1. What is the value of

$$\int_0^1 \frac{1}{\sqrt{1-x}}\,dx?$$

(a) 0

(b) $\sqrt{2}$

(c) 1

(d) -1

(e) 2

(f) It diverges.

$$\int_0^1 \frac{1}{\sqrt{1-x}}\,dx = \lim_{t\to 1^-} \int_0^t \frac{1}{\sqrt{1-x}}\,dx = \lim_{t\to 1^-} -2(1-x)^{\frac{1}{2}}\Big|_0^t$$

$$= \lim_{t\to 1^-} \left(-2(1-t)^{\frac{1}{2}}+2\right) = 2$$

2. Evaluate

$$\int_0^4 \frac{2}{3x-2}\,dx.$$

(a) $\frac{1}{\ln 2}$

(b) 0

(c) 1

(d) $-\ln 5$

(e) $\ln 4$

(f) It diverges.

$3x-2=0$ if $x=\frac{2}{3}$:

$$\int_0^4 \frac{2}{3x-2}\,dx = \int_0^{\frac{2}{3}} \frac{2}{3x-2}\,dx + \int_{\frac{2}{3}}^4 \frac{2}{3x-2}\,dx$$

$$\int_0^{\frac{2}{3}} \frac{2}{3x-2}\,dx = \lim_{t\to\frac{2}{3}^-} \int_0^t \frac{2}{3x-2}\,dx = \lim_{t\to\frac{2}{3}^-} \frac{2}{3}\ln|3x-2|\Big|_0^t$$

$$= \lim_{t\to\frac{2}{3}^-} \frac{2}{3}\ln|3t-2| - \frac{2}{3}\ln 2 = -\infty$$

The integral diverges

3. The average time to answer a phone call at a call center is 4 minutes. What is the probability that the phone is answered in more than 1 minute?

(a) $\frac{3}{4}$

(b) $e^{-\frac{1}{4}}$

(c) $e^{\frac{1}{4}}$

(d) $1 - \frac{1}{e}$

(e) $1 - \frac{4}{e}$

(f) $\frac{4}{e}$

$\mu = 4$, so $c = \frac{1}{4}$, so $f(x) = \frac{1}{4} e^{-\frac{x}{4}}$

$P(X > 1) = \int_1^\infty \frac{1}{4} e^{-\frac{x}{4}}\, dx = \lim_{t \to \infty} - e^{-\frac{x}{4}} \Big|_1^t = \lim_{t \to \infty} +e^{-\frac{1}{4}} - e^{-\frac{t}{4}} = e^{-\frac{1}{4}}$

4. What is the mean for the following probability density function?

$$f(x) = \begin{cases} 4x(1-x^2) & \text{if } \ 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) $\frac{1}{4}$

(b) $\frac{1}{2}$

(c) $\frac{1}{8}$

(d) 1

(e) $\frac{4}{21}$

(f) $\frac{8}{15}$

$$\text{mean} = \int_{-\infty}^{\infty} x\, f(x)\, dx = \int_0^1 4x^2(1-x^2)\, dx = \frac{4x^3}{3} - \frac{4x^5}{5} \Big|_0^1$$

$$= \frac{4}{3} - \frac{4}{5} = 4\, \frac{2}{15} = \frac{8}{15}$$

5. For what value of k is the following function a probability density function?

$$f(x) = \begin{cases} k\sin(\frac{\pi x}{2}) & \text{if } 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) $\frac{\pi}{2}$

(b) $\frac{\pi}{4}$

(c) $\frac{1}{\pi}$

(d) 1

(e) $\frac{2}{\pi}$

(f) 2π

$$1 = \int_{-\infty}^{\infty} f(x)\,dx = \int_0^2 k\sin\left(\frac{\pi x}{2}\right) dx = -k\cos\left(\frac{\pi x}{2}\right)\frac{2}{\pi}\Big|_0^2,$$

$$= -k\cancel{\cos}(\pi)\frac{2}{\pi} + k\cancel{\cos}0\frac{2}{\pi}$$

$$= \frac{2}{\pi}2k = \frac{4k}{\pi} \Rightarrow k = \frac{\pi}{4}$$

~~for $k=\pi/4$~~ if $0 \le x \le 2$ and $k > 0$ $k\sin\left(\frac{\pi x}{2}\right) \ge 0$.

6. If the sequence $\{a_n\}$ is defined by $a_n = n^2 e^{-5n}$, then what is

$$\lim_{n\to\infty} a_n?$$

(a) 0

(b) 1

(c) 5

(d) ∞

(e) e

(f) The limit does not exist.

$$f(x) = \frac{x^2}{e^{5x}}$$

$$\lim_{x\to\infty} f(x) \underset{\text{l'Hospital}}{=} \lim_{x\to\infty} \frac{2x}{5e^{5x}} \underset{\text{l'Hospital}}{=} \lim_{x\to\infty} \frac{2}{25e^{x}} = 0$$

so $\lim_{n\to\infty} \cancel{f}(n) = 0$

7. The sequence $\{a_n\}$ is defined by the recursive formula:
$a_1 = \sqrt{6}$, $a_{n+1} = \sqrt{6 + a_n}$. Suppose that

$$\lim_{n \to \infty} a_n = L.$$

What is L?

