Math 132
Fall 2015 Exam 1

Formulas

$\ln(1) = 0$, $\ln(e) = 1$, $\ln(x\,y) = \ln(x) + \ln(y)$, $\ln(x^p) = p\ln(x)$

$\sin\left(\frac{\pi}{6}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$, $\sin\left(\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$, $\sin\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$,

$\sin(0) = \sin(\pi) = \cos\left(\frac{\pi}{2}\right) = 0$, $\sin\left(\frac{\pi}{2}\right) = \cos(0) = \tan\left(\frac{\pi}{4}\right) = 1$, $\sin\left(\frac{3\pi}{2}\right) = \cos(\pi) = -1$

$\int \frac{1}{x}\,dx = \ln(|x|) + C$, $\int \ln(x)\,dx = x\ln(x) - x + C$, $\int u\,dv = u\,v - \int v\,du$

$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$, $\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \arcsin\left(\frac{x}{a}\right) + C$,

$\int \frac{1}{|x|\sqrt{x^2 - a^2}}\,dx = \frac{1}{a}\operatorname{arcsec}\left(\frac{x}{a}\right) + C$

$\int \sec(x)^2\,dx = \tan(x) + C$, $\int \csc(x)^2\,dx = -\cot(x) + C$

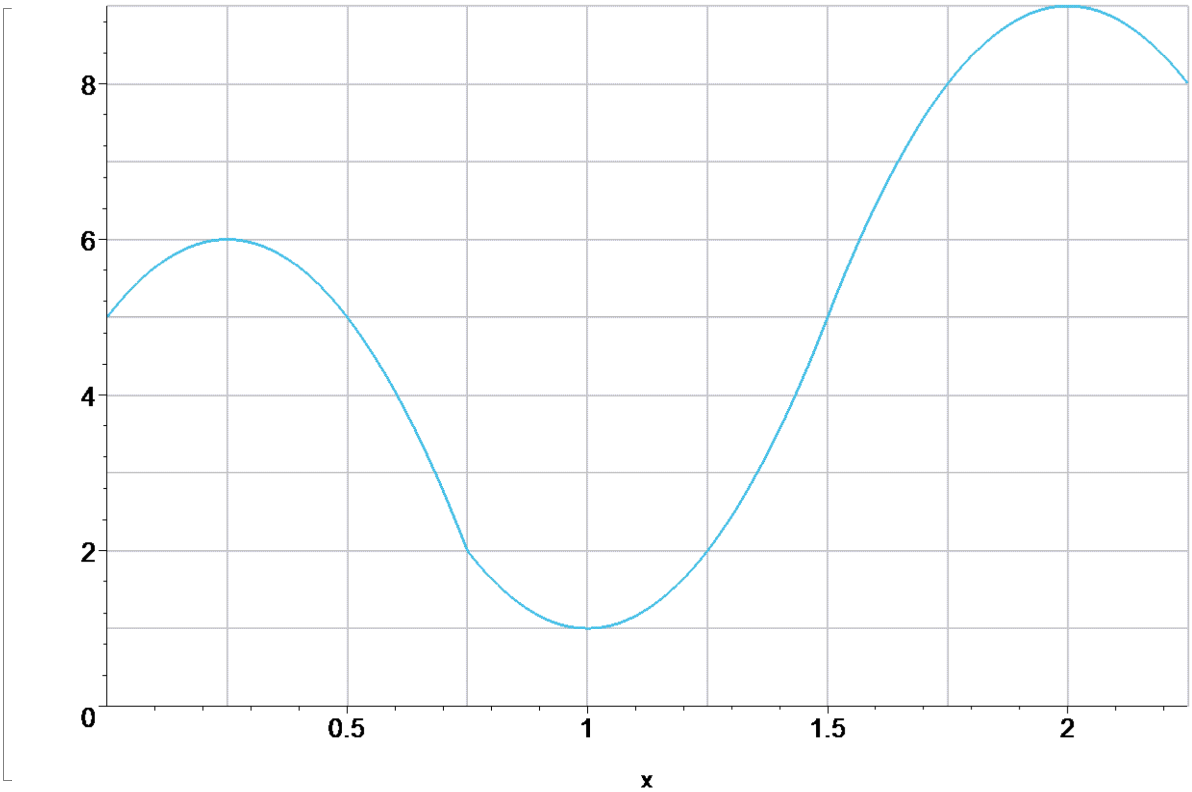
$\int \sec(x)\tan(x)\,dx = \sec(x) + C$, $\int \csc(x)\cot(x)\,dx = -\csc(x) + C$

$\int \tan(x)\,dx = \ln(|\sec(x)|) + C$, $\int \sec(x)\,dx = \ln(|\sec(x) + \tan(x)|) + C$

1. In the figure below, a function f has been plotted over an interval [a,b].

Approximate $\int_a^b f(x)\,dx$ **by the Riemann sum** $\sum_{j=1}^{3} f(s_j)\,\Delta x$,

where the sample points at which f is evaluated are chosen so that the sum is an upper Riemann sum.



a) 9   b) 10   c) 11   d) 12   e) 13

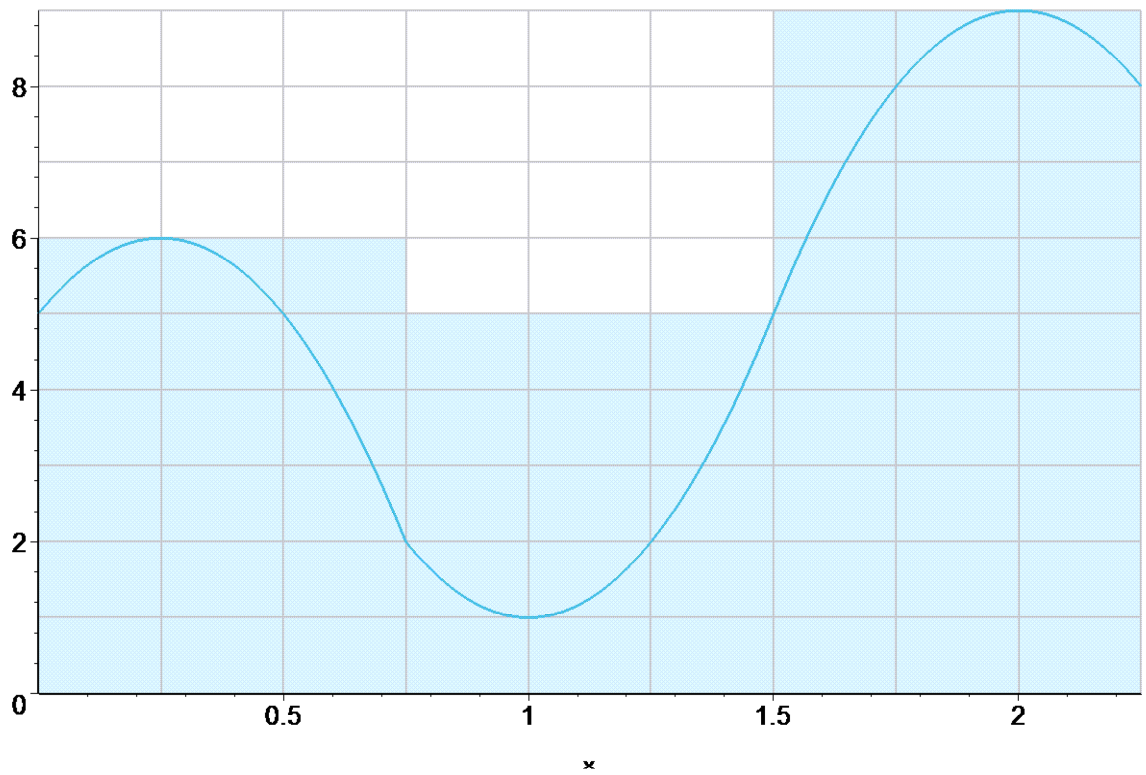
f) 14   g) 15   h) 16   i) 17   j) 18

Solution (g)

