{1}{2}\right)x^2 \\
&\qquad + \left(-\frac{1}{2}\cdot\frac{1}{16} - \frac{1}{4}\cdot\frac{1}{8} - \frac{1}{8}\cdot\frac{1}{4} - \frac{1}{16}\cdot\frac{1}{2}\right)x^3 + \cdots \\
&= \frac{1}{4} - \frac{1}{4}\,x + \frac{3}{16}\,x^2 - \frac{1}{8}\,x^3 \pm \cdots\ .
\end{aligned}$$

8. We have

$$\begin{aligned}
\frac{1}{(2+x)^2} &= -\frac{d}{dx}\left(\frac{1}{2+x}\right) \\
&= -\frac{d}{dx}\left(\frac{1}{2} - \frac{1}{4}\,x + \frac{1}{8}\,x^2 - \frac{1}{16}\,x^3 + \frac{1}{32}\,x^4 \pm \cdots\right) \\
&= -\left(0 - \frac{1}{4} + \frac{1}{4}\,x - \frac{3}{16}\,x^2 + \frac{1}{8}\,x^3 \pm \cdots\right) \\
&= \frac{1}{4} - \frac{1}{4}\,x + \frac{3}{16}\,x^2 - \frac{1}{8}\,x^3 \pm \cdots\ .
\end{aligned}$$

9. We have

$$\begin{aligned}
\frac{1}{2-x-x^2} &= \frac{1}{1-x} \cdot \frac{1}{2+x} \\
&= \left(1 + x + x^2 + x^3 + x^4 + \cdots\right)\left(\frac{1}{2} - \frac{1}{4}\,x + \frac{1}{8}\,x^2 - \frac{1}{16}\,x^3 + \frac{1}{32}\,x^4 \pm \cdots\right) \\
&= \frac{1}{2} + \left(-\frac{1}{4} + \frac{1}{2}\right)x + \left(\frac{1}{8} - \frac{1}{4} + \frac{1}{2}\right)x^2 + \left(-\frac{1}{16} + \frac{1}{8} - \frac{1}{4} + \frac{1}{2}\right)x^3 + \cdots \\
&= \frac{1}{2} + \frac{1}{4}\,x + \frac{3}{8}\,x^2 + \frac{5}{16}\,x^3 + \cdots\ .
\end{aligned}$$

10.

$$\begin{aligned}\ln(2+x) &= \int \frac{1}{2+x}\,dx \\ &= \int \left(\frac{1}{2} - \frac{1}{4}\,x + \frac{1}{8}\,x^2 - \frac{1}{16}\,x^3 + \frac{1}{32}\,x^4 \pm \cdots\right) dx \\ &= C + \frac{1}{2}\,x - \frac{1}{8}\,x^2 + \frac{1}{24}\,x^3 - \frac{1}{64}\,x^4 + \frac{1}{160}\,x^5 \pm \cdots\end{aligned}$$

Setting $x = 0$, we see that $C = \ln(2)$.

11. The method of multiplication follows problems already written up. We turn to the second method:

$$\begin{aligned}x + 2\,x^2 + 3\,x^3 + 4\,x^4 + \cdots &= (2+x)\left(a_0 + a_1\,x + a_2\,x^2 + a_3\,x^3 + a_4\,x^4 + \cdots\right) \\ &= 2\,a_0 + (2\,a_1 + a_0)\;x + (2\,a_2 + a_1)\;x^2 + (2\,a_3 + a_2)\;x^3 + (2\,a_4 + a_3)\;x^4 + \cdots\;.\end{aligned}$$

Equating like coefficients of powers of x on each side, we obtain the equations $0 = 2\,a_0$, $1 = 2\,a_1 + a_0$, $2 = 2\,a_2 + a_1$, $3 = 2\,a_3 + a_2$, $4 = 2\,a_4 + a_3$. One by one, we solve $a_0 = 0$, $a_1 = 1/2$, $a_2 = 3/4$, $a_3 = 9/8$, $a_4 = 23/16$. That is $q(x) = x/2 + 3x^2/4 + 9x^3/8 + 23x^4/16 + \cdots$.

12. We have

$$\begin{aligned}2 + x &= \left(2 + x + 2\,x^2 + 3\,x^3 + \cdots\right)\left(a_0 + a_1\,x + a_2\,x^2 + a_3\,x^3 + \cdots\right) \\ &= 2a_0 + (2a_1 + a_0)\;x + (2a_2 + a_1 + 2a_0)\;x^2 + (2a_3 + a_2 + 2a_1 + 3a_0)\;x^3 + \cdots\;.\end{aligned}$$

Writing the left side of this equation as $2 + x + 0x^2 + 0x^3$ and equating like powers of x on the two sides of the equation, we obtain $2 = 2a_0, 1 = 2a_1 + a_0, 0 = 2a_2 + a_1 + 2a_0, 0 = 2a_3 + a_2 + 2a_1 + 3a_0$. We solve successively, $a_0 = 1, a_1 = 0, a_2 = -1, a_3 = 0$. Thus, $(2+x)/f(x) = 1 - x^2 + \cdots$. If you wish to carry the calculation further, you will get $(2+x)/f(x) = 1 - x^2 - x^3 - x^4/2 + x^5/4 + \cdots$.