(a) 2

(b) $\sqrt{6}$

(c) 10

(d) $\sqrt{5}$

(e) 3

(f) $\frac{3}{2}$

If $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} a_{n+1} = L$, so $\lim_{n\to\infty} \sqrt{6+a_n} = L$

so $\sqrt{6+L} = L$ so $6+L = L^2$, so $L^2 - L - 6 = 0$, so

$(L-3)(L+2) = 0$ so $L=3$ or $L=-2$

not possible since all the terms are positive, ~~then~~ their limit is also positive.

8. What is

$$\sum_{n=1}^{\infty} \left(\frac{1}{2^n} - \frac{1}{4^{n-1}}\right)?$$

(a) 2

(b) $\frac{1}{4}$

(c) $\frac{1}{2}$

(d) $-\frac{1}{3}$

(e) 3

(f) $\frac{3}{2}$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$$

$a = \frac{1}{2}$
$r = \frac{1}{2}$

$$\sum_{n=1}^{\infty} \frac{1}{4^{n-1}} = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}$$

$a = 1$
$r = \frac{1}{4}$

$$1 - \frac{4}{3} = \frac{-1}{3}$$

9. Which of the following 3 series is convergent?

$$(1)\sum_{n=1}^{\infty}\frac{n}{\sqrt{n^4+n+2}},\qquad (2)\sum_{n=1}^{\infty}\sqrt[n]{15},\qquad (3)\sum_{n=1}^{\infty}\frac{\ln n}{n}$$

(a) only 1

(b) only 2

(c) only 3

(d) 1 and 2

(e) 1 and 3

(f) none of the series is convergent.

- $\frac{\ln n}{n} > \frac{1}{n}$ if $n \geq 3$ so ~~also~~ (3) is divergent
- $\lim_{n\to\infty} 15^{\frac{1}{n}} = 1 \Rightarrow$ (2) is divergent.
- Comparing $\overset{a_n}{\frac{n}{\sqrt{n^4+n+2}}}$ with $\overset{b_n}{\frac{1}{n}}$

$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\frac{n}{n^2\sqrt{1+\frac{n}{n^4}+\frac{2}{n^4}}}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n^3}+\frac{2}{n^4}}} = 1$, so (1) divergent.

10. Find all values of a for which the series

$$\sum_{n=1}^{\infty}(-1)^n\frac{a^{n+1}}{3^n}$$

converges.

(a) $-3 < a < 3$

(b) $\frac{-1}{3} < a < \frac{1}{3}$

(c) $-3 < a < 0$

(d) $-1 < a < 1$

(e) $-9 < a < 9$

(f) $\frac{-1}{9} < a < \frac{1}{9}$

$$\sum_{n=1}^{\infty}(-1)^n\frac{a^{n+1}}{3^n} = \sum_{n=1}^{\infty} -\frac{a^2}{3}\left(\frac{-a}{3}\right)^{n-1} = -\frac{a^2}{3}\sum_{n=1}^{\infty}\left(-\frac{a}{3}\right)^{n-1}$$

So the series is convergent if $-1 < -\frac{a}{3} < 1$, so

$-3 < -a < 3$, so $-3 < a < 3$

11. Which of the following 3 series is convergent?

$$(1) \sum_{n=1}^{\infty} \frac{1}{5^n - 1} \qquad (2) \sum_{n=1}^{\infty} \frac{n^2}{n^3 - 1} \qquad (3) \sum_{n=1}^{\infty} \frac{1}{n!}$$

(a) only 1

(b) only 2

(c) only 3

(d) 1 and 2

(e) 1 and 3

(f) 2 and 3

$\frac{1}{n!}$ $\quad n! = 1 \times 2 \times \cdots \times n > 2^{n-1}$ so $\frac{1}{n!} < \frac{1}{2^{n-1}}$

so $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges

b_n $\frac{1}{5^n - 1}$ $\quad a_n = \frac{1}{5^n}$ $\quad \lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{\frac{1}{5^n}}{\frac{1}{5^n - 1}} = \lim_{n \to \infty} \frac{5^n - 1}{5^n}$

$= \lim_{n \to \infty} 1 - \frac{1}{5^n} = 1$

so $\sum_{n=1}^{\infty} \frac{1}{5^n - 1}$ converges

$\lim_{n \to \infty} \cancel{\left(\frac{\frac{1}{n}}{\frac{1}{n^3 - 1}} \right)}$ $\quad \frac{n^2}{n^3 - 1} > \frac{1}{n}$ $\quad$ so $\sum_{n=1}^{\infty} \frac{n^2}{n^3 - 1}$ diverges

