Formulas

$\pi = 3.141592654, \quad \ln(1) = 0, \quad \ln(e) = 1, \quad \ln(x\,y) = \ln(x) + \ln(y), \quad \ln(x^p) = p\,\ln(x)$

$\sin\left(\dfrac{\pi}{6}\right) = \cos\left(\dfrac{\pi}{3}\right) = \dfrac{1}{2}, \quad \sin\left(\dfrac{\pi}{3}\right) = \cos\left(\dfrac{\pi}{6}\right) = \dfrac{\sqrt{3}}{2}, \quad \sin\left(\dfrac{\pi}{4}\right) = \cos\left(\dfrac{\pi}{4}\right) = \dfrac{\sqrt{2}}{2},$

$\sin(0) = \sin(\pi) = \cos\left(\dfrac{\pi}{2}\right) = 0, \quad \sin\left(\dfrac{\pi}{2}\right) = \cos(0) = \tan\left(\dfrac{\pi}{4}\right) = 1, \quad \sin\left(\dfrac{3\pi}{2}\right) = \cos(\pi) = -1$

$\sin(2\,x) = 2\,\sin(x)\,\cos(x), \quad 2\,\cos(x)^2 = 1 + \cos(2\,x), \quad 2\,\sin(x)^2 = 1 - \cos(2\,x),$

$\displaystyle\int \dfrac{1}{x}\,dx = \ln(|x|) + C, \quad \int \ln(x)\,dx = x\,\ln(x) - x + C, \quad \int a^x\,dx = \dfrac{a^x}{\ln(a)} + C,$

$\displaystyle\int u\,dv = u\,v - \int v\,du$

$\displaystyle\int \dfrac{1}{a^2 + x^2}\,dx = \dfrac{1}{a}\,\arctan\left(\dfrac{x}{a}\right) + C, \qquad \int \dfrac{1}{\sqrt{a^2 - x^2}}\,dx = \arcsin\left(\dfrac{x}{a}\right) + C,$

$\displaystyle\int \dfrac{1}{|x|\,\sqrt{x^2 - a^2}}\,dx = \dfrac{1}{a}\,\operatorname{arcsec}\left(\dfrac{x}{a}\right) + C, \quad 1 - \sin(x)^2 = \cos(x)^2, \quad 1 + \tan(x)^2 = \sec(x)^2$

$\displaystyle\int \sec(x)^2\,dx = \tan(x) + C, \qquad \int \csc(x)^2\,dx = -\cot(x) + C,$

$\displaystyle\int 2\,\sec(x)^3\,dx = \sec(x)\,\tan(x) + \ln(|\sec(x) + \tan(x)|) + C$

$\displaystyle\int \sec(x)\,\tan(x)\,dx = \sec(x) + C, \qquad \int \csc(x)\,\cot(x)\,dx = -\csc(x) + C$

$\displaystyle\int \tan(x)\,dx = \ln(|\sec(x)|) + C, \qquad \int \sec(x)\,dx = \ln(|\sec(x) + \tan(x)|) + C$

$\displaystyle\int \cos(x)^n\,dx = \dfrac{1}{n}\,\sin(x)\,\cos(x)^{(n-1)} + \dfrac{n-1}{n}\,\int \cos(x)^{(n-2)}\,dx$

Integrating factor for $\dfrac{dy}{dx} + p(x)\,y = q(x):\quad e^{P(x)}$ **where** $P(x) = \displaystyle\int p(x)\,dx$

1. Let R be the region that is bounded above by $y = 2\sqrt{x}$ and below by $y = x$.

The area of R is $\dfrac{8}{3}$ and the mass density of R is the constant δ at every point.

What is the x-coordinate of the center of mass of R ?

a) $\dfrac{7}{5}$ b) $\dfrac{22}{15}$ c) $\dfrac{23}{15}$ d) $\dfrac{8}{5}$ e) $\dfrac{5}{3}$

f) $\dfrac{26}{15}$ g) $\dfrac{9}{5}$ h) $\dfrac{28}{15}$ i) $\dfrac{29}{15}$ j) 2

Solution: d

Mass

```
> A := Int( 2*sqrt(x)-x, x=0..4);
  #
  # The area of R, unevaluated
```

$$A := \int_0^4 2\sqrt{x} - x \, dx$$

```
> A := int( 2*sqrt(x)-x, x=0..4);
  #
  # The area of R, evaluated
```

$$A := \frac{8}{3}$$

```
> M := delta*A;
  #
  # The mass of R. (The constant delta will soon cancel and
  disappear.)
```

$$M := \frac{8\,\delta}{3}$$

Moment about y-axis (denoted here by My)

```
> My := Int( x*(2*sqrt(x)-x)*delta, x = 0 .. 4);
  #
  # The moment about the y-axis, unevaluated
```

$$My := \int_0^4 x\,(2\sqrt{x} - x)\,\delta \, dx$$

```
> My := int( x*(2*sqrt(x)-x)*delta, x = 0 .. 4);
  #
  # The moment about the y-axis, evaluated
```

$$My := \frac{64\,\delta}{15}$$

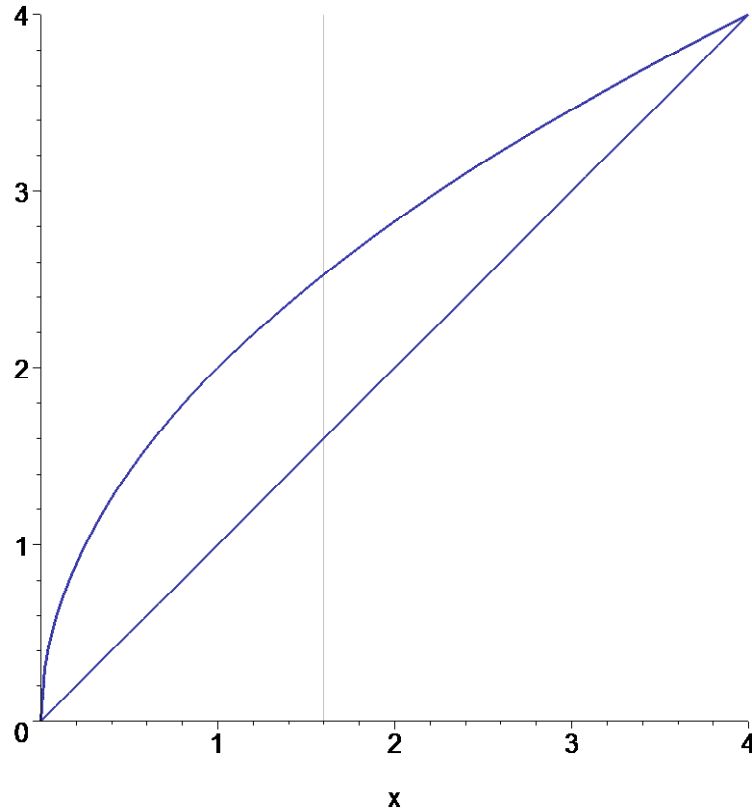**x-coordinate of center of mass (denoted here by x_bar)**

```
> x_bar := My/M;
```

```
#
# The x-coordinate of the center of mass is the moment about
the y-axis divided by the area.
```

$$x\_bar := \frac{8}{5}$$

The center of mass lies on the vertical line drawn:

2. As in the preceding problem, let R be the region that is bounded above by $y = 2\sqrt{x}$ and below by $y = x$. The area of R is $\frac{8}{3}$ and the mass density of R is the constant δ at every point. What is the y-coordinate of the center of mass of R ?

a) $\frac{8}{5}$ b) $\frac{5}{3}$ c) $\frac{26}{15}$ d) $\frac{9}{5}$ e) $\frac{28}{15}$

f) $\frac{29}{15}$ g) 2 h) $\frac{31}{15}$ i) $\frac{32}{15}$ j) $\frac{11}{5}$

Solution: g

Moment about the x-axis (denoted here by Mx)

```
> Mx := Int( ((2*sqrt(x))^2 - (x)^2)*delta, x = 0..4);
  #
  # The moment about the x-axis, unevaluated
```

$$Mx := \int_{0}^{4} (4\,x - x^2)\,\delta\,dx$$

```
> Mx := int( ((2*sqrt(x))^2 - (x)^2)*delta, x = 0..4);
  #
  # The moment about the x-axis, evaluated
```
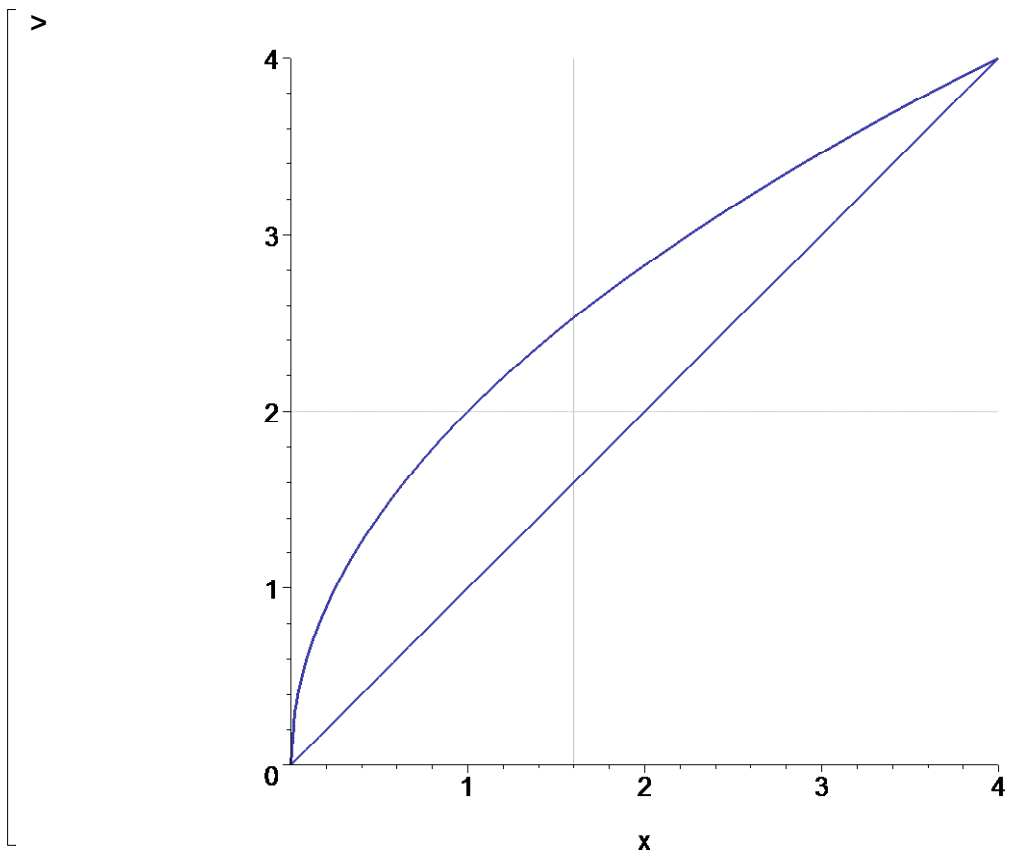
$$Mx := \frac{32\,\delta}{3}$$

y-coordinate of center of mass (denoted here by y_bar)

```
> y_bar := Mx/(2*M);
```

$$y\_bar := 2$$

The y-center of mass lies on the horizontal line drawn. The center of mass is in the cross-hairs.

```
>
```

3. A function y(x) that has positive values for $x > 0$ satisfies the initial value problem

$$2x^2\frac{dy}{dx} = \frac{x^2 - 2}{y + 1}, \quad y(1) = 3.$$

What is y(2)?

a) 3   b) 4   c) 5   d) 6   e) 7
f) 8   g) 9   h) 10   i) 11   j) 12

Solution: a

After separating variables, the integrated equation is

```
> eqn1 := Int( 2*(y + 1), y) = Int( (x^2-2)/x^2, x);
```

$$eqn1 := \int 2\,y + 2\,dy = \int \frac{x^2 - 2}{x^2}\,dx$$

```
> eqn2 := lhs(eqn1) = expand(rhs(eqn1));
```

$$eqn2 := \int 2\,y + 2\,dy = \int 1 - \frac{2}{x^2}\,dx$$

```
> eqn3 := value(lhs(eqn2)) = value(rhs(eqn2)) + C;
```

$$eqn3 := y^2 + 2\,y = x + \frac{2}{x} + C$$

```
> eqn4 := lhs(eqn3) + 1 = rhs(eqn3);
```

$$eqn4 := y^2 + 2\,y + 1 = x + \frac{2}{x} + C$$

```
> eqn5 := factor(lhs(eqn4)) = rhs(eqn4);
```

$$eqn5 := (y + 1)^2 = x + \frac{2}{x} + C$$

```
> initialCondition := subs({x=1,y=3}, eqn5);
```

$$initialCondition := 16 = 3 + C$$

```
> eqn_for_C := C = solve(initialCondition,C);
```

$$eqn\_for\_C := C = 13$$

```
> eqn6 := subs(eqn_for_C,eqn5);
```

$$eqn6 := (y + 1)^2 = x + \frac{2}{x} + 13$$

```
> eqn7 := map(simplify, subs(x=2,eqn6));
```

$$eqn7 := (y + 1)^2 = 16$$

```
> solve(eqn7,y);
```

$$3, -5$$

The function y(x) we seek takes positive values, we have been told, and, in any event, -5 is not among the answer choices.