With $a = 0$, $b = \frac{9}{4}$, and $N = 3$, we have $\Delta x = \frac{b-a}{N} = \frac{3}{4}$.

The nodes of the partition are $x_0 = 0$, $x_1 = \frac{3}{4}$, $x_2 = \frac{3}{2}$, and $x_3 = \frac{9}{4}$.

The specified sample points are $s_1 = \frac{1}{4}$, $s_2 = \frac{3}{2}$, and $s_3 = 2$, with $f(s_1) = 6$, $f(s_2) = 5$, and $f(s_3) = 9$.




```
> requestedRiemannSum = ( 6 + 5 + 9 )*3/4;
```

$$requestedRiemannSum = 15$$

2. If a Riemann sum based on nine equal length subintervals is used to approximate $\int_a^b f(x)\,dx$ for the function f and interval [a,b] shown in the preceding problem, then what approximation of the integral is the worst possible underestimate that might result?

(Note: The question asks for a particular approximation of the integral. The word "underestimate" is Escher-like. Suppose the actual value of the integral were 1000 and we estimated the value to be 800. Our underestimate of 800 underestimates the integral by 200. This problem refers to the underestimate 800, not the underestimate 200.)

a) 9 **b)** $\frac{37}{4}$ **c)** $\frac{19}{2}$ **d)** $\frac{39}{4}$ **e)** 10

f) $\frac{41}{4}$ **g)** $\frac{21}{2}$ **h)** $\frac{43}{4}$ **i)** 11 **j)** $\frac{45}{4}$

Solution (b)

With $a = 0,\ b = \frac{9}{4},$ and $N = 9$, we have $\Delta x = \frac{b-a}{N} = \frac{1}{4}.$

The nodes of the partition are

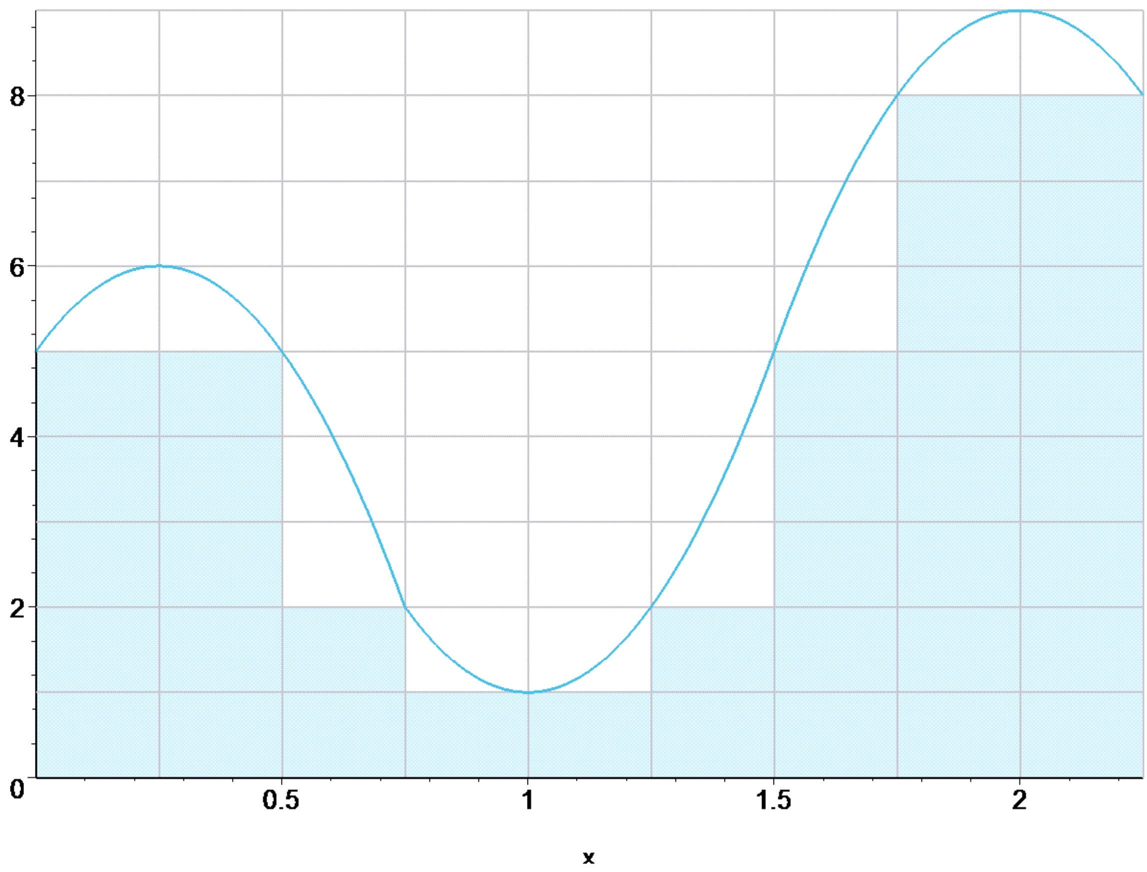
$$x_0 = 0,\ x_1 = \frac{1}{4},\ x_2 = \frac{1}{2},\ x_3 = \frac{3}{4},\ x_4 = 1,\ x_5 = \frac{5}{4},\ x_6 = \frac{3}{2},\ x_7 = \frac{7}{4},\ x_8 = 2,\ x_9 = \frac{9}{4}$$

