1. (1 pt) Library/Dartmouth/setMTCh7S1/problem_1.pg

Let C be the positively oriented square with vertices (0,0), (2,0), (2,2), (0,2). Use Green's Theorem to evaluate the line integral \( \int_C 8y^2 x\,dx + x^3y\,dy \).

2. (1 pt) Library/Dartmouth/setMTCh7S1/problem_1.pg

Let C be the positively oriented circle \( x^2 + y^2 = 1 \). Use Green's Theorem to evaluate the line integral \( \int_C 18xy\,dx + 8x\,dy \).


A)
Use Green's theorem to compute the area inside the ellipse \( \frac{x^2}{19^2} + \frac{y^2}{19^2} = 1 \).
Use the fact that the area can be written as
\[
\iint_D dx\,dy = \frac{1}{2} \int_{AB} -y\,dx + x\,dy.
\]
Hint: \( x(t) = 10\cos(t) \).
The area is __________.

B)
Find a parametrization of the curve \( x^{2/3} + y^{2/3} = 5^{2/3} \) and use it to compute the area of the interior.
Hint: \( x(t) = 5\cos^3(t) \).

From Rogawski ET section 16.3, exercise 17.
Determine whether the vector field is conservative and, if so, find the general potential function.

\[ F = \langle \cos z, 2y^3, -x\sin z \rangle \]
\[ \varphi = _______ + c \]
Note: If the vector field is not conservative, write "DNE".

From Rogawski ET section 17.1, exercise 23.
Evaluate \( I = \int_C (\sin x + 4y)\,dx + (12x + y)\,dy \) for the non-closed path ABCD in the figure.

Write down the iterated integral which expresses the surface area of \( z = y^3\cos^2 x \) over the triangle with vertices (-1,1), (1,1), (0,2):

\[
\int_a^b \int_{f(y)}^{g(y)} \sqrt{R(x,y)} \,dx\,dy
\]
\[ a = _______ \]
\[ b = _______ \]
\[ f(y) = _______ \]
\[ g(y) = _______ \]
\[ h(x,y) = _______ \]
7. (1 pt) Library/ASU-topics/sectCalculus/sec16.6p4.png

A sphere of radius 2 is centered at the origin. It may be viewed as a parametrized surface: \( r(\theta, \phi) = (2 \cos \theta \sin \phi, 2 \sin \theta \sin \phi, 2 \cos \phi) \), a level surface of the function \( f(x, y, z) = x^2 + y^2 + z^2 \), or as the graph of the function \( g(x, y) = \sqrt{4 - x^2 - y^2} \).

Consider the sphere at the point (1.00000,1.00000,1.41421) (corresponding to \((\theta, \phi) = (\pi/4, \pi/4)\)).

A) Find the normal vector \( n \times r_0 \) at the given point: 
(______________________________)

B) Find the gradient of \( f \) at the indicated point: 
(______________________________)

They should be parallel ...

Find the surface area of that part of the plane \( 8x + 10y + z = 8 \) that lies inside the elliptic cylinder \( \frac{x^2}{16} + \frac{y^2}{9} = 1 \)

Surface Area = _____________________

Show that \( \Phi(u, v) = (3u + 7, u - v, 5u + v) \) parametrizes the plane \( 2x - y - z = 14 \). Then:
(a) Calculate \( T_u, T_v, \) and \( n(u, v). \)
(b) Find the area of \( S = \Phi(D) \), where \( D = (u, v) : 0 \leq u \leq 8, 0 \leq v \leq 7. \)
(c) Express \( f(x, y, z) = yz \) in terms of \( u \) and \( v \) and evaluate \( \iiint_S f(x, y, z) \, dS \).
(a) \( T_u = \) __________, \( T_v = \) __________, \( n(u, v) = \) __________
(b) \( Area(S) = \) __________
(c) \( \iiint_S f(x, y, z) \, dS = \) __________