13. With $u = -x/2$ we have

$$\begin{aligned}\ln(2-x) &= \ln\big(2(1+u)\big) \\ &= \ln(2) + \ln(1+u) \\ &= \ln(2) + u - \frac{1}{2}u^2 + \frac{1}{3}u^3 - \frac{1}{4}u^4 + \cdots \\ &= \ln(2) + \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{24}x^3 - \frac{1}{64}x^4 - \cdots\end{aligned}$$

14. With $u = 4x^2/3$ we have

$$\begin{aligned}x\,\ln(3+4x^2) &= x\,\ln\big(3(1+u)\big) \\ &= x\,\big(\ln(3) + \ln(1+u)\big) \\ &= x\left(\ln(3) + u - \frac{1}{2}u^2 + \frac{1}{3}u^3 - \frac{1}{4}u^4 + \cdots\right) \\ &= x\left(\ln(3) + \frac{4}{3}x^2 - \frac{16}{18}x^4 + \cdots\right) \\ &= \ln(3)\cdot x + \frac{4}{3}x^3 - \frac{8}{9}x^5 + \cdots\end{aligned}$$

Solutions: Taylor Polynomials (Corresponds to Stewart 11.9)

1.

$$T_3(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3$$

2.

$$T_4(x) = 2 - 4(x-1/2) + 8(x-1/2)^2 - 16(x-1/2)^3 + 32(x-1/2)^4$$

3.

$$T_4(x) = -(x-\pi) + \frac{1}{6}(x-\pi)^3$$

4.

$$T_4(x) = 1 + \frac{1}{2}(x+2) + \frac{1}{8}(x+2)^2 + \frac{1}{48}(x+2)^3 + \frac{1}{384}(x+2)^4$$

5.

$$T_3(x) = 1 - \frac{3}{2}(x+3) + \frac{15}{8}(x+3)^2 - \frac{35}{16}(x+3)^3$$

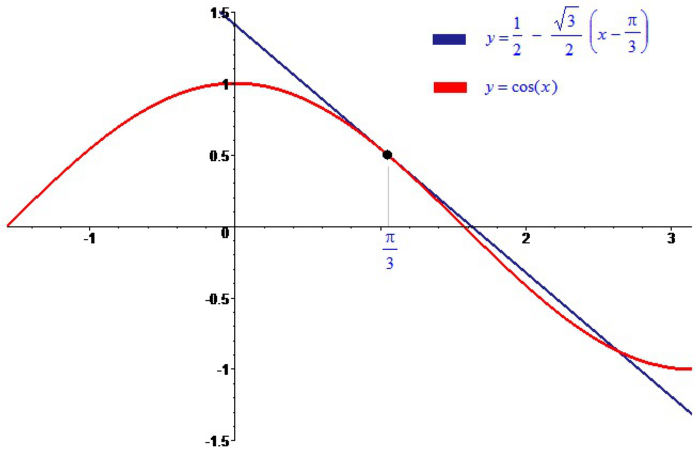
6.

$$T_2(x) = 2 - \frac{7}{4}(x-1) + \frac{111}{64}(x-1)^2$$

7. If a Taylor polynomial of a certain order has been calculated, then one need not begin at the beginning in order to calculate a higher order Taylor polynomial. If T_N is the Taylor polynomial of order N of a function f with base point c, then the order $N+1$ Taylor polynomial of f with base point c is given by

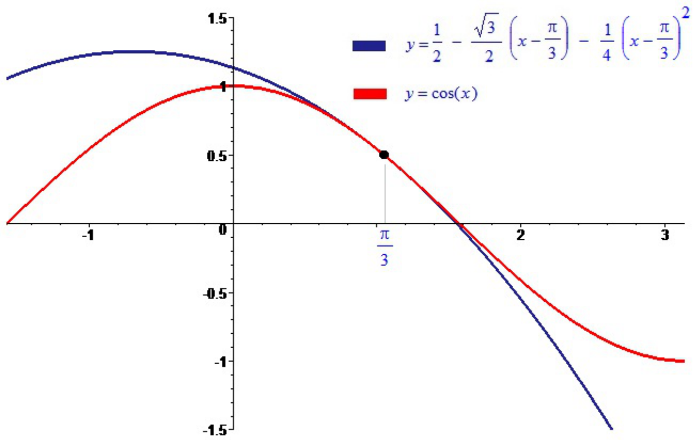
$$T_{N+1}(x) = T_N(x) + \frac{f^{(N+1)}(c)}{(N+1)!}(x-c)^{N+1}.$$

The order 1 Taylor polynomial T_1 of $\cos(x)$ with base point $\pi/3$ is written out in the following figure. The graph of T_1 is the tangent line to the graph of $y = \cos(x)$ at $(\pi/3, \cos(\pi/3))$.