We now use Maple's differential equation solver for confirmation of our work.

```
> eqn := dsolve( {2*x^2*diff(y(x),x) = (x^2-2)/(y(x)+1), y(1) = 3}, y(x));
```
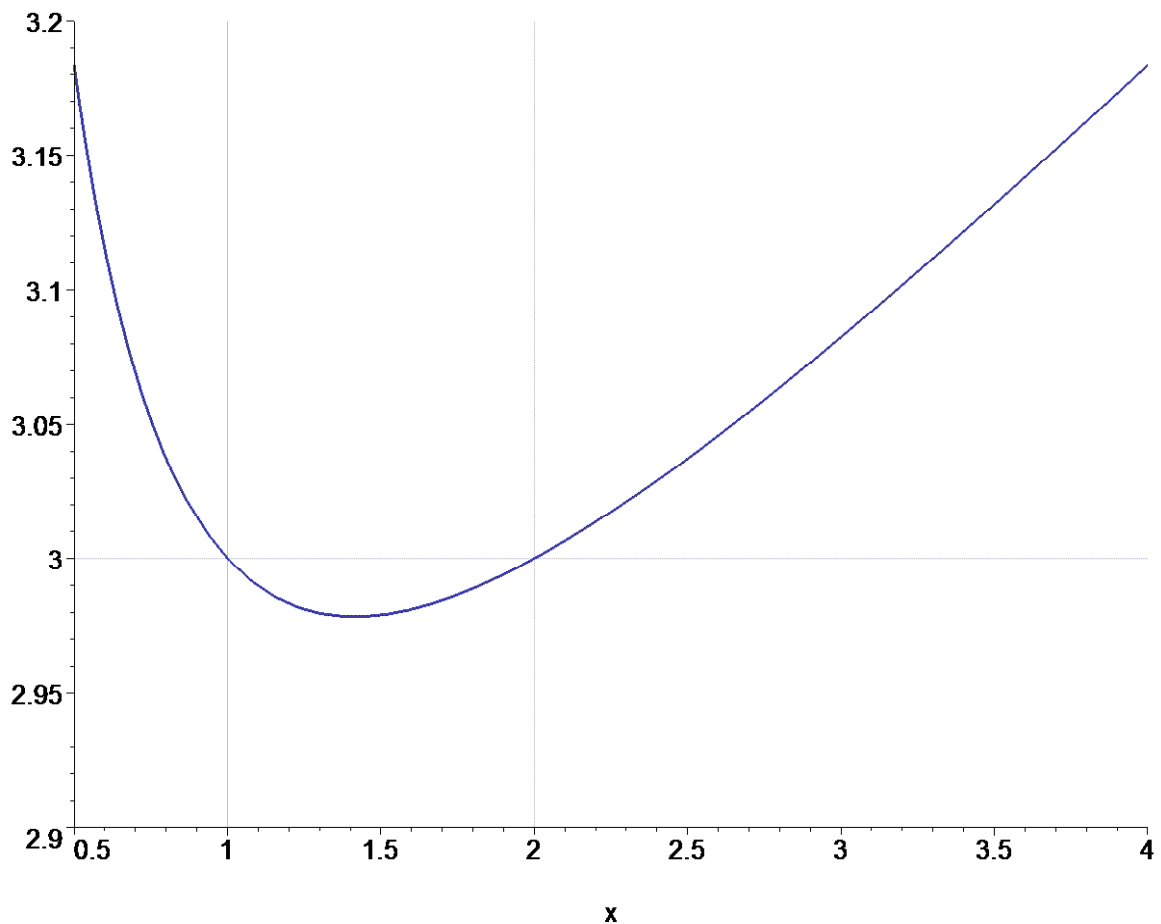
$$eqn := y(x) = \frac{-2\,x + 2\,\sqrt{13\,x^2 + x^3 + 2\,x}}{2\,x}$$

```
> y := unapply( rhs(eqn) , x);
```

$$y := x \rightarrow \frac{1}{2}\,\frac{-2\,x + 2\,\sqrt{13\,x^2 + x^3 + 2\,x}}{x}$$

```
> simplify( y(2) );
```

$$3$$

```
> solutionCurve := plot(y(x), x = 1/2 .. 4 , color = NAVY,
  thickness = 3, view = [1/2..4 , 2.9..3.2]):
  vert1 := plot([1,y,y=2.9..3.2], color = GREY):
  vert2 := plot([2,y,y=2.9..3.2], color = GREY):
  horiz := plot(3, x = 1/2 .. 4, color = GREY):
  plots[display](solutionCurve, vert1, vert2, horiz);
```

4. A mixing tank initially held 120 liters of water in which q kilograms of a solute were dissolved. In order to reduce the strength of the solution, valves were opened so that there was an inflow of water at the rate of 15 liters per minute and an outflow of the thoroughly mixed solution at the rate of 15 liters per minute. The valves were closed as soon as the mass of the solute in the tank had been reduced to $q\,e^{(-3)}$ kilograms. For how many minutes were the valves open?

a) 12   b) 15   c) 16   d) 18   e) 20
f) 24   g) 27   h) 28   i) 30   j) 32

Solution: f

Let y(t) be the mass of the solute in the tank at time t, (in minutes).

```
> ode := diff(y(t),t) = 0 - (y(t)/120)*15;
```

$$ode := \frac{d}{dt}\,y(t) = -\frac{1}{8}\,y(t)$$

```
> linear_ode := map( z -> z + y(t)/8, ode);
```

$$linear\_ode := \left(\frac{d}{dt}\, \mathrm{y}(t)\right) + \frac{1}{8}\, \mathrm{y}(t) = 0$$

```
> linear_ode_times_integrating_factor := map( z ->
  expand(exp(int(1/8, t))*z), linear_ode);
```

$$linear\_ode\_times\_integrating\_factor := \mathbf{e}^{\left(\frac{t}{8}\right)}\left(\frac{d}{dt}\, \mathrm{y}(t)\right) + \frac{1}{8}\, \mathbf{e}^{\left(\frac{t}{8}\right)}\, \mathrm{y}(t) = 0$$

```
> soln_eqn1 := Diff( exp(t/8)*y(t), t) = 0;
```

$$soln\_eqn1 := \frac{d}{dt}\left(\mathbf{e}^{\left(\frac{t}{8}\right)}\, \mathrm{y}(t)\right) = 0$$

```
> soln_eqn2 := exp(t/8)*y(t) = C;
```

$$soln\_eqn2 := \mathbf{e}^{\left(\frac{t}{8}\right)}\, \mathrm{y}(t) = C$$

```
> eqn_for_C := C = subs(y(0)=q,simplify(subs(t=0,
  lhs(soln_eqn2))));
```