Find the surface area of the part of the sphere \( x^2 + y^2 + z^2 = 100 \) that lies above the cone \( z = \sqrt{x^2 + y^2} \)

\[ \int_S f(x, y, z) \, dS = \]
1. Green's Theorem implies: \[
\oint_C 8y^2 \, dx + x^2 y \, dy = \iint_R \frac{\partial}{\partial x} (xy) - \frac{\partial}{\partial y} (8y^2) \, dA
\]
\[
= \iint_R 2xy - 16yx \, dA = \iint_R -14xy \, dA = \int_0^2 \int_0^2 -14xy \, dy \, dx
\]
\[
= \int_0^2 \left. -7xy^2 \right|_0^2 \, dx = \int_0^2 -28x \, dx = -14x^2 \bigg|_0^2 = -56
\]

2. \( c: \) positively oriented unit circle.

Green's Theorem: \[
\oint_C 18y \, dx + 8x \, dy = \iint_R \frac{\partial}{\partial x} (8x) - \frac{\partial}{\partial y} (18y) \, dA
\]
\[
= \iint_R (8 - 18) \, dA = \iint_R -10 \, dA = -10 \text{ area of } R = -10 \pi
\]

3. A) A parametrization for ellipse \( \frac{x^2}{10^2} + \frac{y^2}{19^2} = 1 \) is

given by \( \mathbf{r}(t) = \langle 10 \cos t, 19 \sin t \rangle \quad 0 \leq t \leq 2\pi \)

\[
\mathbf{r}'(t) = \langle -10 \sin t, 19 \cos t \rangle
\]
area of $R = \iint_\mathcal{R} \, dA$

Green's Theorem: \[ \text{area} = \frac{1}{2} \oint_C (-y \, dx + x \, dy) = \frac{1}{2} \int_0^{2\pi} \left< -19 \sin t, 19 \cos t \right> \cdot \left< -10 \sin t, 19 \cos t \right> \, dt = \frac{1}{2} \int_0^{2\pi} 190 \sin^2 t + 190 \cos^2 t \, dt \]

\[ = \frac{1}{2} \int_0^{2\pi} 190 \, dt = \frac{1}{2} \times 2\pi \times 190 = 190 \pi \]

B) If $x(t) = 5 \cos^3 t$, then $x = 5 \frac{3}{3} \cos^2 t = 5 \frac{2}{3}$

so $5 \frac{2}{3} \cos^2 t + y = 5 \frac{2}{3} \sin^2 t$ so $y = 5 \frac{2}{3} (1 - \cos^2 t) = 5 \frac{2}{3} \sin^2 t$

so $y' = 5 \sin t$.

So a parametrization is given by $r(t) = <5 \cos^3 t, 5 \sin^3 t>$, $0 \leq t \leq 2\pi$

$v(t) = <-15 \sin t \cos t, 15 \cos^2 t \sin^2 t>$

area = $\frac{1}{2} \oint_C (-y, x) \cdot dr = \frac{1}{2} \int_0^{2\pi} \left< 5 \sin^3 t, 5 \cos^3 t \right> \cdot \left< -15 \sin t \cos t, 15 \cos^2 t \sin^2 t \right> \, dt$

$= \frac{1}{2} \int_0^{2\pi} (75 \sin t \cos^2 t + 75 \cos^2 t \sin^2 t) \, dt$
\[ \frac{1}{2} \int_{0}^{2\pi} 75 \left( \cos^2 t \sin^2 t \right) \, dt = \frac{1}{2} \int_{0}^{2\pi} 75 \frac{\sin(2t)}{4} \, dt \]
\[ = \frac{75}{2} \int_{0}^{2\pi} \frac{1 - \cos 4t}{8} \, dt = \left. \frac{75}{2} \left( \frac{1}{8} t + \frac{\sin 4t}{32} \right) \right|_{0}^{2\pi} \]
\[ = \frac{75}{2} \times \frac{1}{8} \times 2\pi = \frac{75}{8} \pi \]

4. \( \vec{F} = \langle \cos z, ay, -x \sin z \rangle \)

\[ \frac{\partial N}{\partial y} = 0 = \frac{\partial N}{\partial x}, \quad \frac{\partial M}{\partial z} = -\sin z = \frac{\partial P}{\partial x}, \quad \frac{\partial N}{\partial z} = 0 = \frac{\partial P}{\partial y} \]

