1. What is the average value of the following function over $[0, \pi]$?

$$f(x) = \sin x - \sin 2x$$

(a) 0

(b) 2

(c) 4

(d) $\frac{1}{\pi}$

(e) $\frac{2}{\pi}$

(f) $\frac{4}{\pi}$

$$\text{average} = \frac{1}{\pi} \int_0^{\pi} (\sin x - \sin 2x)\, dx = \frac{1}{\pi} \left(-\cos x + \frac{\cos 2x}{2} \Big|_0^{\pi} \right)$$

$$= \frac{1}{\pi} \left(-\cos \pi + \frac{\cos 2\pi}{2} - \left(-\cos 0 + \frac{\cos 0}{2} \right) \right)$$

$$= \frac{1}{\pi} \left(1 + \frac{1}{2} + 1 - \frac{1}{2} \right) = \frac{2}{\pi}$$

2. Find c in $[0,2]$ such that $f(c)$ is the average value of the function $f(x) = x^2$ over $[0,2]$.

(a) $\frac{2}{3}$

(b) $\frac{4}{3}$

(c) 1

(d) $\sqrt{\frac{4}{3}}$

(e) $\sqrt{\frac{8}{3}}$

(f) $\sqrt{\frac{2}{3}}$

average $= \frac{1}{2} \int_0^2 x^2 \, dx = \frac{1}{2} \left(\frac{x^3}{3} \Big|_0^2 \right) = \frac{1}{2} \left(\frac{8}{3} \right) = \frac{4}{3}$

$f(c) = \frac{4}{3}$ $\quad c^2 = \frac{4}{3}$ $\quad c = \sqrt{\frac{4}{3}}$

3. What is the volume of the solid generated by rotating the region between the curve $y = \sqrt{x-1}$, $x = 5$, and $y = 0$ about the x-axis?

(a) $\frac{2\pi}{5}$

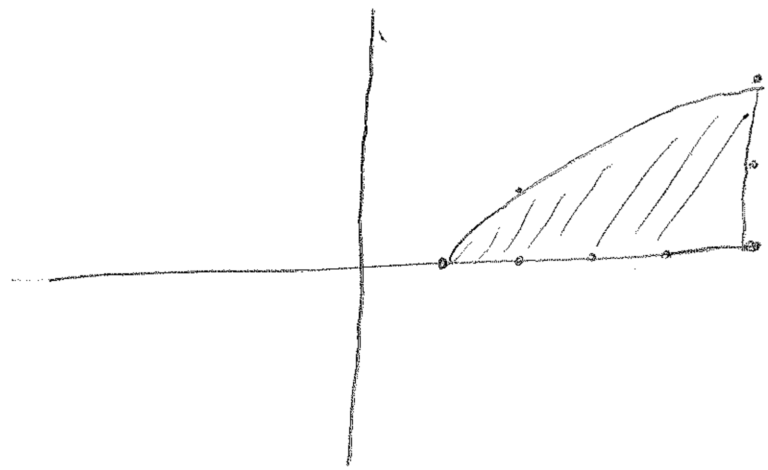
(b) $\frac{\pi}{2}$

(c) $\frac{2\pi}{3}$

(d) 2π

(e) 4π

(f) 8π



$$\text{volume} = \int_1^5 \pi\left(\sqrt{x-1}\right)^2 dx = \int_1^5 \pi(x-1)\,dx = \frac{\pi x^2}{2} - \pi x \Big|_1^5$$

$$= \frac{25\pi}{2} - 5\pi - \frac{\pi}{2} + \pi = 8\pi$$

4. Find the volume of the solid obtained by rotating the region bounded by the curves $y = \frac{1}{4}x^2$, $x = 2$, and $y = 0$ about the y-axis.

(a) $\frac{\pi}{6}$

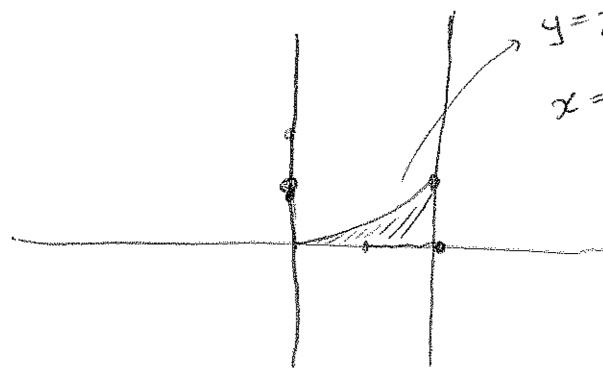
(b) $\frac{5\pi}{6}$

(c) $\frac{\pi}{2}$

(d) 2π

(e) 6π

(f) $\frac{10\pi}{3}$



Cross-section over y, $0 \le y \le 1$ is a washer.

