Introduction to Algebraic Geometry, Spring 2018

Solutions to Selected Problems, HW 7–9

HW8, Problem 5. Let $C = \mathbf{V}(x^2yz + xy^3 - x^4 - y^4$. (There was a typo in the statement and the second term in $x^2yz + xy^3 - x^4 - y^4$ was written as xy^z.) Since the intersection of C and D consists of only one point: $(0:0:1)$, the multiplicity has to be $1 \times 4 = 4$ by Bezout's theorem.

HW8, Problem 7. Let L be the common tangent line to C and D at p. Note that after a linear change of coordinates, we can assume that $p = (1:0:0)$, $L = \mathbf{V}(z)$, and $q = (0:0:1)$ is not on L, C, D, or any of the lines joining 2 points of intersection of C and D. (to do so, first pick a point q with the given property, and pick a point r on L other than p. Then the three points p,q, and r give 3 linearly independent vectors in $\mathbf{C}^3$, and therefore, we can find an invertible matrix sending q to $(0:0:1)$, p to $(1:0:0)$, and r to $(0:1:0)$. Then L is sent to the unique line through $(1:0:0)$ and $(0:1:0)$ which is $\mathbf{V}(z)$.)

Now we write

$$f = a_n z^n + \cdots + a_1 z + a_0$$

and

$$g = b_m z^m + \cdots + b_1 z + b_0$$

where a_i is homogeneous of degree $n-i$ in x, y, and b_i is homogeneous of degree $m-i$ in x, y. Form the matrix A where $\det A = \mathrm{Res}(f, g, z)$. The only non-zero elements in the last column of A are $A_{m,n+m} = a_0$ and $A_{n+m,n+m} = b_0$. Since C and D pass through $(1:0:0)$, we have a_0 and b_0 are multiples of y.

Since the tangent line to C at p is given by

$$f_x(1,0,0)x + f_y(1,0,0)y + f_z(1,0,0)z = 0$$

and since we are assuming the tangent line is given by $z = 0$, we conclude, $f_y(1,0,0) = 0$. This implies that the coefficient of yx^{n-1} in f is zero (since it is he only possible term which could contribute to $f_y(1,0,0)$). This shows that a_0 is a multiple of y^2. Similarly b_0 is a multiple of y^2. Therefore, when we expand the determinant of A

with respect to the last column , we see that the determinant is a sum of polynomials which are multiples of y^2, and therefore the multiplicity at p is at least 2.

HW 9, Question 2. It follows from what we proved n class that the points of inflection of a curve C in $\mathbf{P}^2$ are the points p of C for which the determinant of the following symmetric matrix is zero:

$$A = \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix}$$

The determinant of A is a homogeneous polynomial g of degree $3(d-2)$ and therefore $D := \mathbf{V}(g)$ is of degree $3(d-2)$. Let $C = \mathbf{V}(f)$ where f is a homogeneous polynomial of degree d, then by Bezout's theorem, the number of points in $C \cap D$ is at most $3d(d-2)$.

HW 9, Question 3. By definition, the degree of a curve C in $\mathbf{P}^3$, is the maximum number of points in the intersection of C and a hyperplane in $\mathbf{P}^3$ (assuming C and the hyperplane intersect at a finite number of points.). A hyperplane H in $\mathbf{P}^3$ is given by equation $ax_0 + bx_1 + cx_2 + dx_3 = 0$ where a, b, c, d are constant. So the number of points in the intersection of C and H is exactly the number of solutions of the degree 3 homogeneous polynomial

$$a\, t^3 + b\, t^2 s + c\, ts^2 + d\, c^3$$

in $\mathbf{P}^1$, and the maximum of the number of solutions is 3. So C has degree 3.

HW9, Question 5. Set $f = y^2 z - x^3 - axz^2 - bz^3$. We know that the inverse of a point with coordinates $(x : y : z)$ is the point with coordinates $(x : -y : z)$. To find the coordinates of $2p$, we look at the line tangent to C at $p = (x_0 : y_0 : 1)$, find the 3rd point of intersection of the line with C, and $2p$ will be the inverse of that point. We should assume that $y_0 \neq 0$ for this problem. The equation of the tangent line at p is given by

$$\alpha x + \beta y + \gamma z = 0$$

where $\alpha = f_x(x_0, y_0, 1) = -3x_0^2 - a, \beta = f_y(x_0, y_0, 1) = 2y_0 \neq 0, \gamma = f_z(x_0, y_0, 1)$. The points of intersection of the tangent line with C is the common zero locus of the above linear polynomial and f. From the linear equation we get

$$y = -\frac{\alpha}{\beta} x - \frac{\gamma}{\beta} z$$

and from f we get

$$y^2 z = (-\frac{\alpha}{\beta}x - \frac{\gamma}{\beta}z)^2 z = x^3 + axz^2 + bz^3.$$

So the coordinates $(x : y : z)$ of the points of intersection satisfy

$$x^3 - \frac{\alpha^2}{\beta^2}x^2 z + (a - \frac{2\alpha\gamma}{\beta^2})xz^2 + (b - \frac{\gamma^2}{\beta^2})z^3 = 0.$$

Note that z cannot be zero, since then x would be zero too, and hence $y = \frac{\alpha}{\beta}x - \frac{\gamma}{\beta}z = 0$. So (after dividing the homogeneous coordinates by z), we can assume $z = 1$ for all the common roots, and x satisfies

$$x^3 - \frac{\alpha^2}{\beta^2}x^2 + (a - \frac{2\alpha\gamma}{\beta^2})x + (b - \frac{\gamma^2}{\beta^2}) = 0.$$

The sum of roots of the above polynomial is negative the coefficient of x^2 which is $\frac{\alpha^2}{\beta^2}$. Since x_0 is a root of multiplicity 2, so the x-coordinate of the the other root should be

$$\frac{\alpha^2}{\beta^2} - 2x_0 = (\frac{3x_0^2 + a}{2y_0})^2 - 2x_0.$$