So \( \vec{F} \) is conservative (and since it is defined everywhere, and the whole space is connected) it is a gradient field.

\[ \vec{F} = \nabla f \]

You can guess what the general form of \( f \) looks like by looking at \( \vec{F} \) or you can take integrals:

\[ \frac{df}{dx} = \cos z, \quad \text{so} \quad f(x, y, z) = \int \cos z \, dx \]

\[ f(x, y, z) = x \cos z + g(y, z) \quad (\star) \]
Since we know \( \frac{\partial f}{\partial y} = 2y^{17} \), we get
\[
\frac{\partial g}{\partial y} = \frac{\partial f}{\partial y} - \frac{\partial}{\partial y} (x \cos z) = 2y^{17}
\]
so
\[
g(y, z) = \int 2y^{17} \, dy = \frac{1}{9} y^{18} + h(z)
\]
so
\[
f(x, y, z) = x \cos(z) + \frac{1}{9} y^{18} + h(z) \tag{**}
\]
Since we know \( \frac{\partial f}{\partial z} = -x \sin(z) \), from the above line, we get
\[
\frac{\partial h}{\partial z} = \frac{\partial f}{\partial z} - \frac{\partial}{\partial z} (x \cos z + \frac{1}{9} y^{18}) = -x \sin(z) - x \cos z = 0
\]
\[\Rightarrow h \text{ is a constant function } c, \text{ so}
\]
\[
f(x, y, z) = x \cos z + \frac{1}{9} y^{18} + c \tag{***}
\]

5. You can either directly parametrize the three pieces and compute the line integral, or you can use Green's theorem in the following way:

If we look at the close curve \( \overline{A-B-C-D-A} \), using Green's Theorem we get:
\[ \vec{F} = \langle \sin x + 4y, 12x + y \rangle \]
\[ \int_{AB} \vec{F} \cdot d\vec{r} + \int_{BC} \vec{F} \cdot d\vec{r} + \int_{CD} \vec{F} \cdot d\vec{r} + \int_{DA} \vec{F} \cdot d\vec{r} \]
\[ = \iint_R \frac{\partial}{\partial x} (12x + y) - \frac{\partial}{\partial y} (\sin x + 4y) \, dA = \iint_R 12 - 4 \, dA \]
\[ = 8 \iint_R 1 \, dA = 8 \text{ area of } R = 8 \left( \frac{1}{a} + 1 + \frac{1}{a} \right) = 16 \]

So \[ \int_{AB} \vec{F} \cdot d\vec{r} + \int_{BC} \vec{F} \cdot d\vec{r} + \int_{CD} \vec{F} \cdot d\vec{r} = 16 - \int_{DA} \vec{F} \cdot d\vec{r} = 16 + \int_{AD} \vec{F} \cdot d\vec{r} \]

And \( \vec{AD} \) is parametrized by \( \vec{r}(t) = \langle 0, t \rangle \) \( 0 \leq t \leq 3 \)

So \( \vec{v}(t) = \langle 0, 1 \rangle \), so

\[ \int_{AD} \vec{F} \cdot d\vec{r} = \int_0^3 \langle \sin (0) + 4t, 12x0 + t \rangle \cdot \langle 0, 1 \rangle \, dt \]
\[ = \int_0^3 t \, dt = \frac{t^2}{2} \bigg|_0^3 = \frac{9}{2} \quad (a) \]

So \[ \iint_{ABCD} (\sin x + 4y) \, dx + (12x + y) \, dy = 16 + \frac{9}{2} = 20.5 \]

by \( a \) and \( (a) \)

6. The surface is parametrized by

\[ \vec{F} (x, y) = \langle x, y, y^3 \cos^5 (x) \rangle \]