$\left.\begin{array}{l}\text{inner radius} = 2\sqrt{y} \\ \text{outer radius} = 2\end{array}\right\}$ so the area $= \pi(4 - 4y)$

$$\text{Volume} = \int_0^1 \pi(4-4y)\,dy = 4\pi y - 2\pi y^2 \Big|_0^1 = 4\pi - 2\pi = 2\pi$$

5. What is the x-coordinate of the center of mass of the region bounded by the curves $y = 4x$, $y = 0$ and $x = 1$?

(a) $\frac{1}{2}$

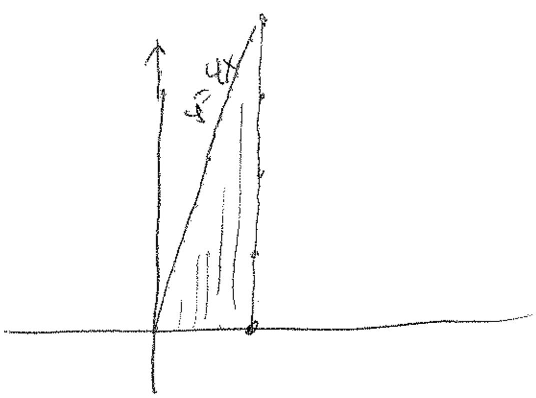
(b) $\frac{2}{3}$

(c) $\frac{1}{3}$

(d) $\frac{3}{4}$

(e) $\frac{5}{6}$

(f) $\frac{5}{8}$



area = 2

$$\bar{x} = \frac{\int_0^1 x(4x)\,dx}{2} = \frac{\int_0^1 4x^2}{2}$$

$$= \frac{\frac{4}{3}x^3\Big|_0^1}{2} = \frac{4}{6} = \frac{2}{3}$$

6. What is the y-coordinate of the center of mass of the region bounded by the curves $y = e^x$, $y = 0$, $x = 0$, and $x = 1$.

(a) $\frac{e+1}{4}$

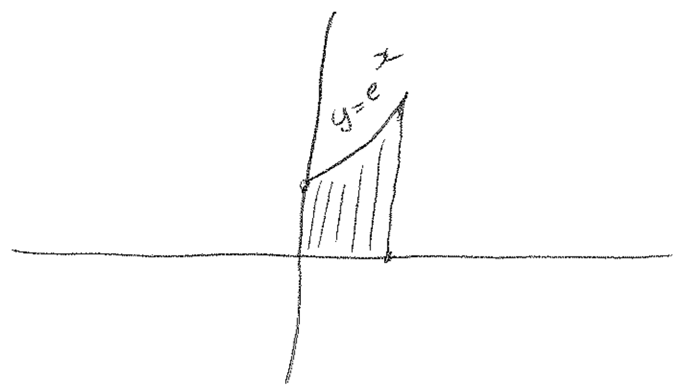
(b) e

(c) $\frac{2e}{3}$

(d) 1

(e) $\frac{2e+1}{2}$

(f) $2e - 1$



$$\text{area} = \int_0^1 e^x \, dx = e^x \Big|_0^1 = e - 1$$

$$\bar{y} = \frac{\int_0^1 \frac{1}{2} e^{2x} \, dx}{e-1} = \frac{\frac{1}{4} e^{2x} \Big|_0^1}{e-1} = \frac{\frac{1}{4}(e^2 - 1)}{e-1} = \frac{1}{4}(e+1)$$

7. Evaluate

$$\int_0^{\frac{\pi}{2}} \sin^3 x \ \cos^4 x \ dx.$$

(a) $\frac{2}{35}$

(b) $\frac{3}{25}$

(c) $\frac{2}{7}$

(d) 0

(e) $\frac{4}{15}$

(f) $\frac{1}{15}$

$u = \cos x \qquad du = -\sin x \, dx$

$\int \sin^3 x \ \cos^4 x \ dx = \int \sin x \ \sin^2 x \ \cos^4 x \ dx$

$= \int -(1-u^2)u^4 \, du = \int u^6 - u^4 \, du = \frac{u^7}{7} - \frac{u^5}{5} + C = \frac{\cos^7 x}{7} - \frac{\cos^5 x}{5} + C$

so $\int_0^{\frac{\pi}{2}} \sin^3 x \ \cos^4 x \ dx = \frac{\cos^7 \frac{\pi}{2}}{7} - \frac{\cos^5 \frac{\pi}{2}}{5} - \frac{\cos^7 0}{7} + \frac{\cos^5 0}{5}$