The specified sample points are

$$s_1 = 0,\ s_2 = \frac{1}{2},\ s_3 = \frac{3}{4},\ s_4 = 1,\ s_5 = 1,\ s_6 = \frac{5}{4},\ s_7 = \frac{3}{2},\ s_8 = \frac{7}{4},\ s_9 = \frac{9}{4}$$

with

$$f(s_1) = 5,\ f(s_2) = 5,\ f(s_3) = 2,\ f(s_4) = 1,\ f(s_5) = 1,\ f(s_6) = 2,\ f(s_7) = 5,\ f(s_8) = 8,\ f(s_9) = 8$$




```
> (5 + 5 + 2 + 1 + 1 + 2 + 5 + 8 + 8)*(1/4);
  #
  # Direct addition to calculate Riemann sum
  # The next three lines repeat this calculation and may be
  skipped.
```

$$\frac{37}{4}$$

```
> s[1] := 0: s[2] := 1/2: s[3] := 3/4: s[4] := 1: s[5] := 1:
  s[6] := 5/4: s[7] := 3/2: s[8] := 7/4: s[9] := 9/4:
> y[1] := f(s[1]): y[2] := f(s[2]): y[3] := f(s[3]): y[4] :=
  f(s[4]): y[5] := f(s[5]): y[6] := f(s[6]): y[7] := f(s[7]):
  y[8] := f(s[8]): y[9] := f(s[9]):
> sum(y[j],j=1..9)*(1/4);
```

$$\frac{37}{4}$$

Note: The upper Riemann sum for this partition is

```
> (6 + 6 + 5 + 2 + 2 + 5 + 8 + 9 + 9)*(1/4);
```

$$13$$

This is the smallest upper bound we can find. So the worst underestimate, namely 37/4, may under-estimate the integral by 13 - 37/4, or 15/4.

3. Calculate $\int_{-12}^{-4} \frac{1}{x}\, dx$

a) $-\ln(12)$ **b)** $-\ln(8)$ **c)** $-\ln(6)$ **d)** $-\ln(4)$ **e)** $-\ln(3)$
f) $\ln(3)$ **g)** $\ln(4)$ **h)** $\ln(6)$ **i)** $\ln(8)$ **j)** $\ln(12)$

Solution (e)

$$\int_{-12}^{(-4)} \frac{1}{x}\, dx = \ln(|-4|) - \ln(|-12|) = \ln(4) - \ln(12) = \ln\left(\frac{4}{12}\right) = \ln\left(\frac{1}{3}\right) =$$
$$\ln(3^{(-1)}) = -\ln(3)$$

Remark: Observe that $-\ln(3) < 0$. Because the integrand 1/x is negative on the entire interval [-12 , -4], and because we are integrating from left to right, we expect a negative value
for the integral.

Verification using Maple's builtin integrator:

```
> Int(1/x,x = -12 .. -4)  = int(1/x,x = -12 .. -4);
```

$$\int_{-12}^{-4} \frac{1}{x}\, dx = -\ln(3)$$

4. Calculate $\int_{1}^{2} \frac{4\, x^3 - 2\, x^2 - 1}{x^2}\, dx$

a) $\frac{1}{2}$ b) 1 c) $\frac{3}{2}$ d) 2 e) $\frac{5}{2}$

f) 3 g) $\frac{7}{2}$ h) 4 i) $\frac{9}{2}$ j) 5

Solution (g)

```
> J := Int( (4*x^3-2*x^2-1)/(x^2), x = 1 .. 2);
```

$$J := \int_1^2 \frac{4x^3 - 2x^2 - 1}{x^2}\, dx$$

```
> J := Int( expand(integrand(J)), x = 1 .. 2);
```

$$J := \int_1^2 4x - 2 - \frac{1}{x^2}\, dx$$

```
> F := unapply( int( integrand(J), x), x);
  'F(x)' = F(x);
```

$$F := x \to 2x^2 - 2x + \frac{1}{x}$$

$$F(x) = 2x^2 - 2x + \frac{1}{x}$$

```
> F(2) - F(1);
```

$$\frac{7}{2}$$

Verification using Maple's builtin integrator:

```
> Int((4*x^3-2*x^2-1)/(x^2),x = 1 .. 2) =
  int((4*x^3-2*x^2-1)/(x^2),x = 1 .. 2);
```

$$\int_1^2 \frac{4x^3 - 2x^2 - 1}{x^2}\, dx = \frac{7}{2}$$

5. Calculate $\int_0^1 7\sqrt{x}\,(x-1)^2\,dx$

a) $\frac{4}{3}$ b) $\frac{4}{15}$ c) $\frac{8}{3}$ d) $\frac{8}{15}$ e) $\frac{16}{3}$
f) $\frac{16}{15}$ g) $\frac{32}{3}$ h) $\frac{32}{15}$ i) $\frac{64}{3}$ j) $\frac{64}{15}$

Solution (f)

```
> J1 :=  Int(7*sqrt(x)*(x-1)^2,x = 0 .. 1);
#
# J1 is the inert (unevaluated) form of the given integral
# In Maple, "Int" prevents the evaluation of the integral,
# whereas "int" calls for an evaluation, if possible
```

$$J1 := \int_0^1 7\sqrt{x}\,(x-1)^2\,dx$$

```
> eqn1 := J1  = map(expand,J1);
#
# "expand" does what it says it does to the integrand of J1
# First it expand the square (x-3)^2, then it multiplies each
# term of the exapnded square by sqrt(x)
```

$$eqn1 := \int_0^1 7\sqrt{x}\,(x-1)^2\,dx = \int_0^1 7\,x^{(5/2)} - 14\,x^{(3/2)} + 7\sqrt{x}\,dx$$

```
> J1 = value(rhs(eqn1));
#
# "value" forces the evaluation of an inert integral
```

$$\int_0^1 7\sqrt{x}\,(x-1)^2\,dx = \frac{16}{15}$$

Verification using Maple's builtin integrator:

```
> Int(7*sqrt(x)*(x-1)^2,x = 0 .. 1) = int(7*sqrt(x)*(x-1)^2,x =
0 .. 1);
```

$$\int_0^1 7\sqrt{x}\,(x-1)^2\,dx = \frac{16}{15}$$

6. Let $F(x) = \displaystyle\int_1^x \frac{7\,t^3 + t + 2}{\sqrt{4 + t^5}}\,dt$. **Calculate** $F'(2)$, **the derivative of** $F(x)$ **at** $x = 2$.

a) 1 b) 2 c) 3 d) 4 e) 5
f) 6 g) 7 h) 8 i) 9 j) 10

Solution (j)

```
> F := x -> Int((7*t^3+t+2)/sqrt(4+t^5),t = 1 .. x);
```

$$F := x \rightarrow \int_1^x \frac{7\,t^3 + t + 2}{\sqrt{4 + t^5}}\,dt$$

```
> D(F)(x);
  #
  # This is calculated using the Fundamental Theorem of
  Calculus - without any integration
```

$$\frac{7\,x^3 + x + 2}{\sqrt{4 + x^5}}$$

```
> derivative := D(F)(2);
  Answer = simplify( derivative );
```

$$derivative := \frac{5\sqrt{36}}{3}$$

$$Answer = 10$$

7. Let $F(x) = \displaystyle\int_x^3 \frac{(7-t)^2}{(2-t)^3}\,dt$. **Calculate** $F'(4)$, **the derivative of** $F(x)$ **at** $x = 4$.

