Surface Integrals

Here is a summary of how to find surface integrals of functions and vector fields.

If we have a surface $S$, then we can integrate a function or a vector field over $S$. To integrate a function or a vector field over $S$, we first parametrize $S$. Since $S$ is two-dimensional, we always need two parameters $u$ and $v$, and we should express $x$, $y$ and $z$ in terms of our parameters.

$$r(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle.$$  

We then compute $r_u \times r_v$.

It might be tricky to find the correct parametrization, but here are a few general rules:

1. Use spherical coordinates only if the surface is a part a sphere.
2. If $S$ is a part of a cylinder, for example $x^2 + y^2 = 1$, then use $\theta$ and $z$ to parametrize $S$. So $S$ is parametrized by

$$r(\theta, z) = \langle \cos(\theta), \sin(\theta), z \rangle.$$  

In this case, $r_{\theta} \times r_z = \langle \cos(\theta), \sin(\theta), 0 \rangle$.

3. If $S$ is a part of a plane with equation $ax + by + cz = d$, then use $x$ and $y$ as parameters

$$r(x, y) = \langle x, y, \frac{d}{c} - \frac{a}{c}x - \frac{b}{c}y \rangle.$$  

In this case, $r_{x} \times r_y = \langle \frac{a}{c}, \frac{b}{c}, 1 \rangle$.

4. If $S$ is a part of a graph of a function $z = f(x, y)$, we can use $x$ and $y$ as parameters, so

$$r(x, y) = \langle x, y, f(x, y) \rangle.$$  

In this case, $r_{x} \times r_y = \langle -f_x, -f_y, 1 \rangle$. If for example, $S$ is the paraboloid $z = x^2 + y^2$, and if we choose $x$ and $y$ as our parameteres, $r_{x} \times r_y = \langle -2x, -2y, 1 \rangle$, and if $S$ is the upper half of the cone $z^2 = x^2 + y^2$, so
\[ z = \sqrt{x^2 + y^2}, \] we can choose \( x \) and \( y \) as our parameters and \( \mathbf{r}_x \times \mathbf{r}_y = \frac{\mathbf{r}_x}{\sqrt{x^2 + y^2}}, \frac{\mathbf{r}_y}{\sqrt{x^2 + y^2}}, 1 >, \]

Of course, in each of these case, you should also find the region where the parameters come from. And you may need to change these sometimes a little bit. For example if we have a surface given by an equation like \( y = g(x, z) \), then it is easier to use \( x \) and \( z \) as parameters.

Once you have the parametrization of \( S \), say
\[ \mathbf{r}(u, v) = < x(u, v), y(u, v), z(u, v) >, \]
then

- You find the surface area of \( S \) by
\[ \text{area} = \int\int_{R} |\mathbf{r}_u \times \mathbf{r}_v| \, dA. \]

- If \( f(x, y, z) \) is a function, then we can take the integral of \( f \) over \( S \). This is called a surface integral, and we have had two different notations: in the book \( \iint_{S} f \, d\sigma \) and in webwork \( \iint_{S} f \, dS \).

\[ \text{integral of } f \text{ over } S = \int\int_{R} f |\mathbf{r}_u \times \mathbf{r}_v| \, dA. \]

So the area is equal to the surface integral of the constant function 1.

- If we have a vector field \( \mathbf{F} \), we can also take the integral of \( \mathbf{F} \) over \( S \). But to do so, we need to first fix an orientation of \( S \) say \( \mathbf{n} \). If we look at \( f = \mathbf{F} \cdot \mathbf{n} \), then we get a function whose value at every point is the scalar component of \( \mathbf{F} \) in the direction of \( \mathbf{n} \).
the integral of \( \mathbf{F} \) with respect to \( \mathbf{n} \) is defined to be the integral of the function \( f \). This is sometimes called the flux of \( \mathbf{F} \) with respect to \( \mathbf{n} \), and sometimes simply the surface integral of \( \mathbf{F} \) with respect to \( \mathbf{n} \).

There are two notations for the integral of a vector field over a surface: in the book \( \int_S \mathbf{F} \cdot \mathbf{n} \, d\sigma \), and in the webwork \( \int_S \mathbf{F} \cdot d\mathbf{S} \).

How do you find the integral of \( \mathbf{F} \)? You look at a parametrization, you look at \( \mathbf{r}_u \times \mathbf{r}_v \). This vector is always normal to the surface. if it was in the same direction as the orientation of \( S \), then \( \mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} \), so

\[
\text{integral of } \mathbf{F} \text{ with respect to } \mathbf{n} = \int \int_R \mathbf{F} \cdot \left( \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} \right) |\mathbf{r}_u \times \mathbf{r}_v| \, dA
\]

\[
= \int \int_R \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) \, dA
\]

If \( \mathbf{r}_u \times \mathbf{r}_v \) is in the opposite direction as the orientation of \( S \), then \( \mathbf{n} = -\frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} \), so

\[
\text{integral of } \mathbf{F} \text{ with respect to } \mathbf{n} = -\int \int_R \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) \, dA.
\]
Finally, Stokes' theorem says that if \( S \) is a surface with an orientation \( \mathbf{n} \), and \( C \) is the boundary of \( S \) which is oriented positively with respect to \( \mathbf{n} \), then for a vector field \( \mathbf{F} \), the line integral of \( \mathbf{F} \) on the boundary is equal to the surface integral of \( \text{curl} \ \mathbf{F} \) over \( S \), with respect to \( \mathbf{n} \).