$= -\frac{1}{7} + \frac{1}{5} = \frac{2}{35}$

8. Use integration by parts to compute

$$\int_1^e x^2 \ln x \, dx.$$

(a) $\frac{1}{4}$

(b) $\frac{1}{4} + \frac{1}{4e}$

(c) $\frac{1}{4} - \frac{3}{4}e^{-2}$

(d) $\frac{1+e^{-4}}{7}$

(e) $\frac{e(e+1)}{7}$

(f) $\frac{2e^3+1}{9}$

$$\int_1^e \underbrace{x^2}_{f'(x)} \underbrace{\ln x}_{g(x)} \, dx = \frac{x^3 \ln x}{3}\Big|_1^e - \int_1^e \frac{x^3}{3} \frac{1}{x} \, dx$$

$$= \frac{e^3 \overset{1}{\cancel{\ln e}}}{3} - \frac{1^3 \overset{0}{\cancel{\ln 1}}}{3} - \frac{x^3}{9}\Big|_1^e$$

$f(x) = \frac{x^3}{3}$

$g'(x) = \frac{1}{x}$

$$= \frac{e^3}{3} - \left(\frac{e^3}{9} - \frac{1}{9}\right) = \frac{2e^3+1}{9}$$

9. What is

$$\int \frac{x^3 - 3x^2 - 9}{x^3 - 3x^2}\,dx?$$

(a) $\ln|x| - \ln|x-3| + x - 3x^{-1} + C$

(b) $\ln|x| - 2\ln|x-3| + x^{-1} + C$

(c) $2\ln|x| - 2\ln|x-3| + x^{-3} + C$

(d) $2\ln|x| + 2\ln|x-3| + x^{-1} + C$

(e) $\ln|x| - 4\ln|x-3| + x^{-2} + C$

(f) $2\ln|x| - 3\ln|x-3| + x^{-2} + C$

$$\frac{x^3-3x^2-9}{x^3-3x^2} = 1 - \frac{9}{x^2(x-3)} \qquad \frac{9}{x^2(x-3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-3}$$

$$= \frac{Ax(x-3) + B(x-3) + Cx^2}{x^2(x-3)}$$

so $9 = (A+C)x^2 + (-3A+B)x - 3B$ so

$A + C = 0 \Rightarrow C = -1$

$-3A + B = 0 \Rightarrow A = 1$

$-3B = 9 \Rightarrow B = 3$

so $\displaystyle\int \frac{x^3-3x^2-9}{x^3-3x^2}\,dx = \int 1 + \frac{1}{x} + \frac{3}{x^2} + \frac{-1}{x-3}\,dx$

$= x + \ln|x| - 3x^{-1} - \ln|x-3| + C$

10. Evaluate

$$\int_0^5 \frac{x^2}{\sqrt{25-x^2}}\,dx.$$

(a) 0

(b) $\frac{1}{5}$

(c) $\frac{2}{5}$

(d) $\frac{5\pi}{4}$

(e) $\frac{25\pi}{4}$

(f) $\frac{5\pi}{2}$

trig substitution $\quad x = 5\sin\theta \quad dx = 5\cos\theta\, d\theta$

$\sqrt{25-x^2} = \sqrt{25-25\sin^2\theta} = 5\cos\theta$

$x=0 \quad \theta = 0$

$x=5 \quad \sin\theta = 1 \quad \theta = \frac{\pi}{2}$

so $\displaystyle\int_0^5 \frac{x^2}{\sqrt{25-x^2}}\,dx = \int_0^{\frac{\pi}{2}} \frac{25\sin^2\theta}{5\cos\theta}\, 5\cos\theta\, d\theta = 25\int_0^{\frac{\pi}{2}} \sin^2\theta\, d\theta$

$\displaystyle = 25\left(-\frac{\cos\theta\sin\theta}{2} + \frac{\theta}{2} \Big|_0^{\frac{\pi}{2}} \right) = 25\,\frac{\pi}{4}$

11. If we write $\frac{3x+1}{(x+3)(x-5)}$ as a sum of partial fractions

$$\frac{A}{x+3} + \frac{B}{x-5},$$

what is B?