a) $\frac{5}{4}$ b) $\frac{9}{8}$ c) 1 d) $\frac{7}{8}$ e) $\frac{3}{4}$

f) $-\frac{3}{4}$ g) $-\frac{7}{8}$ h) -1 i) $-\frac{9}{8}$ j) $-\frac{5}{4}$

Solution (b)

```
> F := x -> Int((7-t)^2/((2-t)^3),t = x .. 3);
```

$$F := x \to \int_x^3 \frac{(7-t)^2}{(2-t)^3}\,dt$$

```
> F := x -> -Int( (7-t)^2/(2-t)^3   , t = 3 .. x);
  #
  # Before applying the Fundamental Theorem of Calculus we
  reverse the direction of integration
  #  so that x is the upper limit of integration. (Maple would
  have done this without our intervention.)
```

$$F := x \to -\int_3^x \frac{(7-t)^2}{(2-t)^3}\,dt$$

```
> D(F)(x);
  #
  # This is calculated using the Fundamental Theorem of
  Calculus - without any integration
```

$$-\frac{(7-x)^2}{(2-x)^3}$$

```
> derivative := D(F)(4);
  Answer = simplify( derivative );
```

$$derivative := \frac{9}{8}$$

$$Answer = \frac{9}{8}$$

8. An alternative name for the inverse sine function is arcsin. Suppose that

$$F(x) = \int_{1}^{\arcsin\left(\frac{x}{5}\right)} \sin(t)^2 \, dt.$$ Calculate $F'(3)$, the derivative of $F(x)$ at $x = 3$.

(You may use one of the given formulas as a shortcut, if you wish.)

a) $\frac{1}{100}$ b) $\frac{1}{25}$ c) $\frac{1}{20}$ d) $\frac{3}{50}$ e) $\frac{2}{25}$

f) $\frac{9}{100}$ g) $\frac{1}{10}$ h) $\frac{3}{25}$ i) $\frac{3}{20}$ j) $\frac{1}{4}$

Solution (f)

```
> F := (x) -> Int(sin(t)^2,t = 1 .. arcsin(x/5));
```

$$F := x \rightarrow \int_{1}^{\arcsin(1/5\,x)} \sin(t)^2 \, dt$$

```
> D(F)(x);
  #
  # Note the factor 1/sqrt(25-x^2), wich is the derivative of
  arcsin(x/5)
  # in view of the given formula, int(1/sqrt(25-x^2),x) =
  arcsin(x/5)+C.
  # This factor arises from the Chain Rule.
  # The other factor, x^2/25, is sin^2(arcsin(x/5)), or
  (sin(arcsin(x/5)))^2, or (x/5)^2
```

$$\frac{x^2}{25\sqrt{25 - x^2}}$$

```
> D(F)(3);
```

$$\frac{9\sqrt{16}}{400}$$

```
> simplify( % );
  #
  # The character % refers to the last Maple output.
  # This line simplifies 9*sqrt(16)/400
```

$$\frac{9}{100}$$

9. Let $F(x) = \int_{3x}^{x^2} \frac{4-t}{t^2+t}\,dt$. **Calculate** $F'(2)$, **the derivative of** $F(x)$ **at** $x = 2$.

a) $\frac{1}{7}$ b) $\frac{2}{7}$ c) $\frac{3}{7}$ d) $\frac{4}{7}$ e) $\frac{5}{7}$

f) $\frac{6}{7}$ g) 1 h) $\frac{8}{7}$ i) $\frac{9}{7}$ j) $\frac{10}{7}$

Solution (a)

```
> F := (x)-> Int( (4-t)/(t^2+t) , t = 3*x .. x^2);
```

$$F := x \to \int_{3x}^{x^2} \frac{4-t}{t^2+t}\,dt$$

```
> D(F)(2);
```

$$\frac{1}{7}$$

To get this answer, Maple has done something like the following:

```
> F := (x)-> Int( (4-t)/(t^2+t), t = 0 .. x^2) - Int(
  (4-t)/(t^2+t) , t = 0 .. 3*x);
```

$$F := x \to \int_{0}^{x^2} \frac{4-t}{t^2+t}\,dt - \int_{0}^{3x} \frac{4-t}{t^2+t}\,dt$$

```
> derivative := D(F)(x);
```

$$derivative := \frac{2\,x\,(4-x^2)}{x^4+x^2} - \frac{3\,(4-3\,x)}{9\,x^2+3\,x}$$

```
> subs(x=2, derivative);
```

$$\frac{1}{7}$$

10. Calculate $\int_0^1 76\,(19\,x+8)^{\left(\frac{1}{3}\right)}\,dx.$

a) 75	b) 90	c) 105	d) 120	e) 135
f) 150	g) 165	h) 180	i) 195	j) 210

Solution (i)

```
> J := Int(76*(19*x+8)^(1/3), x = 0 .. 1);
```

$$J := \int_0^1 76\,(19\,x+8)^{(1/3)}\,dx$$

```
> F := Int(integrand(J), x);
```

$$F := \int 76\,(19\,x+8)^{(1/3)}\,dx$$

```
> G := changevar(u = 19*x+8, F, u);
```

$$G := \int 4\,u^{(1/3)}\,du$$

```
> G := value(G);
```

$$G := 3\,u^{(4/3)}$$

```
> F := subs(u = 19*x+8, G);
```

$$F := 3\,(19\,x+8)^{(4/3)}$$

```
> answer := subs(x = 1, F) - subs(x = 0, F);
```

$$answer := 81\;27^{(1/3)} - 24\;8^{(1/3)}$$

```
> simplify( answer );
```

$$195$$

Verification using Maple's builtin integrator:

```
> Int(76*(19*x+8)^(1/3), x = 0 .. 1) = simplify(
  int(76*(19*x+8)^(1/3), x = 0 .. 1) );
```

$$\int_0^1 76\,(19\,x+8)^{(1/3)}\,dx = 195$$

11. Calculate $\displaystyle\int_{e}^{e^2} \frac{1}{x\sqrt{\ln(x)}}\,dx.$

a) $\sqrt{2}-1$ b) $2-\sqrt{2}$ c) $2\sqrt{2}-1$ d) $2(\sqrt{2}-1)$ e) $2(2-\sqrt{2})$

f) $4\sqrt{2}-1$ g) $2(2\sqrt{2}-1)$ h) $\frac{1}{\sqrt{2}}-\frac{1}{2}$ i) $1-\frac{\sqrt{2}}{2}$ j) $\sqrt{2}-\frac{1}{2}$

Solution (d)

```
> J1 := Int(1/x/sqrt(ln(x)), x = exp(1) .. exp(2));
#
# Gives the name J1 to the inert integral of the problem
# In Maple, "Int" tells Maple to set up an integral that is
not
# to be evaluated immediately. For the eventual evaluation,
# the command "value" will be used
```

$$J1 := \int_{e}^{e^2} \frac{1}{x\sqrt{\ln(x)}}\,dx$$

```
> J2 := changevar(u = ln(x), J1, u);
```

$$J2 := \int_{1}^{2} \frac{1}{\sqrt{u}}\,du$$

```
> J1 = value(J2);
#
# Forces the evaluation of the inert integral
```

$$\int_{e}^{e^2} \frac{1}{x\sqrt{\ln(x)}}\,dx = 2\sqrt{2}-2$$

Verification using Maple's builtin integrator:

```
> J1 = simplify( value(J1) );
```

$$\int_{e}^{e^2} \frac{1}{x\sqrt{\ln(x)}}\,dx = 2\sqrt{2} - 2$$

12. Calculate $\int_{-1}^{0} 35\,x^2\sqrt{x+1}\,dx$.

a) $\frac{4}{3}$ b) $\frac{4}{15}$ c) $\frac{8}{3}$ d) $\frac{8}{15}$ e) $\frac{16}{3}$

f) $\frac{16}{15}$ g) $\frac{32}{3}$ h) $\frac{32}{15}$ i) $\frac{64}{3}$ j) $\frac{64}{15}$