$$eqn\_for\_C := C = q$$

```
> soln_eqn3 := subs(eqn_for_C, soln_eqn2);
```

$$soln\_eqn3 := \mathbf{e}^{\left(\frac{t}{8}\right)}\, \mathrm{y}(t) = q$$

```
> soln_eqn4 := y(t) = solve(soln_eqn3,y(t));
```

$$soln\_eqn4 := \mathrm{y}(t) = \frac{q}{\mathbf{e}^{\left(\frac{t}{8}\right)}}$$

```
> answer := solve(subs(y(t) = q*exp(-3), soln_eqn4), t);
```

$$answer := 24$$

```
> answer := 24;
```

Confirmation using Maple's differential equation solver.

```
> eqn := dsolve( {diff(y(t),t) = 0 - 15*y(t)/120, y(0) =
  q},y(t));
```

$$eqn := \mathrm{y}(t) = q\, \mathbf{e}^{\left(-\frac{t}{8}\right)}$$

```
> eqn := y(t) = q*exp(-1/8*t);
> solve(q*exp(-3) = rhs(eqn), t);
```

$$24$$

5. A mixing tank initially held 120 liters of water in which 15 kilograms of salt were dissolved. Valves were opened so that there was an inflow

of a salt water solution at the rate of **20 liters per minute** and an outflow
of the thoroughly mixed solution at the same rate of **20 liters per minute**.
The concentration of the solution entering the mixing tank was **1 kilogram
of salt per 5 liters of water**. For how many minutes had the valves been
open at the moment the mass of salt in the mixing tank was **21 kilograms**?

a) $2\ln(2)$ b) $3\ln(2)$ c) $4\ln(2)$ d) $5\ln(2)$ e) $6\ln(2)$

f) $2\ln(3)$ g) $3\ln(3)$ h) $4\ln(3)$ i) $5\ln(3)$ j) $6\ln(3)$

Solution: j

```
> eqn1 := dsolve( {diff(y(t),t) = (1/5)*20 - (y(t)/120)*20,
  y(0) = 15}, y(t));
```
$$eqn1 := y(t) = 24 - 9\,\mathbf{e}^{\left(-\frac{t}{6}\right)}$$
```
> eqn2 := subs( y(t) = 21 , eqn1 );
```
$$eqn2 := 21 = 24 - 9\,\mathbf{e}^{\left(-\frac{t}{6}\right)}$$
```
> t = solve( eqn2 , t);
```
$$t = 6\ln(3)$$

6. A large (250 liter capacity), partially-filled mixing tank initially held 120 liters
 of water in which 16 kilograms of a solute were dissolved. Two valves were
simultaneously opened. The opening of one valve resulted in the flow of a solution
into the mixing tank at the rate of **2 liters per minute**. The solution entering the mixing
tank contained the same solute as was in the tank. The concentration of the solution
entering the mixing tank was **1 kilogram per 2 liters**. The opening of the other valve
allowed the thoroughly mixed solution to flow out of the tank at the rate of **1 liter per
minute**. What was the mass in kilograms of the solute in the mixing tank **2 hours** after
the valves were opened?

a) 82   b) 84   c) 86   d) 88   e) 90

f) 92   g) 94   h) 96   i) 98   j) 100

Solution: i

```
> ode := diff(y(t),t) = (1/2)*2 - (y(t)/(120+t))*1;
```
$$ode := \frac{d}{dt}\,y(t) = 1 - \frac{y(t)}{120 + t}$$
```
> ic := y(0) = 16:
```

```
> IVP := {ode, ic};
```

$$IVP := \{\, y(0) = 16, \frac{d}{dt} y(t) = 1 - \frac{y(t)}{120 + t} \,\}$$

```
> dsolve(IVP,y(t));
```

$$y(t) = \frac{120\,t + \dfrac{t^2}{2} + 1920}{120 + t}$$

```
> subs(t = 120, rhs(%));
```

$$98$$

7. A population $P(t)$ grew according to the differential equation

$$\frac{dP}{dt} = k\,P$$

for some positive constant k. The initial population was 800. Sixteen years later, the population was 1600. What was the instantaneous rate of change of the population (in hundreds of population units per year) thirty-two years after the

initial count?

a) $\dfrac{1}{2}\ln(2)$ **b)** $\ln(2)$ **c)** $2\ln(2)$ **d)** $4\ln(2)$ **e)** $8\ln(2)$

f) $16\ln(2)$ **g)** $32\ln(2)$ **h)** $64\ln(2)$ **i)** $128\ln(2)$ **j)** $256\ln(2)$

Solution: c

```
> T := 16;
  #
  # T is the doubling time
```

$$T := 16$$

```
> eqn1 := k = solve(k*T = ln(2), k);
  #
  # Use the doubling time to solve for the growth constant
```

$$eqn1 := k = \frac{1}{16}\ln(2)$$

```
> eqn2 := P(t) = 800*exp(k*t);
  #
  # The equation for the poulation size (Alternatively, P(t) =
  800*2^(t/16) )
```

$$eqn2 := P(t) = 800\,\mathbf{e}^{(k\,t)}$$

```
> eqn3 := D(P)(t) = k*P(t);
  #
  # equation for the instantaneous rate of change (measured in
  population units per year)
```

$$eqn3 := \mathrm{D}(P)(t) = k\,\mathrm{P}(t)$$

```
> eqn4 := subs(eqn2,eqn3);
  #
  # Substitutes the formula for P(t) in the equation for
  D(P)(t)
```

$$eqn4 := \mathrm{D}(P)(t) = 800\,k\,\mathbf{e}^{(k\,t)}$$

```
> eqn5 := subs(eqn1,eqn4);
  #
  # Substitutes the value of k in the equation for D(P)(t);
```

$$eqn5 := \mathrm{D}(P)(t) = 50\,\ln(2)\,\mathbf{e}^{(1/16\,\ln(2)\,t)}$$

```
> eqn6 := subs(t=32,eqn5);
  #
  # Substitutes t = 32 in the equation for D(P)(t);
```

$$eqn6 := \mathrm{D}(P)(32) = 50\,\ln(2)\,\mathbf{e}^{(2\,\ln(2))}$$

```
> eqn7 := lhs(eqn6) = simplify(rhs(eqn6));
```

$$eqn7 := \mathrm{D}(P)(32) = 200\,\ln(2)$$

```
> eqn7 := lhs(eqn7) = simplify(rhs(eqn7)/100);
  #
  # Change from population units per year to 100 population
  units per year
```

$$eqn7 := \mathrm{D}(P)(32) = 2\,\ln(2)$$

8. A population $P(t)$ with initial population 300 grew according to the differential equation

$$\frac{dP}{dt} = 7\,\mathrm{P}(t)\,(2100 - \mathrm{P}(t))\,.$$

What is

$$\lim_{t \to \infty}\; \mathrm{P}(t)\; + \;\lim_{t \to \infty}\; \frac{dP}{dt}\; ?$$

a)   600   b) 700   c)   900   d)   1000   e)   1400
f)   1600   g) 2100   h)   2400   i)   2700   j)   2800

Solution: g

The differential equation models logistic growth. (In the presentation found in

Stewart's text, the 2100 would be factored from the parentheses and combined with the 7 to form the equivalent equation $\dfrac{dP}{dt} = 14700\,P\left(1 - \dfrac{P}{2100}\right)$.

