1. What is the average value of the function

$$f(x) = \cos x \ \sin^2 x$$

over $[0, \frac{\pi}{2}]$?

(a) $\frac{2}{3\pi}$

(b) $\frac{4\pi}{3}$

(c) $\frac{3}{2}$

(d) 0

(e) $\frac{\pi}{2}$

(f) $\frac{1}{2}$

$$\int_0^{\frac{\pi}{2}} \cos x \ \sin^2 x \ dx = \left.\frac{\sin^3 x}{3}\right|_0^{\frac{\pi}{2}} = \frac{1}{3}$$

$$\text{average value} = \frac{\frac{1}{3}}{\frac{\pi}{2}} = \frac{2}{3\pi}$$

2. Find the area of the region enclosed by $y = (x-1)^2$ and $y = 1 - x$.

(a) 1

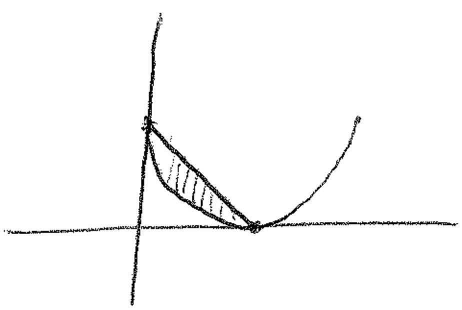
(b) $\frac{1}{2}$

(c) $\frac{3}{7}$

(d) $\frac{3}{8}$

(e) $\frac{1}{6}$

(f) $\frac{1}{8}$



$$\text{area} = \int_0^1 (1-x) - (x-1)^2 \, dx$$

$$= \int_0^1 x - x^2 \, dx = \frac{x^2}{2} - \frac{x^3}{3} \Big|_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

3. Compute the following integral

$$\int_1^2 x\sqrt{x-1}\,dx.$$

(a) 20

(b) 24

(c) $\frac{16}{15}$

(d) 32

(e) $\frac{3}{16}$

(f) $\frac{1}{8}$

$u = x - 1$
$du = dx$
$x = u + 1$

$$\int_0 x\sqrt{x-1}\,dx = \int (u+1)\sqrt{u}\,du = \int u^{\frac{3}{2}} + u^{\frac{1}{2}}\,du$$

$$= \frac{2}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}} = \frac{2}{5}(x-1)^{\frac{5}{2}} + \frac{2}{3}(x-1)^{\frac{3}{2}}$$

so $$\int_1^2 x\sqrt{x-1}\,dx = \frac{2}{5}(x-1)^{\frac{5}{2}} + \frac{2}{3}(x-1)^{\frac{3}{2}}\Big|_1^2 = \frac{2}{5} + \frac{2}{3} = \frac{16}{15}$$

4. What is the volume of the solid obtained by rotating the region bounded by the curves $y = x^3$, $y = x$, $x \geq 0$, about the x-axis?

(a) 14

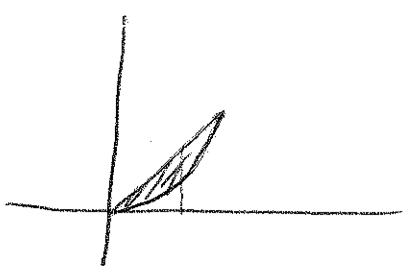
(b) 7π

(c) $\frac{4\pi}{3}$

(d) $\frac{14\pi}{3}$

(e) $\frac{4\pi}{21}$

(f) 14π



$$\text{volume} = \int_0^1 (x^2 - x^6)\,\pi\,dx$$

$$= \left(\frac{x^3}{3} - \frac{x^7}{7}\right)\pi \Big|_0^1 = \left(\frac{1}{3} - \frac{1}{7}\right)\pi = \frac{4\pi}{21}$$