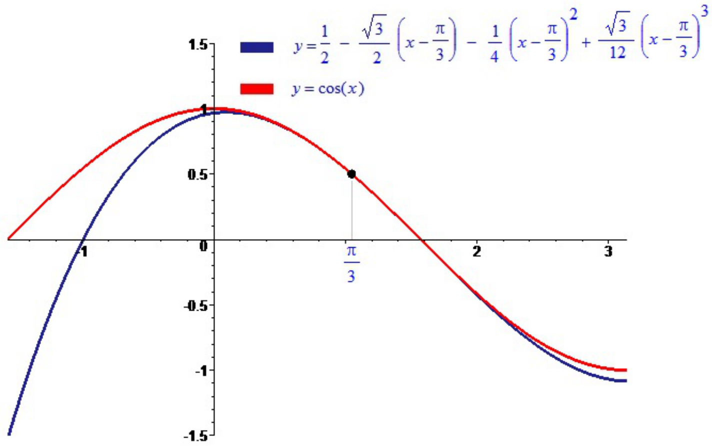
Order 1 Taylor Polynomial of cos(x) with base point $\pi/3$

The order 2 Taylor polynomial T_2 of $\cos(x)$ with base point $\pi/3$ is written out in the following figure. You will note that T_2 is obtained from T_1 by adding one more term. You will also note that the addition of this term results in a quadratic function that approximates the graph of $\cos(x)$ better than the linear graph of T_1.



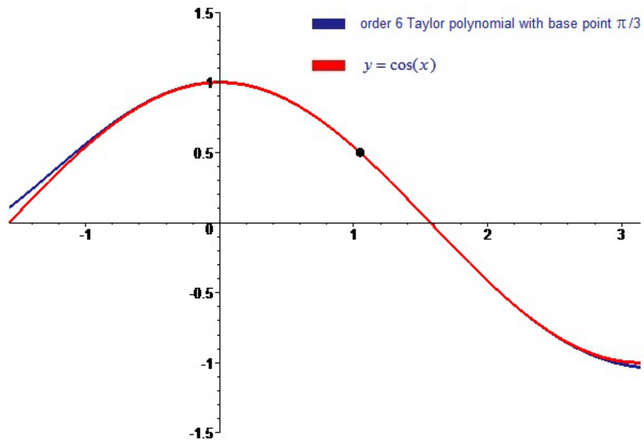
Order 2 Taylor Polynomial of cos(x) with base point $\pi/3$

The order 3 Taylor polynomial T_3 of $\cos(x)$ with base point $\pi/3$ is written out in the following figure. You will note that T_3 is obtained from T_2 by adding one more term. You will also note that the addition of this term results in a cubic polynomial that approximates the graph of $\cos(x)$ better than the graph of the quadratic polynomial T_2.



Order 3 Taylor Polynomial of cos(x) with base point $\pi/3$

The order 6 Taylor polynomial T_6 of $\cos(x)$ with base point $\pi/3$ is graphed in the following figure. This exercise did not request the calculation of T_6. It is graphed here to show that Taylor polynomials can be very accurate approximations of their generating function if the order is sufficiently large.



Order 6 Taylor Polynomial of cos(x) with base point $\pi/3$

Solutions: Taylor and Maclaurin Series (Corresponds to Stewart 11.9)

1.
$$\frac{\sin\left(x^2\right)}{x} = \frac{1}{x}\left(\left(x^2\right) - \frac{1}{3!}\left(x^2\right)^3 + \cdots\right) = x - \frac{1}{6}x^5 + \cdots$$

2.
$$\frac{1-\cos(2x)}{x} = \frac{1}{x}\left(1 - \left(1 - (2x)^2/2 + (2x)^4/24 - (2x)^6/720 + \cdots\right)\right) = 2x - \frac{2}{3}x^3 + \frac{4}{45}x^5 - \cdots$$

3.
$$\begin{aligned}
\frac{x}{\exp\left(x^2\right)} &= x\exp\left(-x^2\right) \\
&= x\left(1 + \left(-x^2\right)\frac{1}{2!}\left(-x^2\right)^2 + \cdots\right) \\
&= x - x^3 + \frac{1}{2}x^5 + \cdots
\end{aligned}$$

4.
$$\begin{aligned}
\exp\left(x^2+2x\right) &= \exp\left(x^2\right)\cdot\exp(2x) \\
&= \left(1 + \left(x^2\right) + \frac{1}{2!}\left(x^2\right)^2 + \frac{1}{3!}\left(x^2\right)^3 + \cdots\right)\left(1 + (2x) + \frac{1}{2!}(2x)^2 + \frac{1}{3!}(2x)^3 + \frac{1}{4!}(2x)^4 + \cdots\right) \\
&= \left(1 + x^2 + \frac{1}{2}x^4 + \cdots\right)\left(1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \cdots\right) \\
&= 1 + 2x + 3x^2 + \frac{10}{3}x^3 + \frac{19}{6}x^4 + \cdots
\end{aligned}$$

5. Use $u = 2x^2$ and $\alpha = -1/2$:
$$1 - x^2 + \frac{3}{2}x^4 - \cdots$$

6. Write $(8+x^3)^{1/3} = 2(1+u)^\alpha$ with $u = x^3/8$ and $\alpha = 1/3$:
$$2 + \frac{1}{12}x^3 + \cdots$$

7.