Solution (e)

```
> J1 := Int(35*x^2*sqrt(x+1),x = -1 .. 0);
```

$$J1 := \int_{-1}^{0} 35\,x^2\sqrt{x+1}\,dx$$

```
> J2 := changevar(u = x+1, J1, u);
  #
  # This line says, Make the substitution u = x+1 in J1 and set
  J2 to be the resulting integral wrt u
  # Note that the new limits of integration for u are
  calculated
```

$$J2 := \int_{0}^{1} 35\,(-1+u)^2\sqrt{u}\,du$$

```
> J3 := map(expand, J2);
  #
  # This expands everything in J2: First the square is
  expanded, then each term is multiplied by sqrt(u)
```

$$J3 := \int_{0}^{1} 35\sqrt{u} - 70\,u^{(3/2)} + 35\,u^{(5/2)}\,du$$

The next line evaluates this integral:

```
> J1 = value(J3);
```

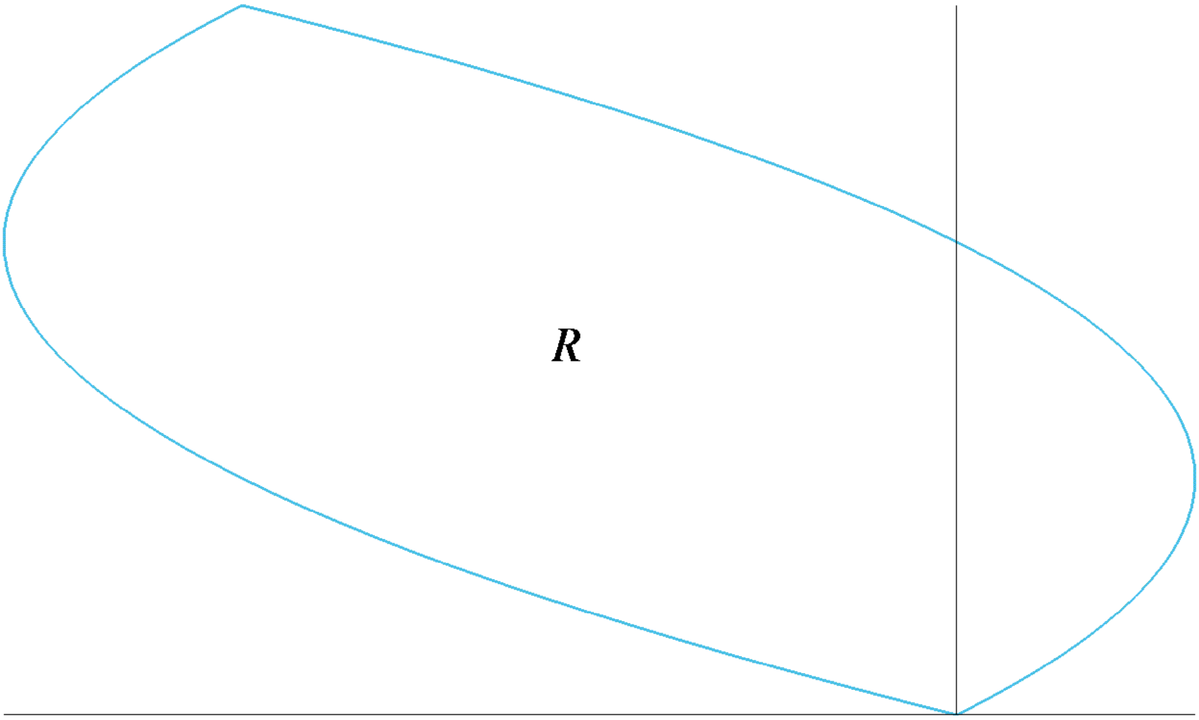
$$\int_{-1}^{0} 35\,x^2\,\sqrt{x+1}\,dx = \frac{16}{3}$$

Verification using Maple's builtin integrator:

```
> J1 = simplify( value(J1) );
```

$$\int_{-1}^{0} 35\,x^2\,\sqrt{x+1}\,dx = \frac{16}{3}$$

13. Calculate the area of the region *R* that is bounded by $x = y^2 - 4\,y$ and $x = 2\,y - y^2$, as shown in the figure below.



a) 5 b) 6 c) 7 d) 8 e) 9
f) 10 g) 11 h) 12 i) 13 j) 14

Solution (e)

```
> f := y -> 2*y - y^2;
  g := y -> y^2 - 4*y;
```

$$f := y \to 2\,y - y^2$$

$$g := y \to y^2 - 4\,y$$

```
> solve(f(y) = g(y) ,y);
  #
  # There are exactly two points of intersection
```

$$0, 3$$

```
> f(1), g(1);
  #
  # These evaluations at a point between x = 1 and x = 2 show
  us that g(y) < f(y) for 0 < y < 3
```

$$1, -3$$

```
> Area  = int( f(y) - g(y), y = 0 .. 3);
```

$$Area = 9$$

Verification by integrations with respect to x:

```
> completesquare(f(y),y);
  completesquare(g(y),y);
  #
  # Shows that the x-interval is [-4,1]
```

$$-(y-1)^2 + 1$$

$$(y-2)^2 - 4$$

```
> f(3);
```

$$-3$$

```
> solve( f(y) = x, y);
```

$$1 + \sqrt{1-x},\ 1 - \sqrt{1-x}$$

```
> solve( g(y) = x, y);
```

$$2 + \sqrt{4+x},\ 2 - \sqrt{4+x}$$

```
> int( ( (2+(4+x)^(1/2)) - (2-(4+x)^(1/2)) ), x = -4 .. -3)
    + int( ( (1+(1-x)^(1/2)) - (2-(4+x)^(1/2)) ), x = -3 .. 0)

      + int( ( (1+(1-x)^(1/2)) - (1-(1-x)^(1/2)) ), x = 0 ..
  1);
```

9

14. Functions f(x) and g(x) satisfy $g(x) \le f(x)$ for $1 \le x \le \frac{4}{3}$ and $f(x) \le g(x)$ for $\frac{4}{3} \le x \le 2$. In the table below, the number in an f(x) or g(x) cell is the value of f or g at the value of x in the x cell above it. For example, f(1) = 2.7 and g(2) = 2.4.