The solution has the properties that

$$\lim_{t \to \infty}\ P(t) = 2100 \quad \text{and} \quad \lim_{t \to \infty}\ D(P)(t) = 0.$$

9. If $y(x)$ is the solution to the initial value problem

$$\frac{dy}{dx} + \frac{3\,y}{x} = \frac{38}{x^2} \quad , \quad y(2) = 5$$

then what is $y(6)$?

a) 1   b) 2   c) 3   d) 4   e) 5
f) 6   g) 7   h) 8   i) 9   j) 10

Solution: c

```
> ode := diff(y(x),x) + 3/x*y(x) = 38/x^2;
  ic := y(2) = 5;
  IVP := {ode,ic};
```

$$ode := \left(\frac{d}{dx}\,y(x)\right) + \frac{3\,y(x)}{x} = \frac{38}{x^2}$$

$$ic := y(2) = 5$$

$$IVP := \{\left(\frac{d}{dx}\,y(x)\right) + \frac{3\,y(x)}{x} = \frac{38}{x^2}, y(2) = 5\}$$

```
> dsolve(IVP,y(x));
```

$$y(x) = \frac{19\,x^2 - 36}{x^3}$$

```
> subs(x=6,(19*x^2-36)/x^3);
```

$$3$$

```
> plot( (19*x^2-36)/x^3 , x = 2..6, thickness=2,
  color=NAVY,view=[2..6,2..5.5]);
```
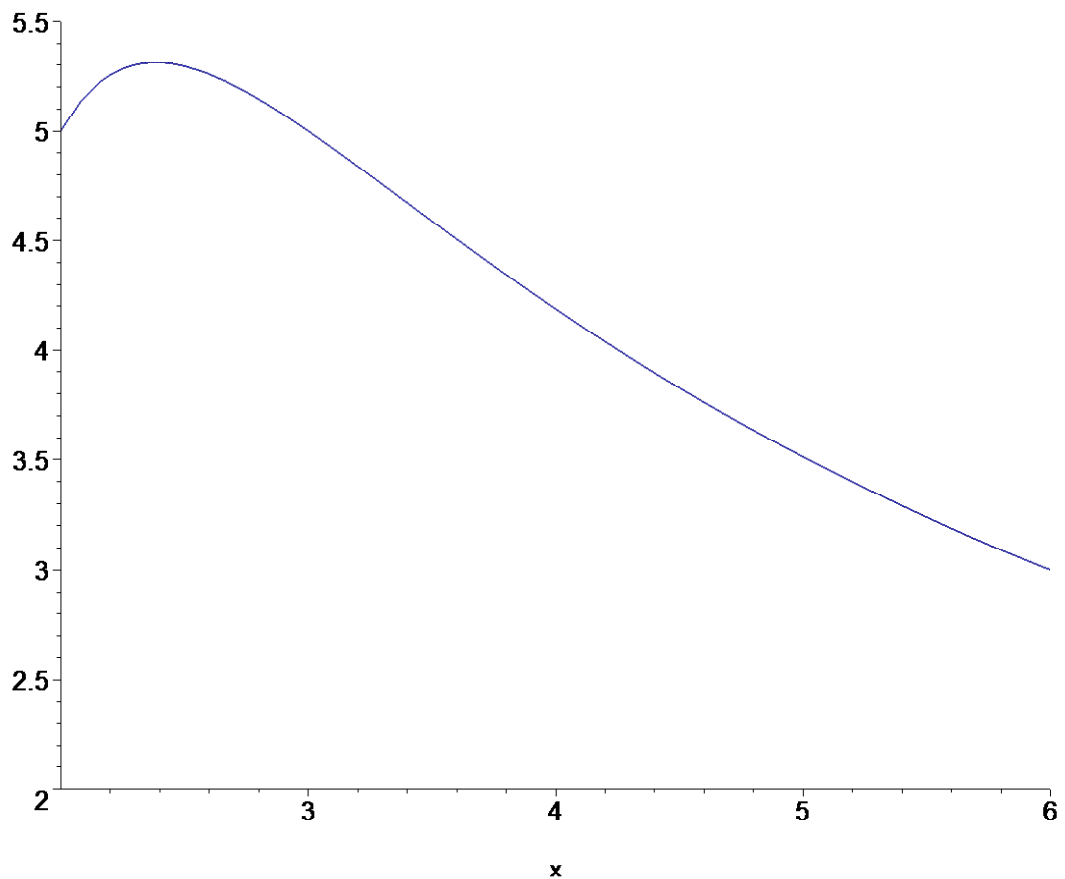
10. If $y(x)$ is the solution to the initial value problem

$$\frac{dy}{dx} = 2\,y(x) + e^{(3\,x)} \quad , \quad y(0) = 0$$

then what is $y(1)$?

a) e b) $e\,(e-1)$ c) $e\,(e+1)$ d) e^2 e) $e^2\,(e-1)$

f) $e^2\,(e+1)$ g) e^3 h) $e^3\,(e-1)$ i) $e^3\,(e+1)$ j) $3\,e$

Solution: e

```
> ode := diff(y(x),x) = 2*y(x)+exp(3*x);
  ic := y(0) = 0;
  IVP := {ode,ic};
```

$$ode := \frac{d}{dx}\,y(x) = 2\,y(x) + e^{(3\,x)}$$

$$ic := y(0) = 0$$

$$IVP := \{ \frac{d}{dx} y(x) = 2\,y(x) + \mathbf{e}^{(3\,x)}, \ y(0) = 0 \}$$

```
> dsolve(IVP,y(x));
```