$$\begin{aligned}
\int_0^{0.2} \sin\left(x^2\right)\,dx &= \int_0^{0.2}\left(x^2 - \frac{1}{3!}x^6 + \frac{1}{5!}x^10 - \cdots\right)dx \\
&= \left(\frac{1}{3}x^3 - \frac{1}{42}x^7 + \frac{1}{1320}x^11 - \cdots\right)\Bigg|_0^{0.2} \\
&= \frac{(0.2)^3}{3} - \frac{(0.2)^7}{42} + \frac{(0.2)^11}{1320} - \cdots
\end{aligned}$$

We calculate $\frac{(0.2)^7}{42} = 0.0000003... < 10^{-5}$. Therefore, the error resulting from dropping the second and all subsequence terms is less than 10^{-5}. In other words, $\int_0^{0.2} \sin\left(x^2\right)\,dx \approx \frac{(0.2)^3}{3} = 0.0027$ with an error less than 10^{-5}. (In fact, the given integral equals 0.002666361920 to 10 correct decimal places.)

8. We calculate,

$$\begin{aligned}
\int_0^{0.2} \frac{1}{1+x^3}\,dx &= \int_0^{0.2}\left(1 - x^3 + x^6 - x^9 + \cdots\right)dx \\
&= \left(x - \frac{x^4}{4} + \frac{x^7}{7} - \frac{x^{10}}{10} + \cdots\right)\Bigg|_0^{1/5} \\
&= (0.2) - \frac{(0.2)^4}{4} + \frac{(0.2)^7}{7} - \frac{(0.2)^{10}}{10} + \cdots \\
&= 0.2 - 0.0004 + 0.1828571429 \times 10^{-5} - \cdots
\end{aligned}$$

Note that the third summand in the last line is less than 10^{-5}. Thus

$$\int_0^{0.2} \frac{1}{1+x^3}\,dx \approx 0.2 - 0.0004 = .1996$$

with an error less than 10^{-5}. In fact, the given integral has value 0.1996018184 to 10 correct decimal places.

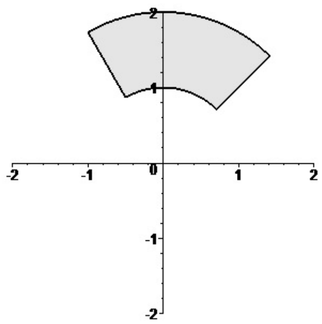
Solutions: Polar Coordinates (Corresponds to Stewart 10.3)

1. $P = (3\sqrt{2}/2, 3\sqrt{2}/2), Q = (0, 4), R = (-5, 5\sqrt{3})$
   Alternative polar coordinates for P: $(3, 9\pi/4)$ and $(-3, 5\pi/4)$
   Alternative polar coordinates for Q: $(4, \pi/2)$ and $(-4, -\pi/2)$
   Alternative polar coordinates for R: $(10, 5\pi/3)$ and $(-10, 2\pi/3)$

2. $P = (3\sqrt{2}, 3\pi/4)$ and $(-3\sqrt{2}, 7\pi/4)$
   $Q = (4, \pi/3)$ and $(-4, 4\pi/3)$
   $R = (2, 5\pi/6)$ and $(-2, 11\pi/6)$

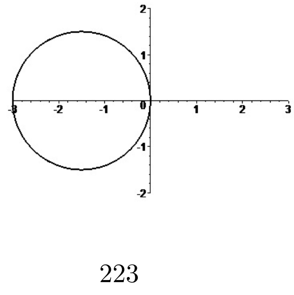
3. The Cartesian coordinates of the points are $(\sqrt{3}, -1)$ and $(-3, -3\sqrt{3})$. The distance between them is $2\sqrt{10}$.

4.

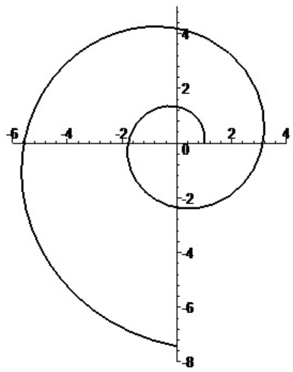


5. (i) $x = 5$
   (ii) $y = 3/x$

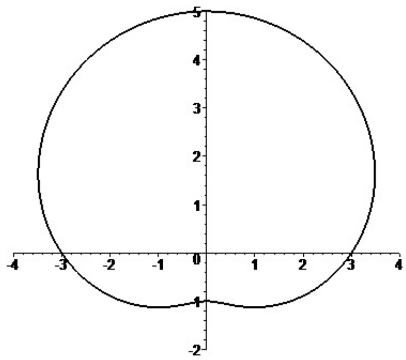
6.



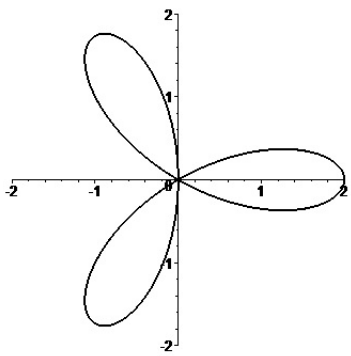
7.