5. Use integration by parts to evaluate the following integral

$$\int_0^1 x^2 e^{-x}\, dx.$$

(a) $e^2 - 3e$

(b) e^{-1}

(c) $e - 3$

(d) $\frac{3}{5e}$

(e) $\ln 2$

(f) $2 - \frac{5}{e}$

$$\int_0^1 x^2 e^{-x} dx = -x^2 e^{-x}\Big|_0^1 + \int_0^1 2x e^{-x} dx$$

$$= -x^2 e^{-x}\Big|_0^1 - 2x e^{-x}\Big|_0^1 + \int_0^1 2 e^{-x} dx$$

$$= -x^2 e^{-x}\Big|_0^1 - 2x e^{-x}\Big|_0^1 - 2 e^{-x}\Big|_0^1$$

$$= -e^{-1} - 2e^{-1} - 2e^{-1} + 2 = 2 - 5e^{-1}$$

6. What is the x-coordinate of the center of mass of the region bounded by the curves $y = \frac{5}{x}$, $y = 0$, $x = 1$, $x = 3$?

(a) 2

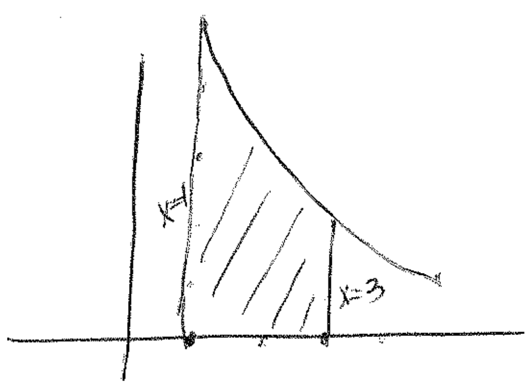
(b) 3

(c) $\frac{1}{\ln 4}$

(d) $\frac{2}{\ln 3}$

(e) $\ln 2$

(f) $\frac{1}{8}$



$$\bar{x} = \frac{\int_1^3 x \frac{5}{x}\, dx}{\int_1^3 \frac{5}{x}\, dx} = \frac{5x\big|_1^3}{5\ln|x|\big|_1^3}$$

$$= \frac{10}{5\ln 3} = \frac{2}{\ln 3}$$

7. Evaluate the following integral.

$$\int \frac{5x+7}{(x+1)(x+2)}\,dx$$

(a) $2\ln|x+1| + 2\ln|x+2| + C$

(b) $2\ln|x+1| + 4\ln|x+2| + C$

(c) $2\ln|x+1| - 4\ln|x+2| + C$

(d) $\ln|x+1| + 3\ln|x+2| + C$

(e) $2\ln|x+1| + 3\ln|x+2| + C$

(f) $2\ln|x+1| + \ln|x+2| + C$

$$\frac{5x+7}{(x+1)(x+2)} = \frac{a}{x+1} + \frac{b}{x+2} = \frac{(a+b)x + (2a+b)}{(x+1)(x+2)}$$

$a+b=5 \quad \Rightarrow \quad a = 5-b$

$2a+b=7 \quad \Rightarrow \quad 2(5-b)+b=7 \Rightarrow 10-b=7 \Rightarrow b=3$

so $a=2$

$$\int \frac{2}{x+1} + \frac{3}{x+2}\,dx = 2\ln|x+1| + 3\ln|x+2| + C$$

8. Evaluate the following integral

$$\int_{-1}^{0} \frac{1}{x^2+2x+2}\,dx.$$

(a) $\frac{\pi}{8}$

(b) $\frac{\pi}{4}$

(c) $\frac{\pi}{2}$

(d) π

(e) $\frac{\pi}{3}$

(f) 0

$$\int_{-1}^{0} \frac{1}{x^2+2x+2}\,dx = \int_{-1}^{0} \frac{1}{(x+1)^2+1}\,dx = \arctan(x+1)\Big|_{-1}^{0}$$

$$= \arctan 1 - \arctan 0$$

$$= \frac{\pi}{4}$$

9. Evaluate the following improper integral

$$\int_0^\infty \frac{x}{x^2+1}\,dx.$$

(a) It diverges.