$$y(x) = (\mathbf{e}^x - 1)\,\mathbf{e}^{(2\,x)}$$

```
> subs(x=1,%);
```

$$y(1) = (\mathbf{e} - 1)\,\mathbf{e}^2$$

```
> curve := plot([(exp(x)-1)*exp(2*x), (exp(1)-1)*exp(2)], x=0
  .. 1.1, color = [NAVY,GREY], thickness=[2,1]):
  vert := plot([1,y,y=0..subs(x=1.1,(exp(x)-1)*exp(2*x))],
  color = GREY):
  plots[display](curve,vert);
```
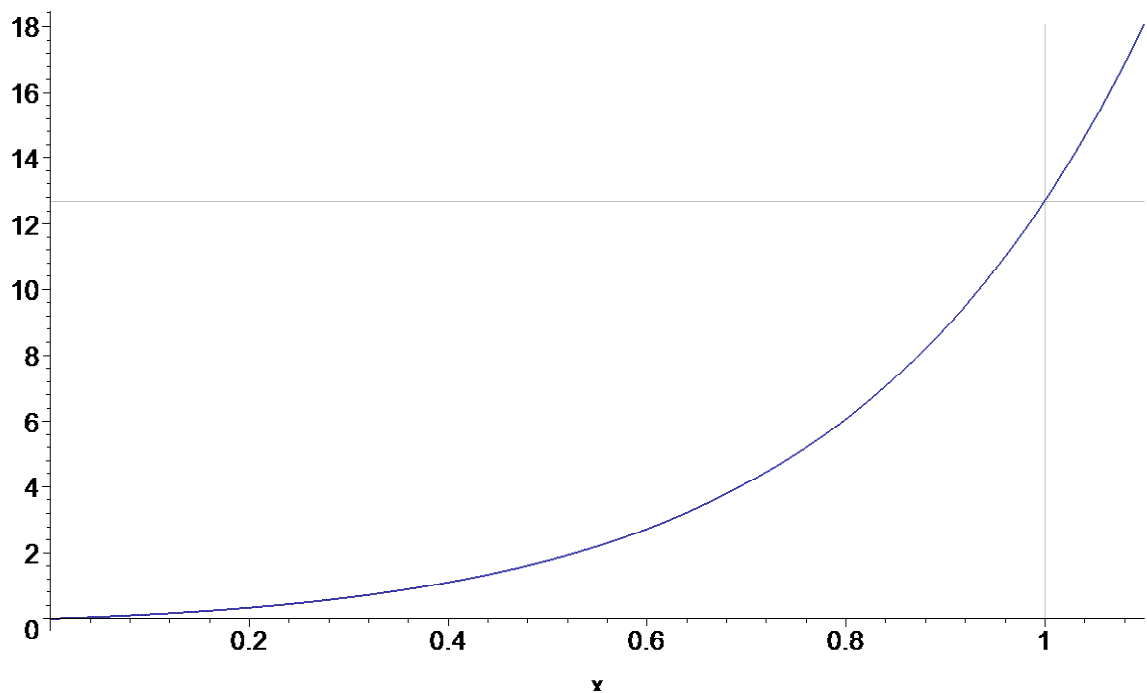

11. For nonzero constants a and b, and for the sequence defined by

$$x_n = \sqrt{\frac{9\,n^{50} + 100^{200}\,b^2}{4\,n^{50} + 200^{100}\,a^2}} \ ,$$

calculate $\lim\limits_{n \to \infty} x_n$.

a) $\dfrac{3}{2}$ b) $\dfrac{9}{4}$ c) $\dfrac{b}{a}$ d) $\dfrac{b^2}{a^2}$ e) $\sqrt{\dfrac{9 + b^2}{4 + a^2}}$

f) $\dfrac{9+b^2}{4+a^2}$ g) $\dfrac{3+b}{2+a}$ h) $\dfrac{3\,b}{2\,a}$ i)   0   j) **Does not exist**

Solution: a

Divide each term in the fraction by n^{50}: $x_n = \sqrt{\dfrac{9+\dfrac{c}{n^{50}}}{4+\dfrac{d}{n^{50}}}}$ where $c = 100^{200}\,b^2$ and

$d = 200^{100}\,a^2$ are constants.

It follows that $\displaystyle\lim_{n\to\infty} x_n = \sqrt{\dfrac{9+0}{4+0}} = \dfrac{3}{2}$.

12. Let r be a constant.   Calculate $\displaystyle\lim_{n\to\infty}\left(1-\dfrac{r}{n}\right)^n$.

a)   1   b) r c) $1-r$ d) $1-\dfrac{1}{r}$ e) $\ln(r)$

f) $\dfrac{1}{\ln(r)}$ g) e^r h) $\dfrac{1}{e^r}$ i)   0   j)   **Does not exist**

Solution: h

An important limit formula is $\displaystyle\lim_{n\to\infty}\left(1+\dfrac{a}{n}\right)^n = e^a$ for any constant a. To obtain this formula,

let $y_n = \ln\!\left(\left(1+\dfrac{a}{n}\right)^n\right) = n\ln\!\left(1+\dfrac{a}{n}\right) = \dfrac{\ln\!\left(1+\dfrac{a}{n}\right)}{\dfrac{1}{n}} = \dfrac{\ln(1+a\,u)}{u}$ where $u = \dfrac{1}{n}$ tends to 0

from the right.

By L'Hopital's Rule,

$$\lim_{n \to \infty} y_n = \lim_{u \to 0+} \frac{\dfrac{\partial}{\partial u} \ln(1 + a\,u)}{\dfrac{d}{du} u} = \lim_{u \to 0+} \frac{a}{1 + a\,u} = a$$

It follows that

$$\lim_{n \to \infty} \left(1 + \frac{a}{n} \right)^n = \lim_{n \to \infty} e^{y_n} = e^a \ . \ \textbf{Now set } a = -r \textbf{ to answer this problem.}$$

13. For positive constants a and b, and for the sequence $\{x_n\}$ defined by

$$x_n = \left(a + e^{(b + n)} \right)^{\left(\frac{1}{n} \right)} \qquad \left(\textbf{ the } n \textbf{'th root of } a + e^{(b + n)} \right),$$

calculate $\displaystyle \lim_{n \to \infty} x_n$.

a) a **b)** e **c)** $a + e$ **d)** e^b **e)** $a + e^b$

f) $a\,e$ **g)** $a\,e^b$ **h)** 1 **i)** 0 **j) Does not exist**

Solution: b

Intuitively, for large values of n we expect the additive constants a and b to have the effect of 0,

leading us to expect x_n to be approximately $(e^n)^{\left(\frac{1}{n} \right)}$, or e.

Let $y_n = \ln(x_n) = \dfrac{\ln(a + e^{(b+n)})}{n}$. By L'Hopital's Rule we have

$$\lim_{n \to \infty} y_n = \lim_{n \to \infty} \frac{\dfrac{\partial}{\partial n} \ln(a + e^{(b+n)})}{\dfrac{d}{dn} n} = \lim_{n \to \infty} \frac{e^{(b+n)}}{a + e^{(b+n)}} = \lim_{n \to \infty} \frac{1}{\dfrac{1}{e^{(b+n)}} + 1} = \frac{1}{0 + 1} = 1.$$

It follows that $\displaystyle \lim_{n \to \infty} x_n = \lim_{n \to \infty} e^{y_n} = e^{\left(\lim_{n \to \infty} y_n \right)} = $ e.