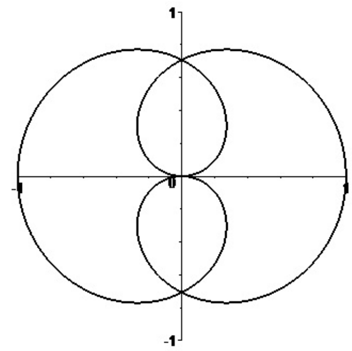
8.



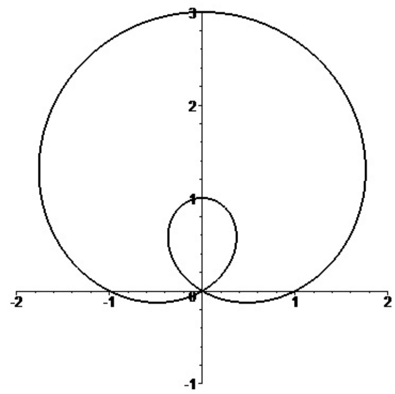
9.



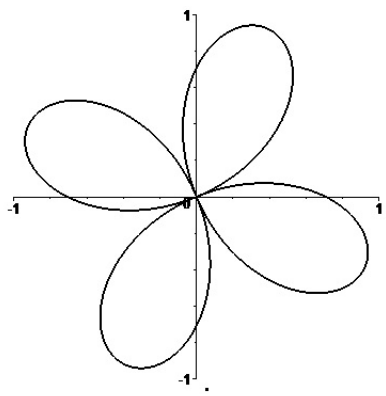
10.



11.



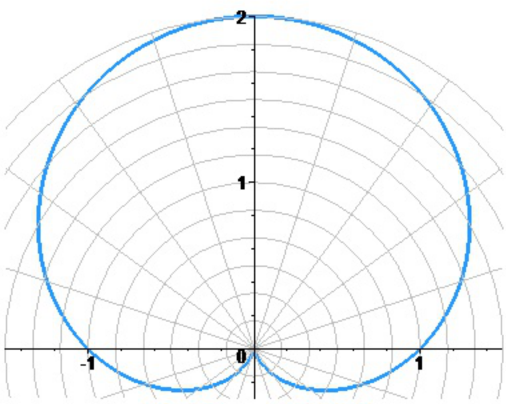
12.



Solutions: Areas in Polar Coordinates (Corresponds to Stewart 10.4)

In the calculations that follow, we will use certain integration facts without elaboration: The integrals of $\sin(\theta)$ and $\cos(\theta)$ are 0 over the θ-interval $[0, 2\pi]$ and the integrals of $\sin(\theta)^2$ and $\cos(\theta)^2$ are $\pi/4$ over each of the θ-intervals $[0, \pi/2], [\pi/2, \pi], [\pi, 3\pi/2]$ and $[3\pi/2, 2\pi]$.

1. Figure for Exercise 1:

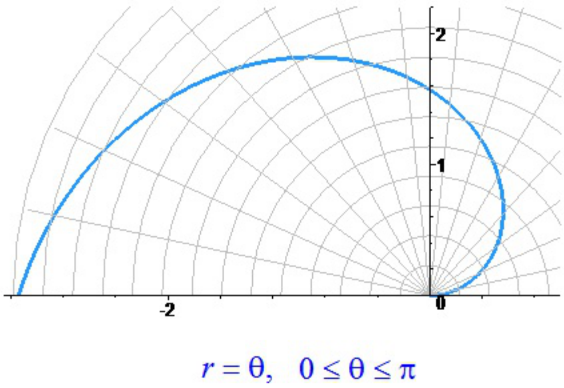


$r = 1 + \sin(\theta)$

The area A is given by

$$
\begin{aligned}
A &= \frac{1}{2}\int_0^{2\pi} (1+\sin(\theta))^2 \, d\theta \\
&= \frac{1}{2}\int_0^{2\pi} \left(1 + 2\sin(\theta) + \sin(\theta)^2\right) \, d\theta \\
&= \frac{1}{2}\left(2\pi + 0 + 4\cdot\frac{\pi}{4}\right) \\
&= \frac{3\pi}{2}.
\end{aligned}
$$

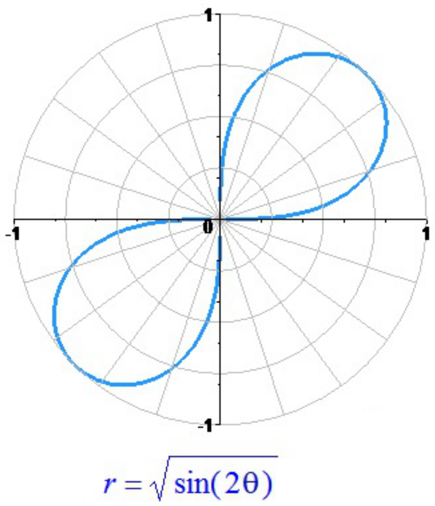
2. Figure for Exercise 2:



The area A is given by

$$A = \frac{1}{2}\int_0^{\pi} \theta^2\, d\theta = \frac{\pi^3}{6}.$$

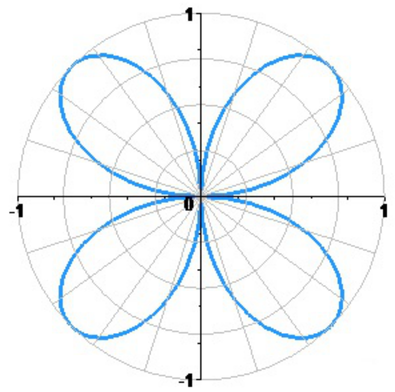
3. Figure for Exercise 3:



Using symmetry, we see that the requested area A is

$$A = 2\left(\frac{1}{2}\int_0^{\pi/2}\left(\sqrt{\sin(2\,\theta)}\right)^2\, d\theta\right) = \int_0^{\pi/2} \sin(2\,\theta)\, d\theta = \left(-\frac{1}{2}\cos(2\,\theta)\right)\Big|_0^{\pi/2} = \frac{1}{2}.$$

4. Figure for Exercise 4:

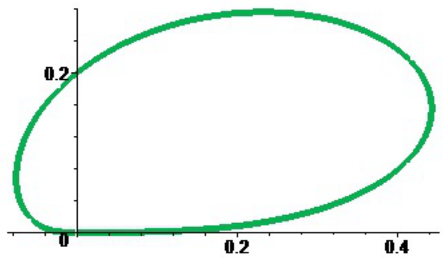


$r = \sin(2\theta)$

Using symmetry, and making the substitution $u = 2\theta, du = 2\,d\theta$ with limits of integration $u = 0$ corresponding to $\theta = 0$ and $u = \pi$ corresponding to $\theta = \pi/2$, we see that the requested area A is

$$A = 4\left(\frac{1}{2}\int_0^{\pi/2} \sin(2\,\theta)^2\,d\theta\right) = 4\left(\frac{1}{4}\int_0^{\pi} \sin(u)^2\,du\right) = 4\cdot\frac{1}{4}\cdot 2\cdot\frac{\pi}{4} = \frac{\pi}{2}.$$

5. Figure for Exercise 5:

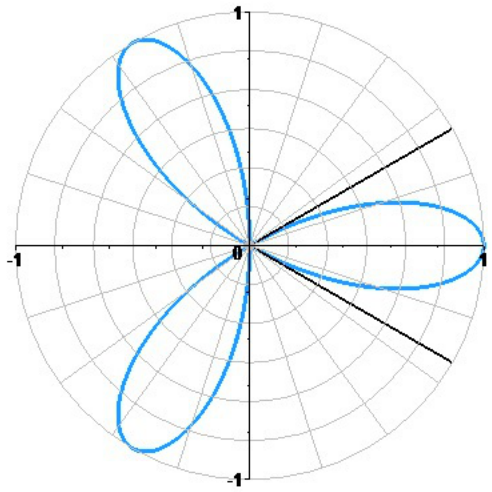


$$r = \frac{\sqrt{\sin(\theta)}}{5 - 4\cos(\theta)}$$

The requested area A is given by

$$A = \frac{1}{2}\int_0^{\pi}\left(\frac{\sqrt{\sin(\theta)}}{5-4\,\cos(\theta)}\right)^2 d\theta = \frac{1}{2}\int_0^{\pi}\frac{\sin(\theta)}{(5-4\,\cos(\theta))^2}\,d\theta = -\frac{1}{8}\,(5-4\,\cos(\theta))^{-1}\Big|_0^{\pi} = \frac{1}{9}.$$

6. Figure for Exercise 6:

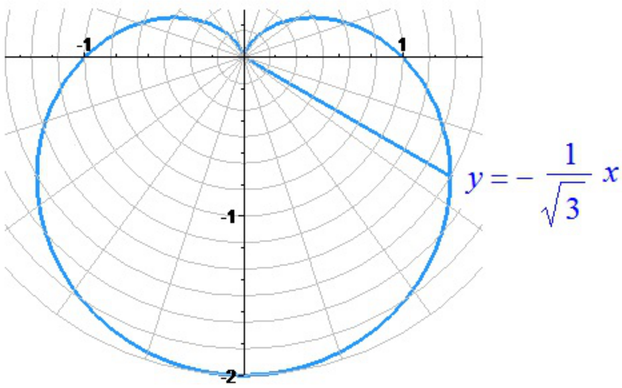


$r = \cos(3\,\theta)$

A polar curve passes through the origin when $r = 0$. The solutions of $r = \cos(3\,\theta) = 0$ in the first and fourth quadrants are $\theta = \pi/6$ and $\theta = -\pi/6$. The polar graphs of these equations are the black rays in the figure. (They are tangent to the rose.) Making the change of variable $u = 3\,\theta$, $du = 3\,d\theta$, we see that the requested area is

$$A = \frac{1}{2}\int_{-\pi/6}^{\pi/6} \cos(3\,\theta)^2\,d\theta = \frac{1}{6}\int_{-\pi/2}^{\pi/2} \cos(u)^2\,du = \frac{1}{6}\cdot 2\cdot \frac{\pi}{4} = \frac{\pi}{12}.$$