(b) $\ln 2$

(c) 0

(d) 2

(e) 1

(f) e^2

$u = x^2+1 \qquad du = 2x\,dx$

$$\int \frac{x}{x^2+1}\,dx = \int \frac{1}{2u}\,du = \frac{1}{2}\ln|u| = \frac{1}{2}\ln|x^2+1|$$

so $$\int_0^\infty \frac{x}{x^2+1}\,dx = \lim_{t\to\infty}\int_0^t \frac{x}{x^2+1}\,dx = \lim_{t\to\infty}\frac{1}{2}\ln|t^2+1| - \cancel{\frac{1}{2}\ln|1|}^{\,0}$$

$$= \lim_{t\to\infty}\frac{1}{2}\ln|t^2+1| = \infty$$

10. Find the sum

$$\sum_{n=1}^{\infty} \frac{1}{3 \cdot 4^n}$$

(a) $\frac{1}{4}$

(b) $\frac{1}{3}$

(c) $\frac{1}{2}$

(d) $\frac{1}{9}$

(e) $\frac{1}{16}$

(f) $\frac{1}{12}$

$$\sum_{n=1}^{\infty} \frac{1}{3 \cdot 4^n} = \frac{\frac{1}{12}}{1-\frac{1}{4}} = \frac{\frac{1}{12}}{\frac{3}{4}} = \frac{1}{9}$$

geometric series

11. Which of the following series is convergent?

$$(1)\sum_{n=1}^{\infty} e^{-n} n! \qquad (2)\sum_{n=1}^{\infty}\left(\frac{-2n}{n+2}\right)^{3n} \qquad (3)\sum_{n=1}^{\infty}\frac{\sin n}{3^n}$$

(a) only (1)

(b) only (2)

(c) only (3)

(d) (1) and (2)

(e) (1) and (3)

(f) (2) and (3)

$\sqrt[n]{\left|\frac{-2n}{n+2}\right|^{3n}} = \left(\frac{2n}{n+2}\right)^3 \quad \lim_{n\to\infty}\left(\frac{2n}{n+2}\right)^3 = 2^3 = 8 > 1$, so (2) diverges

$\left|\frac{\sin n}{3^n}\right| < \frac{1}{3^n}$, so by the comparison Theorem $\sum \frac{\sin n}{3^n}$ is absolutely convergent, and therefore convergent.

$\lim_{n\to\infty} \frac{e^{-(n+1)}(n+1)!}{e^{-n}\, n!} = \lim_{n\to\infty} \frac{n+1}{e} = \infty > 1$, so (1) diverges

12. Which of the below statements is correct about the following two series?

$$(1) \sum_{n=2}^{\infty} (-1)^n \frac{1}{\ln n} \qquad (2) \sum_{n=1}^{\infty} (-1)^n \frac{1}{n^3 + n}$$

(a) both series are absolutely convergent.

(b) both series are divergent.

(c) both series are convergent, and none of them is absolutely convergent.

(d) (1) is absolutely convergent and (2) is divergent

(e) (1) is divergent and (2) is absolutely convergent.

(f) both series are convergent, and (2) is absolutely convergent.

$\sum_{n=1}^{\infty} \left| (-1)^n \frac{1}{n^3+n} \right| = \sum_{n=1}^{\infty} \frac{1}{n^3+n}$ $\quad \frac{1}{n^3+n} < \frac{1}{n^3}$, so by the comparso theorem (2) is absolutely convergent.

$\sum_{n=2}^{\infty} \frac{1}{\ln n}$ $\quad \frac{1}{\ln n} > \frac{1}{n}$, so by the comparison theorem (1) is NOT absolutely convergent, but it is convergent by Alternating series test.

13. If we write the Taylor series of $f(x) = \sqrt{x}$ at 1 as

$$f(x) = \sum_{n=0}^{\infty} c_n (x-1)^n,$$

then what is c_2?