\( (x, y) \) belongs to the triangle
So \( 1 \leq y \leq 2 \quad \text{and} \quad -2 \leq x \leq 2 - y \)

\[
\vec{r}_x = \langle 1, 0, -5 \sin(x) \cos(x) \rangle, \quad y^3
\]

\[
\vec{r}_y = \langle 0, 1, 3y^2 \cos(x) \rangle
\]

\[
\vec{r}_x \times \vec{r}_y = \langle +5 \sin(x) \cos(x) \rangle, \quad y^3, -3y^2 \cos(x), 1 \rangle
\]

\[
|\vec{r}_x \times \vec{r}_y| = \sqrt{1 + 25 \sin^2(x) \cos^2(x) y^6 + 9y^4 \cos^2(x)}
\]

\[
\text{Surface area} = \iint \sqrt{h(x, y)} \, dA = \int_1^2 \int_{y=2}^{y^2+2} \sqrt{h(x, y)} \, dA
\]

7. \( r(\theta, \phi) = \langle 2 \cos \theta \sin \phi, 2 \sin \theta \sin \phi, 2 \cos \phi \rangle \)

\[
\vec{r}_\theta = \langle -2 \sin \theta \sin \phi, 2 \cos \theta \sin \phi, 0 \rangle
\]

\[
\vec{r}_\phi = \langle 2 \cos \theta \cos \phi, 2 \sin \theta \cos \phi, -2 \sin \phi \rangle
\]

\[\vec{r}_\theta \cdot \vec{r}_\phi = \langle -\frac{\pi}{4}, \frac{\pi}{4} \rangle \]

\[
\vec{r}_\theta \times \vec{r}_\phi = \langle -\sqrt{2}, -\sqrt{2}, 0 \rangle
\]

\[
\vec{r}_\phi = \langle 1, 0, 1 \rangle
\]
\[ f(x, y, z) = x^2 + y^2 + z^2 \]
\[ \nabla f = \left< 2x, 2y, 2z \right> \]

at \( \theta = \frac{\pi}{4} \) \( \phi = \frac{\pi}{4} \) \[ \mathbf{r}(\theta, \phi) = \left< \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \sqrt{2} \right> \]

so \( \nabla f \) at \( \left( 1, 1, \sqrt{2} \right) = \left< 2, 2, 2\sqrt{2} \right> \)

they are parallel because they are both normal to the sphere (i.e. they are orthogonal to the tangent plane of the sphere).

8.

The points on the plane are parametrized by

\[ \mathbf{r}(x, y) = \left< x, y, 8 - 8x - 10y \right> \]

If we look at the points which are inside the elliptic cylinder (this means that the base of the cylinder is an ellipse instead of a circle), then we are saying that our parameters come from the base \( \mathcal{R} \) region \( \mathcal{R} \) enclosed by the ellipse \( \frac{x^2}{16} + \frac{y^2}{9} = 1 \).
\[ \vec{r}_x = \langle 1, 0, -8 \rangle \quad \vec{r}_x \times \vec{r}_y = \langle 8, 10, 1 \rangle \quad |\vec{r}_x \times \vec{r}_y| = \sqrt{165} \]

Therefore, the surface area of the part of the plane in the question is

\[
\text{area} = \iint_S \, dA = \iint_R |\vec{r}_x \times \vec{r}_y| \, dA = \iint_R \sqrt{165} \, dA = \sqrt{165} \iint_R \, dA
\]

\[ = \sqrt{165} \text{ area of } R = \sqrt{165} \times 11 \times 4 \times 3 = 12\sqrt{165} \text{ m}^2 \]

We showed in class using Cauchy's Theorem that the area enclosed by the ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \) is \( \pi ab \).

9. (a) \[ \vec{r}_u = \vec{r}_u = \langle 3, 1, 5 \rangle \]
\[ \vec{r}_v = \vec{r}_v = \langle 0, -1, 1 \rangle \]
\[ \vec{r}_u \times \vec{r}_v = \langle 6, -3, -3 \rangle \]
\[ |\vec{r}_u \times \vec{r}_v| = \sqrt{36 + 9 + 9} = \sqrt{54} \]