7. Figure for Exercise 7:

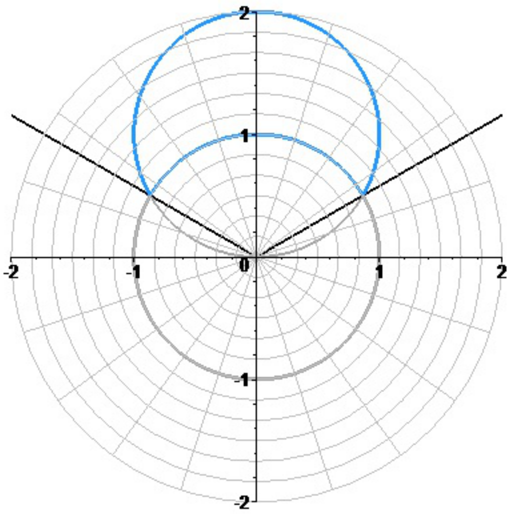


$r = 1 - \sin(\theta)$

The slope of the given line segment is $-1/\sqrt{3}$, or $\tan(-\pi/6)$. The graph of the given line segment therefore has polar equation $\theta = -\pi/6$, $0 \le r \le 1 - \sin(-\pi/6)$. The requested area A is given by

$$
\begin{aligned}
A &= \frac{1}{2}\int_{-\pi/6}^{\pi/2} (1-\sin(\theta))^2 \, d\theta \\
&= \frac{1}{2}\int_{-\pi/6}^{\pi/2} \left(1 - 2\sin(\theta) + \sin^2(\theta)\right) d\theta \\
&= \frac{1}{2}\int_{-\pi/6}^{\pi/2} \left(1 - 2\sin(\theta) + \frac{1}{2}(1-\cos(2\theta))\right) d\theta \\
&= \int_{-\pi/6}^{\pi/2} \left(\frac{3}{4} - \sin(\theta) - \frac{1}{4}\cos(2\theta)\right) d\theta \\
&= \left(\frac{3}{4}\theta + \cos(\theta) - \frac{1}{8}\sin(2\theta)\right)\Bigg|_{-\pi/6}^{\pi/2} \\
&= \left(\frac{3\pi}{8} + 0 + 0\right) - \left(-\frac{\pi}{8} + \frac{\sqrt{3}}{2} - \frac{1}{8}\frac{\sqrt{3}}{2}\right) \\
&= \frac{\pi}{2}
\end{aligned}
$$

8. Figure for Exercise 8:

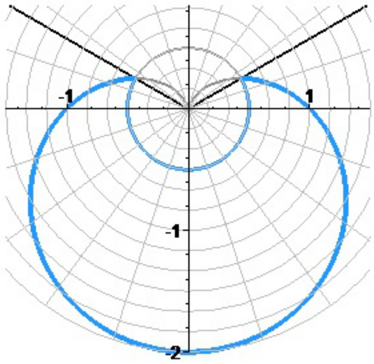


The circles $r = 2\sin(\theta)$ and $r = 1$

The points of intersection occur where $2\sin(\theta) = r = 1$, or $\sin(\theta) = 1/2$, or $\theta = \pi/6$ and $\theta = 5\pi/6$: The rays corresponding to these two polar equations have been drawn in black. The requested area is

$$A = \frac{1}{2}\int_{\pi/6}^{5\pi/6} \left((2\sin(\theta))^2 - 1^2\right) d\theta = \sqrt{3} - \frac{\pi}{3}.$$

9. Figure for Exercise 9:

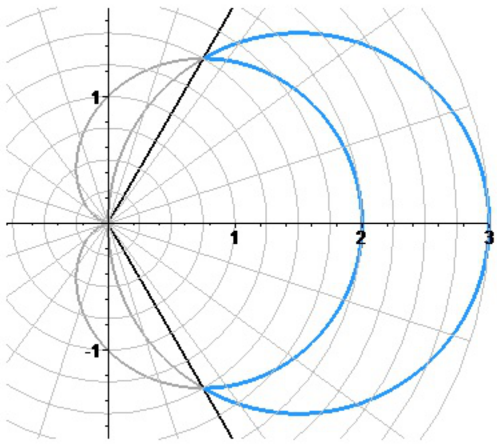


The cardioid $r = 1 - \sin(\theta)$ and the circle $r = \frac{1}{2}$

The points of intersection occur where $1 - \sin(\theta) = r = 1/2$, or $\sin(\theta) = 1/2$, or $\theta = \pi/6$ and $\theta = 5\,\pi/6$: In setting up the integral, it will be convenient to use the angle $\theta = \pi/6 + 2\,\pi$ instead of $\theta = \pi/6$. The rays corresponding to these two polar equations have been drawn in black. The requested area is

$$A = \frac{1}{2}\int_{5\,\pi/6}^{\pi/6+2\,\pi} \left(\left(1 - \sin(\theta)\right)^2 - \left(\frac{1}{2}\right)^2\right)\,d\theta = \frac{7\,\sqrt{3}}{8} + \frac{5\,\pi}{6}.$$