Maple answers this question with a one-line command:

```
> Limit( (a+exp(b + n))^(1/n), n = infinity) = limit( (a+exp(b
  + n))^(1/n), n = infinity);
```

$$\lim_{n \to \infty} (a + e^{(b+n)})^{\left(\frac{1}{n}\right)} = \mathbf{e}$$

14. Let $\{s_N\}$ denote the sequence of partial sums of the series $\sum\limits_{n=1}^{\infty} a_n$.

If $s_N = \left(\dfrac{\dfrac{\ln(N)}{\sqrt{N}} + 192\, N^5}{N^{\left(\frac{1}{N}\right)} + 3\,N^5} \right)^{\left(\frac{1}{3}\right)}$, what is the value of $\sum\limits_{n=1}^{\infty} a_n$?

a) 1   b) 2   c) 3   d) 4   e) 5
f) 6   g) 7   h) 8   i) 0   j) The series diverges

Solution: d

Divide every term in the fraction by n^5: $s_N = \left(\dfrac{\dfrac{\ln(N)}{N^{\left(\frac{11}{2}\right)}} + 192}{\dfrac{N^{\left(\frac{1}{N}\right)}}{N^5} + 3} \right)^{\left(\frac{1}{3}\right)}$.

Because $\lim\limits_{N \to \infty} \dfrac{\ln(N)}{N^p} = 0$ for any positive value of p, and $\lim\limits_{N \to \infty} N^{\left(\frac{1}{N}\right)} = 1$, we have

$$\lim_{N \to \infty} s_N = \left(\frac{0 + 192}{0 + 3}\right)^{\left(\frac{1}{3}\right)} = 64^{\left(\frac{1}{3}\right)} = 4 .$$

15. **Wacko and Dot are calculating the sequence $\{s_N\}$ of partial sums of the series**

$$\sum_{n=1}^{\infty} \frac{n}{2^n} .$$

They split up the labor with Wacko calculating s_N for odd values of N and Dot calculating s_N for even N. Almost immediately after Wacko announces $s_9 = \dfrac{1013}{512}$, Dot correctly follows with what value for s_{10}?

a) $\dfrac{2027}{1024}$ b) $\dfrac{507}{256}$ c) $\dfrac{2029}{1024}$ d) $\dfrac{1015}{512}$ e) $\dfrac{2031}{1024}$

f) $\dfrac{127}{64}$ g) $\dfrac{2033}{1024}$ h) $\dfrac{1017}{512}$ i) $\dfrac{2035}{1024}$ j) $\dfrac{509}{256}$

Solution: j

```
> eqn1 := S[9] = sum(k/2^k,k=1..9);
```
$$eqn1 := S_9 = \frac{1013}{512}$$

```
> eqn2 := S[10] = S[9] + 10/2^10;
```
$$eqn2 := S_{10} = S_9 + \frac{5}{512}$$

```
> subs(eqn1, eqn2);
```

$$S_{10} = \frac{509}{256}$$

16. Poetry Corner

Big fleas have little fleas,
　　Upon their backs to bite 'em.
And little fleas have lesser fleas,
　　And so *ad infinitum*.

**If the mass of each biter is 0.82 times the mass of its bitee,
and if the entire mass of the flea chain is 0.572 grams,
then, to three decimal places, what is the mass in grams of the**

largest flea?

a) 0.058   b) 0.067   c) 0.076   d) 0.085   e) 0.094
f) 0.103   g) 0.112   h) 0.121   i) 0.130   j) 0.138

Solution: f

**Let *m* be the mass of the largest flea. Then the mass of the second largest flea in
the biting chain is 0.82*m*. The mass of the thirdd largest flea in the biting chain is $.82^2 m$.
And so on.**

```
> eqn1 := 0.572 = Sum( m*0.82^n, n=0..infinity);
```

$$eqn1 := 0.572 = \sum_{n=0}^{\infty} m\, 0.82^n$$

```
> eqn2 := 0.572 = sum( m*0.82^n, n=0..infinity);
  #
  # The sum on the right side of eqn1 is m/(1-r)
```

$$eqn2 := 0.572 = 5.555555556\, m$$

```
> m = solve(eqn2, m);
```

$$m = 0.1029600000$$

17. To which of the following series does the Test for Divergence lead
to a conclusion?

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ (ii) $\displaystyle\sum_{n=1}^{\infty} \frac{1+\sqrt{n}}{n^2}$ (iii) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\left(\frac{1}{n}\right)^n}$ (iv) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n+1}{2^n+n^3}$

a) (i) only   b) (ii) only   c) (iii) only   d) (iv) only   e) (i) & (ii) only

f) (i) & (iii) only   g) (i) & (iv) only   h) (ii) & (iii) only   i) (ii) & (iv) only   j) (iii) & (iv) only

Solution: j

For $a_n = \dfrac{1}{n}$ and $a_n = \dfrac{1+\sqrt{n}}{n^2}$, we have $\displaystyle\lim_{n\to\infty} a_n = 0$.

The Test for Divergence therefore does not apply to Series (i) and (ii).

For $a_n = \dfrac{1}{\left(\frac{1}{n}\right)^n}$ and $a_n = \dfrac{2^n+1}{2^n+n^3}$, we have $\displaystyle\lim_{n\to\infty} a_n = 1$.

The Test for Divergence therefore does apply to Series (iii) and (iv)
(with a conclusion of "Diverges")..

18. The Integral Test can be used to show that the series

$$\sum_{n=1}^{\infty} \frac{2\,n}{(n^2+4)^2} \quad \text{(Note: } (n^2+4) \text{ is squared.)}$$

converges. Let s be the value of this series. Adding the first 6 terms of the series results in the sum 0.2173935650 (to ten decimal places). The Integral Test

provides a number β such that $s \le 0.2173935650 + β$. What is β?
(The question does not ask for *some* number β that makes the inequality
correct. It asks for *the specific* number that arises from the Integral Test.
If the desired value for β is replaced with a larger number, then the inequality

remains true. But the replacement value does not correctly answer the question.)

a) $\dfrac{1}{25}$ b) $\dfrac{1}{30}$ c) $\dfrac{1}{35}$ d) $\dfrac{1}{40}$ e) $\dfrac{1}{45}$

f) $\dfrac{1}{50}$ g) $\dfrac{1}{55}$ h) $\dfrac{1}{60}$ i) $\dfrac{1}{65}$ j) $\dfrac{1}{70}$

Solution: d

The Integral Test gives the value $\beta = \displaystyle\int_6^\infty \dfrac{2\,x}{\left(x^2+4\right)^2}\,dx.$

```
> evalf(1/40);
```
$$0.02500000000$$
```
> evalf( sum( 2*n/((n^2+4)^2),n=1..6) ,20);
```
$$0.21739356500080912411$$
```
> Int( 2*x/((x^2+4)^2),x=6..infinity ) = int(
  2*x/((x^2+4)^2),x=6..infinity );
```
$$\int_6^\infty \dfrac{2\,x}{\left(x^2+4\right)^2}\,dx = \dfrac{1}{40}$$

To ten decimal places, we have $\displaystyle\sum_{n=1}^{\infty} \dfrac{2\,n}{\left(n^2+4\right)^2} = 0.2389127750$, whereas $0.2173935650 +$

$1/40$ is 0.2423935650.