x	1	7/6	8/6	9/6	10/6	11/6	2
f(x)	2.7	3.1	3.2	3.0	2.6	2.5	2.4
g(x)	2.7	2.9	3.2	3.4	3.5	3.1	2.4

Use a Riemann sum with equal length subintervals and with midpoints for sample points to estimate the area between the graphs of $y = f(x)$ and $y = g(x)$ for $1 \le x \le 2$.

a) 0.2 b) 0.3 c) 0.4 d) 0.5 e) 0.6
f) 0.7 g) 0.8 h) 0.9 i) 1.0 j) 1.1

Solution (c)

The.

```
> Delta := ((2-1)/6)*2;
```

$$\Delta := \frac{1}{3}$$

```
> ( (3.1-2.9) + (3.4-3.0) + (3.1-2.5) )*Delta;
```

0.4000000000

15. Let R be the region that lies below $y = \sqrt{x}$, above the x-axis, and to the left of $x = 6$. Calculate the volume of the solid that results when R is rotated about the x-axis.

a) 3π b) 6π c) 8π d) 10π e) 12π
f) 15π g) 16π h) 18π i) 20π j) 24π

Solution (h)

Method of Disks (Method of Discs in British Commonwealth countries) (An excellent choice)

```
> Volume_by_disks := Pi*Int(sqrt(x)^2, x = 0 .. 6);
```

$$Volume\_by\_disks := \pi \int_0^6 x \, dx$$

```
> value(Volume_by_disks);
```

$$18\pi$$

Method of Shells (A reasonably good choice)

```
> Volume_by_shells := 2*Pi*Int(y*(6-y^2), y = 0 .. sqrt(6));
```

$$Volume\_by\_shells := 2\pi \int_0^{\sqrt{6}} y\,(6 - y^2)\, dy$$

```
> value(Volume_by_shells);
```

$$18\pi$$

16. Let R be the region that lies above $y = \sqrt{x}$ and below $y = 2$. The boundary of R on the left is the y-axis. for x between 2 and 3. What is the volume of the solid that results when R is rotated about the y-axis?

a) $\frac{64\pi}{3}$ b) $\frac{64\pi}{5}$ c) $\frac{32\pi}{3}$ d) $\frac{32\pi}{5}$ e) $\frac{16\pi}{3}$
f) $\frac{16\pi}{5}$ g) $\frac{8\pi}{3}$ h) $\frac{8\pi}{5}$ i) $\frac{4\pi}{3}$ j) $\frac{4\pi}{5}$

Solution (d)

Method of Disks (a fine choice)

```
> Volume_by_discs := Pi*Int( ( y^2 )^2 , y = 0 .. 2);
```

$$Volume\_by\_discs := \pi \int_0^2 y^4 \, dy$$

```
> value(Volume_by_discs);
```

$$\frac{32\,\pi}{5}$$

Method of Shells (another fine choice)

```
> Volume_by_shells := 2*Pi*Int(x*(2-sqrt(x)), x = 0 .. 4);
```

$$Volume\_by\_shells := 2\,\pi \int_0^4 x\,(2-\sqrt{x})\,dx$$

```
> value(Volume_by_shells);
```

$$\frac{32\,\pi}{5}$$

17. Let R be the region that lies to the right of the arc of the parabola $y = x^2$, $1 \le x \le \sqrt{2}$, to the left of the line segment $y = x - 2$, $3 \le x \le 4$, above $y = 1$, and below $y = 2$. Using washers, the volume of the solid that results when R is rotated about the axis $x = 5$ is

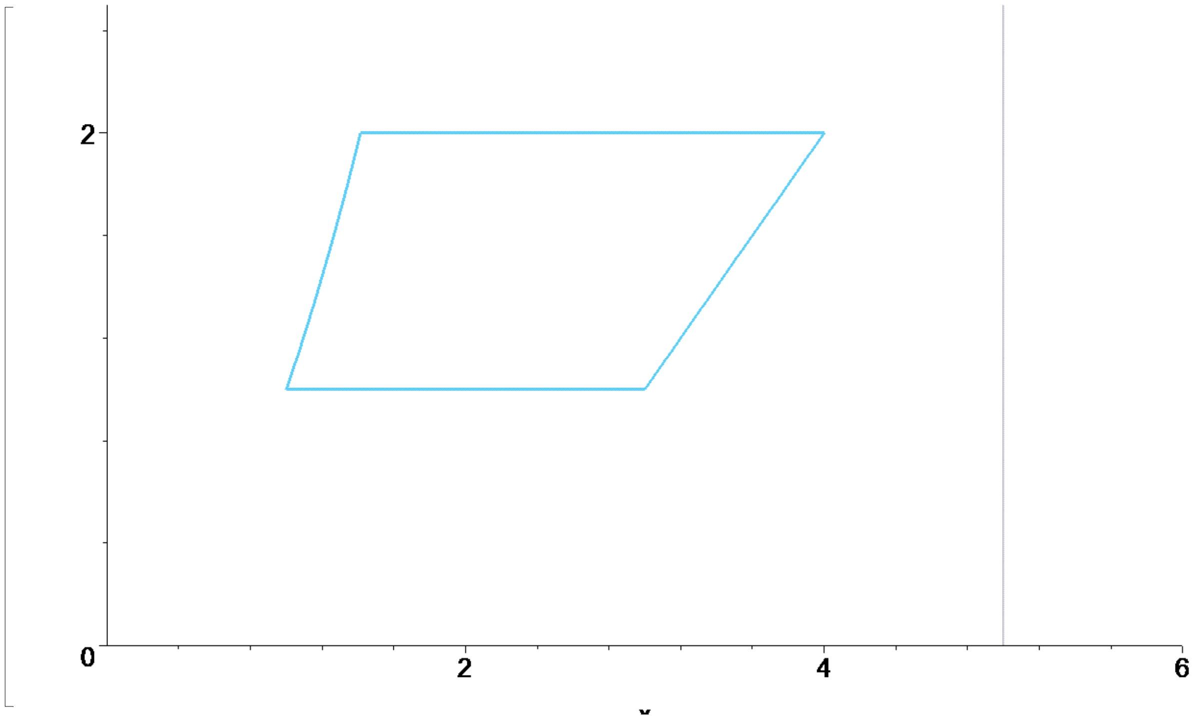
$$\pi \int_1^2 W(y)\,dy$$

for a function W. Find an expression for W(y). What is the value of that expression when $y = 4$?

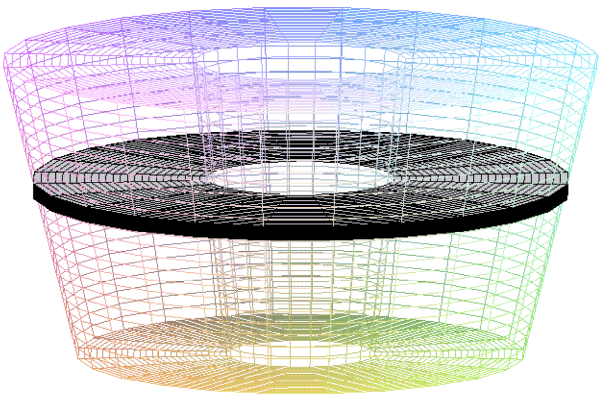
a)	8	b)	9	c)	10	d)	11	e)	12
f)	13	g)	14	h)	15	i)	16	j)	17

Solution (a)

The figure below shows the region R and the axis of rotation.