(a) -1

(b) 1

(c) 2

(d) -2

(e) 3

(f) -3

$$\frac{3x+1}{(x+3)(x-5)} = \frac{A(x-5)+B(x+3)}{(x+3)(x-5)}$$

$$3x+1 = A(x-5)+B(x+3) = (A+B)x + (-5A+3B)$$

$3 = A+B \Rightarrow A = -B+3$

$1 = -5A+3B \longrightarrow 1 = -5(-B+3) + 3B$

so $16 = 8B$

so $B = 2$

12. Evaluate the following integral
$$\int_{-2}^{1} \frac{1}{x^2+4x+13}\,dx.$$

(a) $\frac{\pi}{12}$

(b) $\frac{\pi}{3}$

(c) 0

(d) $\frac{1}{13}$

(e) $\frac{1}{3}$

(f) $\frac{\pi}{6}$

$x^2+4x+13 = (x+2)^2+9 \qquad u = x+2 \qquad du = dx$

$$\int \frac{1}{x^2+4x+13}\,dx = \int \frac{1}{u^2+9}\,du = \frac{1}{3}\arctan\frac{u}{3} = \frac{1}{3}\arctan\frac{x+2}{3}$$

so $$\int_{-2}^{1} \frac{1}{x^2+4x+13}\,dx = \frac{1}{3}\underbrace{\cancel{\arctan 1}}_{\frac{\pi}{4}} - \frac{1}{3}\underbrace{\cancel{\arctan 0}}_{0} = \frac{\pi}{12}$$

13. Evaluate

$$\int_1^4 \frac{e^{-\sqrt{x}}}{\sqrt{x}}\, dx.$$

(a) $\frac{1}{e^2}$

(b) $1 - e$

(c) $2e + 2$

(d) $\frac{2e-2}{e^2}$

(e) $\frac{2}{e}$

(f) $\frac{2}{e} + \frac{1}{e^2}$

$u = \sqrt{x}$ $\quad u^2 = x \quad 2u\,du = dx$

~~du~~

$$\int \frac{e^{-\sqrt{x}}}{\sqrt{x}}\, dx = \int \frac{e^{-u}}{u}\, 2u\, du = \int 2e^{-u}\, du = -2e^{-u} + C = -2e^{-\sqrt{x}} + C$$

So $\int_1^4 \frac{e^{-\sqrt{x}}}{\sqrt{x}}\, dx = -2e^{-\sqrt{x}}\Big|_1^4 = -2e^{-2} + 2e^{-1} = \frac{-2}{e^2} + \frac{2}{e} = \frac{+2e-2}{e^2}$

14. What is

$$\int_0^\infty xe^{-2x}\,dx?$$

(a) It diverges.

(b) $\frac{1}{8}$

(c) $\frac{1}{4}$

(d) e

(e) 0

(f) $\frac{1}{2}$

integration by parts

$$\int_0^t \underbrace{x}_{f(x)}\, \underbrace{e^{-2x}}_{g(x)}\, dx = -\left.\frac{xe^{-2x}}{2}\right|_0^t + \int_0^t \frac{e^{-2x}}{2}\, dx$$

$$g(x) = -\frac{1}{2}e^{-2x}$$

$$= -\frac{te^{-2t}}{2} + \left.\frac{-e^{-2x}}{4}\right|_0^t = -\frac{te^{-2t}}{2} - \frac{e^{-2t}}{4} + \frac{1}{4}$$

$$\lim_{t\to\infty} \frac{e^{-2t}}{4} = 0 \qquad \lim_{t\to\infty} \frac{te^{-2t}}{2} = \lim_{t\to\infty} \frac{t}{2e^{2t}} = \lim_{t\to\infty} \frac{1}{4e^{2t}} = 0$$

l'Hospital

$$\Rightarrow \int_0^\infty xe^{-2x}\,dx = \lim_{t\to\infty} \int_0^t xe^{-2x}\,dx = \frac{1}{4}$$

15. What is

$$\int_{-\infty}^{0} \frac{1}{6-5x}\,dx?$$

(a) It diverges.

(b) $\frac{1}{6}$

(c) $\frac{-1}{5}$

(d) $\frac{5}{6}$

(e) 1

(f) 0

$$\int_{-\infty}^{0} \frac{1}{6-5x}\,dx = \lim_{t\to-\infty} \int_{t}^{0} \frac{1}{6-5x}\,dx = \lim_{t\to-\infty} -\ln|6-5x|\Big|_{t}^{0}$$

$$= \lim_{t\to-\infty} -\ln 6 + \ln|6-5t|$$

$\lim_{t\to-\infty} \ln|6-5t| = \infty \Rightarrow$ the integral diverges