10. Figure for Exercise 10:

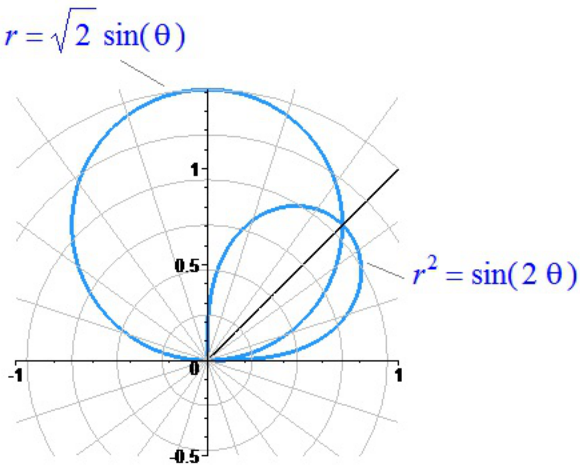


The cardioid $r = 1 + \cos(t)$ and the circle $r = 3\,\cos(t)$

Equate the two formulas for r to find the points of intersection: $3\,\cos(\theta) = 1 + \cos(\theta)$, or $\cos(\theta) = 1/2$, or $\theta = \pm\pi/3$. The requested area is

$$A = \frac{1}{2}\int_{-\pi/3}^{\pi/3} \left(\left(3\,\cos(\theta)\right)^2 - \left(1 + \cos(\theta)\right)^2\right)\,d\theta = \pi.$$

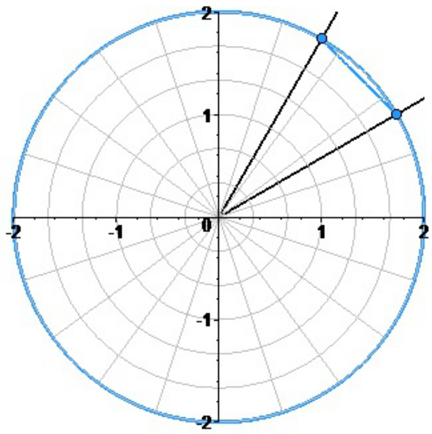
11. Figure for Exercise 11:



Setting the two formulas for r^2 equal to each other, we have $\left(\sqrt{2}\,\sin(\theta)\right)^2 = \sin(2\,\theta)$, or $2\,\sin(\theta)^2 = 2\,\sin(\theta)\cos(\theta)$. For the nonzero solution divide each side by $\sin(\theta)$ to obtain $\sin(\theta) = \cos(\theta)$. This yields $\theta = \pi/4$, which is the equation for the line drawn in black in the figure. The area is

$$\frac{1}{2}\int_0^{\pi/4}\left(\sqrt{2}\,\sin(\theta)\right)^2 d\theta + \frac{1}{2}\int_{\pi/4}^{\pi/2}\left(\sqrt{\sin(2\,\theta)}\right)^2 d\theta = \frac{\pi}{8}.$$

12. Figure for Exercise 12:



a) The slope is $(\sqrt{3}-1)/(1-\sqrt{3})$, or -1. The Cartesian equation of the line is therefore $y = -(x-1)+\sqrt{3}$, or $y = -x + 1 + \sqrt{3}$.
b) From part (a), we have $x + y = 1 + \sqrt{3}$. Replacing x with $r\,\cos(\theta)$ and y with $r\,\sin(\theta)$ results in $r\,\cos(\theta) + r\,\sin(\theta) = 1 + \sqrt{3}$, or $r = (1 + \sqrt{3})/(\cos(\theta) + \sin(\theta))$.
c) Because $\cos(\pi/4) = \sin(\pi/4) = 1/\sqrt{2}$, and in view of the addition formula $\sin(A+B) = \sin(A)\cos(B)+$

$\cos(A)\sin(B)$, we have

$$\begin{aligned} r &= \frac{1+\sqrt{3}}{\sin(\theta)+\cos(\theta)} \\ &= \frac{1+\sqrt{3}}{\sqrt{2}\,(\sin(\theta)\cos(\pi/4)+\cos(\theta)\sin(\pi/4))} \\ &= \frac{1+\sqrt{3}}{\sqrt{2}\,\sin(\theta+\pi/4)} \\ &= \frac{1+\sqrt{3}}{\sqrt{2}}\csc\left(\theta+\frac{\pi}{4}\right). \end{aligned}$$

d) The requested area A is given by

$$\begin{aligned} A &= \frac{1}{2}\int_{\pi/6}^{\pi/3}\left(2^2-\left(\frac{1+\sqrt{3}}{\sqrt{2}}\csc\left(\theta+\frac{\pi}{4}\right)\right)^2\right)d\theta \\ &= \left(2\theta+\frac{(1+\sqrt{3})^2}{4}\cot\left(\theta+\frac{\pi}{4}\right)\right)\Bigg|_{\pi/6}^{\pi/3} \\ &= \frac{\pi}{3}-(2+\sqrt{3})\cot\left(\frac{5\pi}{12}\right). \end{aligned}$$