The solid of revolution in see-through rendering, with washer:



Method of Washers

```
> Volume_by_washers := Pi*Int( (5-sqrt(y))^2 - (5-(y+2))^2, y =
  1..2);
```

$$Volume\_by\_washers := \pi \int_{1}^{2} (5-\sqrt{y})^2 - (3-y)^2 \, dy$$

```
> value(Volume_by_washers);
  evalf(Volume_by_washers);
```

$$\pi \left(\frac{185}{6} - \frac{40\sqrt{2}}{3} \right)$$

$$37.62733431$$

Verification using cylindrical shells

```
> 2*Pi*int( (5-x)*(x^2-1), x = 1 .. sqrt(2)) + 2*Pi*int(
  (5-x)*(2-1), x = sqrt(2)..3) + 2*Pi*int( (5-x)*(2-(x-2)), x =
  3 .. 4);
```

$$2\pi \left(\frac{37}{12} - \frac{5\sqrt{2}}{3} \right) + 2\pi \left(\frac{23}{2} - 5\sqrt{2} \right) + \frac{5\pi}{3}$$

```
> evalf( % );
```

$$37.62733432$$

```
> W := y -> (5-y^(1/2))^2-(3-y)^2;
```

$$W := y \rightarrow (5-\sqrt{y})^2 - (3-y)^2$$

```
> expand(W(y));
```

$$16 - 10\sqrt{y} + 7y - y^2$$

```
> expand(W(4));
```

$$28 - 10\sqrt{4}$$

```
> simplify( % );
```

$$8$$

Method of Shells

The Method of Shells provides a more efficient calculation:

```
> Volume_by_shells := 2*Pi*Int((5-x)*(x^2-1), x = 1 ..
  sqrt(2))
    + 2*Pi*Int((5-x)*(2-1), x = sqrt(2) .. 3)
       +  2*Pi*Int((5-x)*(2-(x-2)), x = 3 .. 4);
```

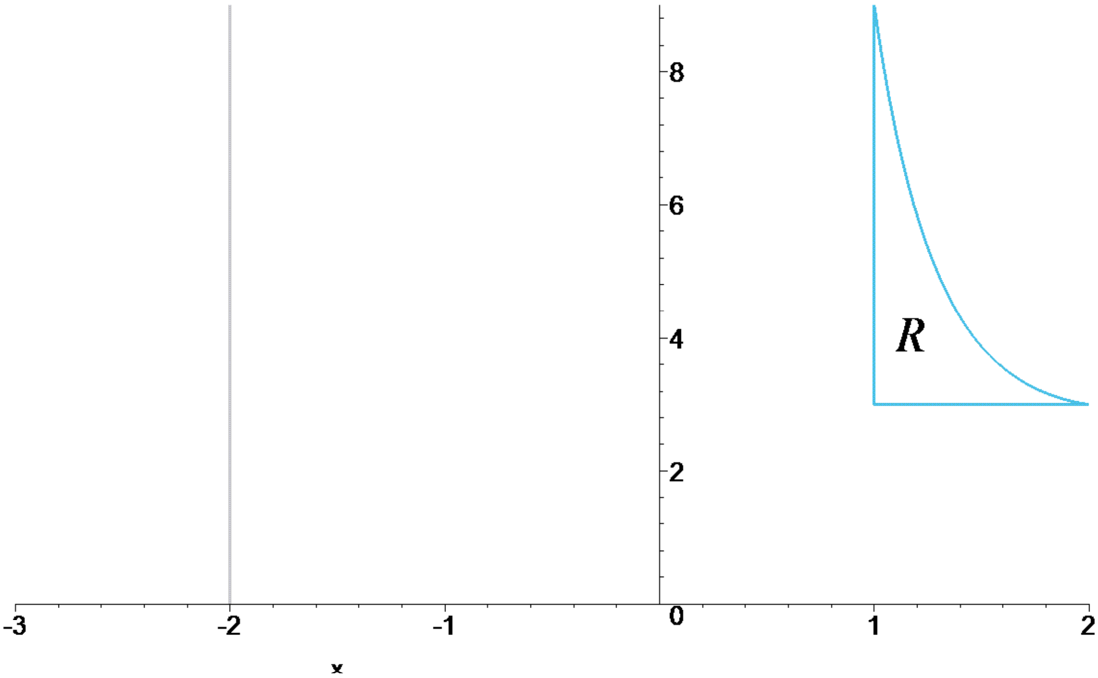
$$Volume\_by\_shells := 2\pi \int_{1}^{\sqrt{2}} (5-x)(x^2-1)\, dx + 2\pi \int_{\sqrt{2}}^{3} 5-x\, dx + 2\pi \int_{3}^{4} (5-x)(4-x)\, dx$$

```
> value(Volume_by_shells);
  evalf(value(Volume_by_shells));
```

$$2\pi\left(\frac{37}{12}-\frac{5\sqrt{2}}{3}\right)+2\pi\left(\frac{23}{2}-5\sqrt{2}\right)+\frac{5\pi}{3}$$

$$37.62733432$$

18. Let R be the "triangular" region that is bounded above by $y = x + \frac{8}{x^3}$ for $1 \le x \le 2$, bounded below by $y = 3$, and bounded on the left by $x = 1$.



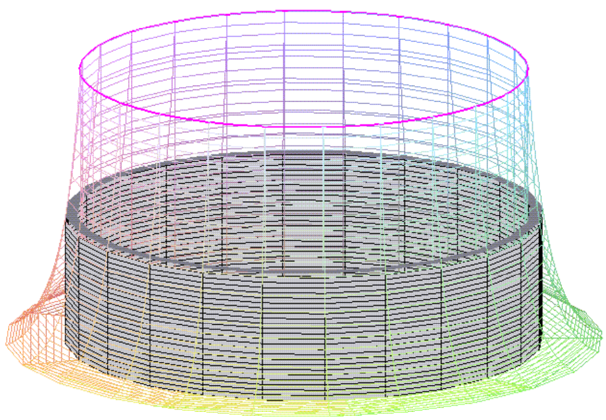
Using cylindrical shells, the volume of the solid that results when R is rotated about the axis $x = -2$ is

$$2\pi \int_{1}^{2} S(x)\, dx$$

for some function S. What is the value $36\,S\left(\frac{4}{3}\right)$?