12. Find $\lim_{n\to\infty}(1+\frac{4}{n})^n$.

(a) $\frac{4}{e}$

(b) 1

(c) e

(d) 4

(e) e^4

(f) ∞

$a_n = (1+\frac{4}{n})^n \qquad \ln a_n = n \ln(1+\frac{4}{n})$

$$\lim_{x\to\infty} x \ln(1+\frac{4}{x}) = \lim_{x\to\infty} \frac{\ln(1+\frac{4}{x})}{\frac{1}{x}} \underset{\text{l'Hospital}}{=} \lim_{x\to\infty} \frac{\frac{1}{1+\frac{4}{x}} \cdot \frac{-4}{x^2}}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{4}{1+\frac{4}{x}} = 4$$

so $\lim_{n\to\infty} \ln a_n = 4$ so $\lim_{n\to\infty} a_n = e^4$

13. What is $\sum_{n=1}^{\infty} \frac{20}{n(n+2)}$?

(a) 13

(b) 14

(c) 15

(d) 16

(e) 17

(f) 18

$$\frac{20}{n(n+2)} = 10\left(\frac{1}{n} - \frac{1}{n+2}\right)$$

so if $S_n = \frac{20}{1\times 3} + \frac{20}{2\times 4} + \cdots + \frac{20}{n(n+2)}$, then

$$S_n = 10\left(\left(\frac{1}{1} - \cancel{\frac{1}{3}}\right) + \left(\frac{1}{2} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{6}}\right) + \cdots + \left(\cancel{\frac{1}{n-2}} - \cancel{\frac{1}{n}}\right) + \left(\cancel{\frac{1}{n-1}} - \frac{1}{n+1}\right) + \left(\cancel{\frac{1}{n}} - \frac{1}{n+2}\right)\right)$$

$$= 10\left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right)$$

So $\lim_{n\to\infty} S_n = 10\left(1 + \frac{1}{2}\right) = \frac{30}{2} = 15$

14. Which of the following sequences is bounded?

$$a_n = \frac{4n+1}{n+1} \qquad b_n = n - \frac{1}{n} \qquad c_n = \frac{\sin n}{n}$$

(a) only $\{a_n\}$

(b) only $\{a_n\}$ and $\{b_n\}$

(c) only $\{c_n\}$

(d) only $\{b_n\}$ and $\{c_n\}$

(e) only $\{a_n\}$ and $\{c_n\}$

(f) all of them

$0 < \frac{4n+1}{n+1} < 4$ bounded

$n < n - \frac{1}{n}$ not bounded

$-1 \leq -\frac{1}{n} < \frac{\sin n}{n} < \frac{1}{n} \leq 1$ bounded

15. If a and b are two positive integers such that
$$\frac{a}{b} = 0.4777777777...,$$
then what is a?

(a) 13

(b) 23

(c) 29

(d) 43

(e) 47

(f) 51

$$0.4777\ldots = \frac{4}{10} + \frac{7}{100} + \frac{7}{1000} + \cdots$$

$$= \frac{4}{10} + \frac{7}{100}\left(1 + \frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \cdots\right) = \frac{4}{10} + \frac{7}{100} \cdot \frac{1}{1 - \frac{1}{10}}$$

$$= \frac{4}{10} + \frac{7}{100} \cdot \frac{1}{\frac{9}{10}} = \frac{4}{10} + \frac{7}{90} = \frac{43}{90}$$

16. Let $S = \sum_{n=1}^{\infty} \frac{1}{n^3}$. Using the Remainder Estimate for the Integral Test, find the smallest value of n that will ensure the error in the approximation $S \approx S_n$ is less than 0.01.

(a) 7

(b) 8

(c) 9

(d) 10

(e) 11

(f) 12

$$R_n = S - S_n < \int_n^\infty \frac{1}{x^3}\,dx = \lim_{t\to\infty} \int_n^t \frac{1}{x^3}\,dx = \lim_{t\to\infty} \left. -\frac{x^{-2}}{2} \right|_n^t$$

$$= \lim_{t\to\infty} \left(-\frac{t^{-2}}{2} + \frac{n^{-2}}{2} \right) = \frac{1}{2n^2}$$

so we want: $\frac{1}{2n^2} < 0.01$, so $\frac{1}{2 \times 0.01} < n^2$

so $50 < n^2$, so $8 \leq n$

17. Let $\{a_n\}$ be a sequence. Which of the following 3 statements is true.

1 . If a_n is positive for all n and $\lim_{n\to\infty} a_n = 0$, then the series $\sum a_n$ converges.

2 . If $0 < a_n < 10$ for all n, then the sequence $\{a_n\}$ converges.

3 . If a_n is positive for all n and the sequence $\{a_n\}$ is decreasing, then $\{a_n\}$ converges.

(a) only 1

(b) only 2

(c) only 3

(d) 1 and 2

(e) 1 and 3

(f) 2 and 3

$a_n = \frac{1}{n}$ $\sum a_n$ divergent shows (1) and (2) are not correct.

If $0 < a_n$ and decreasing, then it is bounded and decreasing, ($0 < a_n \le a_1$) , so it has a limit.