---

19. **For which of the following series does the Limit Comparison Test, applied with a *p*-series for the comparison series, lead to a conclusion of convergence?**

(i) $\displaystyle\sum_{n=1}^{\infty} \dfrac{1}{n+\ln(n)}$ (ii) $\displaystyle\sum_{n=1}^{\infty} \dfrac{1+\sqrt{n}}{n^2}$ (iii) $\displaystyle\sum_{n=1}^{\infty} \dfrac{2^n+n}{4^n+n^3}$ (iv) $\displaystyle\sum_{n=1}^{\infty} \dfrac{(-1)^n\,(n+1)}{\left(\dfrac{5}{2}\right)^n}$

a) (i) only   b) (ii) only   c) (iii) only   d) (iv) only   e)   (i) & (ii) only

f) (i) & (iii) only   g) (i) & (iv) only h) (ii) & (iii) only i) (ii) & (iv) only   j) (iii) & (iv) only

Solution: b

Series (i) is divergent: compare with $\displaystyle\sum_{n=1}^{\infty} \dfrac{1}{2\,n}$. No conclusion of convergence is possible.

Series (ii) is proven to be convergent by using the Limit Comparison Test with the p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^{\left(\frac{3}{2}\right)}} :$$

for $a_n = \dfrac{1 + \sqrt{n}}{n^2}$ and $b_n = \dfrac{1}{n^{\left(\frac{3}{2}\right)}}$, we have

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{n^{\left(\frac{3}{2}\right)} + n^2}{n^2} = \lim_{n \to \infty} \frac{\dfrac{n^{\left(\frac{3}{2}\right)}}{n^2} + \dfrac{n^2}{n^2}}{\dfrac{n^2}{n^2}} = \lim_{n \to \infty} \frac{1}{\sqrt{n}} + 1 = 1.$$

The Limit Comparison Test does work with Series (iii), but not using a p-series for comparison.

For $a_n = \dfrac{2^n + n}{4^n + n^3}$ and $b_n = \dfrac{1}{n^p}$, we have

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{(2^n + n)\, n^p}{4^n + n^3} = \lim_{n \to \infty} \frac{2^n n^p + n^{(p+1)}}{4^n + n^3} = \lim_{n \to \infty} \frac{\dfrac{n^p}{2^n} + \dfrac{n^{(p+1)}}{4^n}}{1 + \dfrac{n^3}{4^n}} = \frac{0 + 0}{1 + 0} = 0.$$

This limit (along with ∞) is not included in the decisive case of the Limit Comparison Test.

Series (iv) is an alternating series. It has infinitely many negative terms and the Limit Comparison Test therefore does not apply: it cannot even be applied to any tail.

$\boxed{-}$ 20. Let $a_n = \dfrac{1}{n}$ and $b_n = \dfrac{2}{n^2}$. Consider the following four alternating series:

(i) $\quad a_1 - b_2 + a_3 - b_4 + a_5 - b_6 + \ ...$

(ii) $\quad a_1 b_1 - a_2 b_2 + a_3 b_3 - a_4 b_4 + \ ...$

(iii) $\dfrac{b_1}{a_1} - \dfrac{b_2}{a_2} + \dfrac{b_3}{a_3} - \dfrac{b_4}{a_4} + \ ...$

(iv) $a_1 - b_2 + \dfrac{a_3}{3} - b_4 + \dfrac{a_5}{5} - b_6 + \ ...$

For which of theses series does the Alternating Series Test lead to a conclusion?

a) (i) only   b) (ii) only   c) (iii) only   d) (iv) only   e) (i) & (ii) only

f) (i) & (iii) only   g) (i) & (iv) only h) (ii) & (iii) only i) (ii) & (iv) only   j) (iii) & (iv) only

Solution: h

To apply the Alternating Series Test to the infinite series $c_k + c_{k+1} + c_{k+2} + \dots$, two conditions must be satisfied:

(i) $c_{n+1} < c_n$ for all $k \le n < \infty$ (i.e. the terms decrease)

and

(ii) $\lim\limits_{n \to \infty} c_n = 0.$

```
> a := n -> 1/n;
  b := n -> 2/n^2;
  x1 := n -> if is(n,odd) then a(n) else b(n) fi;
  x2 := n -> a(n)*b(n);
  x3 := n -> b(n)/a(n);
  x4 := n -> if is(n,odd) then a(n)/n else b(n) fi;
  #
  # x[i], x[2], x[3], x[4] are the four sequences of terms that
  generate the four series under consideration
```

$$a := n \to \frac{1}{n}$$

$$b := n \to \frac{2}{n^2}$$

$x1 :=$ **proc**(n) **option** *operator, arrow;* **if** is(n, odd) **then** a(n) **else** b(n) **end if end proc**

$$x2 := n \to \text{a}(n)\,\text{b}(n)$$

$$x3 := n \to \frac{\text{b}(n)}{\text{a}(n)}$$

$x4 :=$

proc(n) **option** *operator, arrow;* **if** is(n, odd) **then** a$(n)\,/\,n$ **else** b(n) **end if end proc**

```
> map(z -> evalf(z,4), [seq(x1(n), n = 1 .. 10)]);
  map(z -> evalf(z,4), [seq(x2(n), n = 1 .. 10)]);
  map(z -> evalf(z,4), [seq(x3(n), n = 1 .. 10)]);
  map(z -> evalf(z,4), [seq(x4(n), n = 1 .. 10)]);
```

```
#
# The next four lines show numerical evaluations of the four
sequences that are to be summed.
```

[1., 0.5000, 0.3333, 0.1250, 0.2000, 0.05556, 0.1429, 0.03125, 0.1111, 0.02000]

[2., 0.2500, 0.07407, 0.03125, 0.01600, 0.009259, 0.005831, 0.003906, 0.002743, 0.002000]

[2., 1., 0.6667, 0.5000, 0.4000, 0.3333, 0.2857, 0.2500, 0.2222, 0.2000]

[1., 0.5000, 0.1111, 0.1250, 0.04000, 0.05556, 0.02041, 0.03125, 0.01235, 0.02000]

The terms in the second and third lines are terms of sequences that decrease to 0. The terms in the first and last line are not steadily decreasing.

⊞ Code and Extra Problems