a) 165 b) 170 c) 175 d) 180 e) 185
f) 190 g) 195 h) 200 i) 205 j) 210

Solution (i)



```
> 2*Pi*Int( (x+2)*(x + 8/x^3 - 3),x=1..2);
```

$$2\pi \int_{1}^{2} (x+2)\left(x+\frac{8}{x^3}-3\right) dx$$

```
> S := x -> (x+2)*(x+8/x^3-3);
```

$$S := x \to (x+2)\left(x+\frac{8}{x^3}-3\right)$$

```
> S(4/3);
```

$$\frac{205}{36}$$

```
> a := 3: b := 9: N := 1000:
  Delta :=
  (b-a)/N:for j from 1 to N do
  x[j] := fsolve( 3+(j-1/2)*Delta = x + 8/x^3, x, 1..2):
  end do:
```

```
Approximation_by_washers := pi*sum( (x['j']+2)^2 - 9 , 'j' =
1 .. N )*Delta:
evalf(Approximation_by_washers, 5);
2*Pi*Int( (x+2)*(x + 8/x^3 - 3),x=1..2) = 2*Pi*int( (x+2)*(x
+ 8/x^3 - 3), x=1..2);
evalf(29*pi/3, 5);
#
# To verify the setup of the cylindrical shell volume
integral,
# the integral is calculated (middle outout) and numerically

# evaluated (final output). Additionally, the volume is
# calculated by using washers (first output). Agreement of
the
# two values is a good sign that no error has been made.
```

$$9.6666\,\pi$$

$$2\,\pi \int_{1}^{2} (x+2)\left(x+\frac{8}{x^3}-3\right) dx = \frac{29\,\pi}{3}$$

$$9.6667\,\pi$$

19. A spring is stretched 2 m beyond equilibrium, at which point a force of 80 N maintains its position. The spring is then allowed to return to equilibrium. From that position at rest, the spring is stretched a second time. How many meters beyond equilibrium has it been stretched that second time if 120 J of work were expended in the course of the second stretching?

a) $\sqrt{2}$ **b)** $\sqrt{3}$ **c)** $\sqrt{6}$ **d)** $2\sqrt{2}$ **e)** $2\sqrt{3}$
f) $2\sqrt{6}$ **g)** $3\sqrt{2}$ **h)** $3\sqrt{3}$ **i)** $3\sqrt{6}$ **j)** $4\sqrt{2}$

Solution (c)

We will use b to denote the unknown number of meters beyond equilibrium of the second stretch.

```
> HookesLaw := F = k*x;
```

$$HookesLaw := F = k\,x$$

```
> eqn1 := subs({F=80, x=2}, HookesLaw);
#
# This substitutes the given data, F = 80 N  and x = 2m, into
Hooke's Law,
# resulting in an equation involving the spring constant k.
```

$$eqn1 := 80 = 2\,k$$

```
> eqn2 := k = solve(eqn1, k);
  #
  # This gives the value for the spring constant
```

$$eqn2 := k = 40$$

```
> eqn3 := W = 120;
  #
  # The value of work for the second stretching
```

$$eqn3 := W = 120$$

```
> eqn4 := W = Int(k*x, x = 0..b);
  #
  # The equation that relates W, k, and b
```

$$eqn4 := W = \int_0^b k\,x\,dx$$

```
> eqn5 := subs( {eqn2, eqn3}, eqn4);
  #
  # Substitutes the known values of W and k into preceding
  equation
```

$$eqn5 := 120 = \int_0^b 40\,x\,dx$$

```
> eqn6 := lhs(eqn5) = value(rhs(eqn5));
  #
  # Calculates integral on right hand side of preceding
  equation
```

$$eqn6 := 120 = 20\,b^2$$

```
> eqn7 := map( z -> z/20, eqn6);
  #
  # Divides each side of preceding equation by 20
```

$$eqn7 := 6 = b^2$$

```
> b = sqrt( 6 );
  #
  # The positive solution of the preceding equation
```

$$b = \sqrt{6}$$

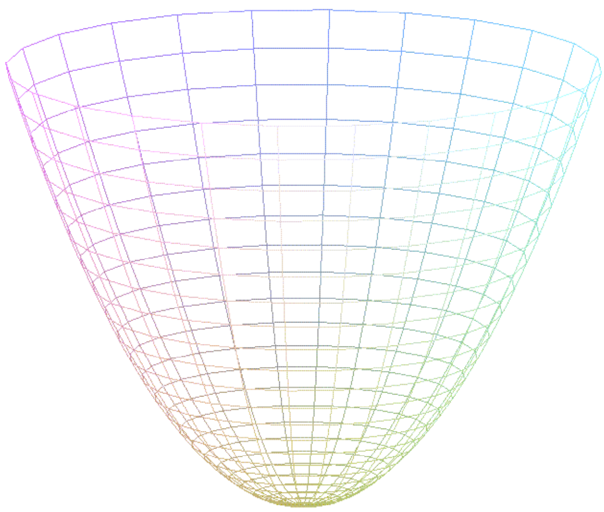
20. **A tank has the shape that results when the curve $y = x^2$, $0 \le x \le 2\ m$,
is rotated about the y-axis. It is partially filled to a depth of 3m with pibegone, a fluid that**

has a weight density of $\frac{3}{\pi}\frac{N}{m^3}$. The fluid is pumped over the top of the tank until the remaining depth is 1m. How many Joules of work have been performed?

a) 4 b) 6 c) 8 d) 10 e) 12
f) 14 g) 16 h) 18 i) 20 j) 22

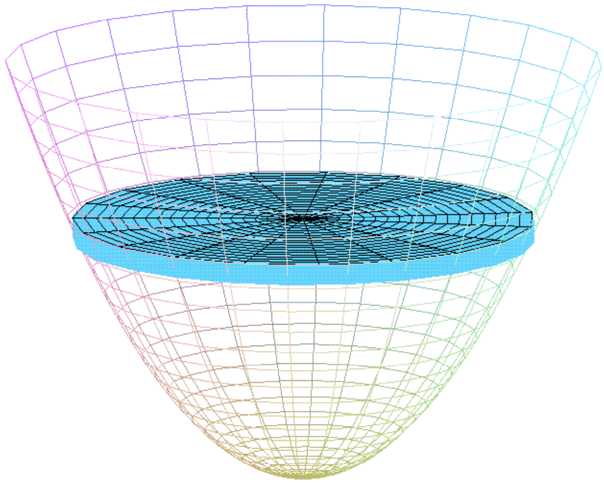
Solution (j)

The tank is shown in the figure below.



The radius of the disk at the top is 2 m. The height of the tank is 4 m.

In the next figure, a "slice" of pibegone at height y is added" in the next figure.



The radius of the disk shown is $x = \sqrt{y}$, and the thickness is dy. The volume of the disk is
$\pi \sqrt{y}^2\ dy$, or $\pi\, y\, dy$.

The weight of the disk shown is $\frac{3}{\pi} \pi\, y\, dy$, or $3\, y\, dy$.

Because the slice is at height y and must be pumped to height 4, the distance it is pumped is $4 - y$. The work done on the
slice is therefore $3\,(4 - y)\, y\, dy$. The total work done is

$$\int_1^3 3\,(4-y)\,y\,dy$$

```
> Int(3*(4-y)*y,y = 1 .. 3) = int(3*(4-y)*y,y = 1 .. 3);
```

$$\int_1^3 3\,(4-y)\,y\,dy = 22$$

Code