(a) $\sqrt{2}$

(b) $\frac{1}{4}$

(c) $\frac{-1}{8}$

(d) $\frac{1}{64}$

(e) $\frac{\sqrt{2}}{64}$

(f) $\frac{1}{16}$

$$c_2 = \frac{f''(1)}{2!} = \frac{-\frac{1}{4}}{2} = \frac{-1}{8}$$

$$f(x) = x^{\frac{1}{2}} \quad f'(x) = \frac{1}{2} x^{-\frac{1}{2}} \quad f''(x) = \frac{-1}{4} x^{-\frac{3}{2}} \Rightarrow f''(1) = \frac{-1}{4}$$

14. Find the sum
$$\sum_{n=0}^{\infty} \frac{3^n}{2^n\, n!}.$$

(a) 1

(b) $\frac{1}{2}$

(c) $e^{\frac{3}{2}}$

(d) e^3

(e) $e^{\frac{-1}{2}}$

(f) It diverges.

$$\sum_{n=0}^{\infty} \frac{3^n}{2^n n!} = \sum_{n=0}^{\infty} \frac{\left(\frac{3}{2}\right)^n}{n!} = e^{\frac{3}{2}}$$

15. Let

$$S = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3}.$$

Using the Alternating Series Estimate, find the smallest n for which the estimate $S \approx S_n$ has an error of at most 0.01?

(a) 1

(b) 2

(c) 4

(d) 6

(e) 8

(f) 9

$\frac{1}{(1+n)^3} < \frac{1}{100}$ $\quad 100 < (n+1)^3$ $\quad n = \cancel{5}\ 4$

if $b_n = \frac{1}{n^3}$, then the error in the estimate

$S \approx S_n$ is at most ~~the~~ b_{n+1}

So $n=4$

16. Use power series to estimate the integral

$$\int_0^1 e^{-x^2}\,dx$$

with error < 0.05.

(a) $\frac{17}{30}$

(b) $\frac{23}{30}$

(c) $\frac{27}{30}$

(d) $\frac{29}{30}$

(e) $\frac{41}{30}$

(f) $\frac{49}{30}$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$$

$$\Rightarrow \int_0^1 e^{-x^2}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)\,n!}\Big|_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)\,n!} = 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \dots$$

$\frac{1}{10} > 0.05$ $\qquad \frac{1}{42} < 0.05$

So $\int_0^1 e^{-x^2}\,dx \approx \underbrace{1 - \frac{1}{3} + \frac{1}{10}}$ has error < 0.05

$$= \frac{23}{30}$$

17. Express the following indefinite integral as a power series.

$$\int \frac{\sin x}{x}\, dx$$

(a) $C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

(b) $C + \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n}}{(2n)!}$

(c) $C + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}$

(d) $C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!(2n+1)}$

(e) $C + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n+1}}{2n!}$

(f) $C + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n}}{(2n+2)!}$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \Rightarrow \frac{\sin x}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}$$

$$\Rightarrow \int \frac{\sin x}{x}\, dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)\ (2n+1)!}$$

18. What is the radius of convergence of the following power series?

$$\sum_{n=1}^{\infty} \frac{9^n x^n}{n^2}$$

(a) 1

(b) $\frac{1}{3}$

(c) $\sqrt{3}$

(d) 3

(e) $\frac{1}{9}$

(f) ∞

$$\lim_{n\to\infty} \left| \frac{\frac{9^{n+1} x^{n+1}}{(n+1)^2}}{\frac{9^n x^n}{n^2}} \right| = \lim_{n\to\infty} \left| 9x \cdot \frac{n^2}{(n+1)^2} \right| = 9|x|,$$

so if $9|x|>1$ the series diverges
if $9|x|<1$ " " converges
$\Rightarrow R = \frac{1}{9}$

19. What is the interval of convergence of the following series?

$$\sum_{n=1}^{\infty} \frac{x^n}{2n}$$

(a) $[-1, 1)$

(b) $(-1, 1)$

(c) $(-1, 1]$

(d) $[-2, 2]$

(e) $(-2, 2)$

(f) $(-2, 2]$

$$\lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{2(n+1)}}{\frac{x^n}{2n}} \right| = \lim_{n\to\infty} \left| x \frac{n}{n+1} \right| = |x| \Rightarrow R=1$$

$x=1$ $\sum_{n=1}^{\infty} \frac{1}{2n}$ diverges

$x=-1$ $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n}$ converges by the Alternating Series test

therefore interval $= [-1, 1)$