(b) The area of \( S = \iint_S \, dA = \iint_D \sqrt{54} \, dA \) where \( D \) is \( uv \)-plane: \( 0 \leq u \leq 8, \ 0 \leq v \leq 7 \)

\[ \text{So area} = \int_0^8 \int_0^7 \sqrt{54} \ dv \, du = \sqrt{54} \cdot 7 \cdot 8 = 56 \sqrt{54} \]
(c): \( f(x,y,z) = yz = (u-v)(5u+v) \)

\[
\int_S f \, dS = \int_0^8 \int_0^7 \sqrt{154} \ (u-v)(5u+v) \, dv \, du
\]

\[
= \sqrt{154} \int_0^8 \int_0^7 5u^2v^2 - 4uv \, dv \, du = \sqrt{154} \int_0^8 5u^2v - \frac{v^3}{3} - 2uv^2 \ |_0^7 du
\]

\[
= \sqrt{154} \int_0^8 35u^2 - \frac{7^3}{3} - 98u \, du = \sqrt{154} \left( \frac{35}{3} u^3 - \frac{7^3}{3} u - 49u^2 \right) |_0^8
\]

\[
= \sqrt{154} \left( \frac{35 \times 8^3}{3} - \frac{7^3 \times 8}{3} - 49 \times 8^2 \right)
\]

10. The cone and the sphere meet along the curve \( z = \sqrt{x^2+y^2}, x^2+y^2+z^2 = 100 \), so \( 2(x^2+y^2) = 100 \), so \( x^2+y^2 = 50 \) and \( z = \sqrt{50} \)

**Solution 1:** We use \( x, y \) as parameters; we get a parametrization of the part of the sphere above the cone \((x,y)\) belongs to the disk of radius \( \sqrt{50} \) around the origin on the xy-plane. Call it \( R \).

\[
\vec{r}(x,y) = \langle x, y, \sqrt{100-x^2-y^2} \rangle
\]

\[
\vec{r}_x = \langle 1, 0, - \frac{2x}{ \sqrt{100-x^2-y^2} } \rangle
\]

\[
\vec{r}_y = \langle 0, 1, - \frac{y}{ \sqrt{100-x^2-y^2} } \rangle
\]

\[
\vec{r}_x \times \vec{r}_y = \langle \frac{x}{ \sqrt{100-x^2-y^2} }, \frac{y}{ \sqrt{100-x^2-y^2} }, 1 \rangle
\]
\[ |r_\theta \times r_\phi| = \sqrt{\frac{x^2}{100-x^2y^2} + \frac{y^2}{100-x^2y^2} + 1} = \frac{10}{\sqrt{100-x^2y^2}} \]

So, the surface area

\[ S = \int_{\mathbb{R}} \frac{10}{\sqrt{100-x^2y^2}} \, dA \quad \text{polar coordinates} \]

\[ = \int_0^{2\pi} \int_0^{\sqrt{100}} \frac{10}{100-r^2} \, r \, dr \]

\[ = \int_0^{2\pi} \frac{10}{10} \, d\theta = \int_0^{2\pi} (100 - 10\sqrt{50}) \, d\theta \]

**Solution 2:** You can also use spherical coordinates:

Points on the sphere of radius 10 are parametrized by

\[ P(\theta, \phi) = <10 \sin \phi \cos \theta, 10 \sin \phi \sin \theta, 10 \cos \phi> \]

\[ |\overrightarrow{r}_\theta \times \overrightarrow{r}_\phi| = 100 \sin \phi \quad \text{(look at page 891 of the book)} \]

The equation of the cone in spherical coordinates is given by

\[ \rho \cos \phi = \frac{\sqrt{x^2 + y^2}}{\rho \sin \phi} \]

So, \( \cos \phi = \sin \phi \) so \( \phi = \frac{\pi}{4} \).
Therefore, the bounds for $s$ are given by
\[ 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \frac{\pi}{4}, \] so we get

\[
\text{surface area} = \int_0^{2\pi} \int_0^{\frac{\pi}{4}} 100 \sin \phi \, d\phi \, d\theta = \int_0^{2\pi} -100 \cos \phi \bigg|_0^{\frac{\pi}{4}} \, d\theta
\]

\[
= \int_0^{2\pi} 100 \left(1 - \frac{\sqrt{2}}{2}\right) \, d\theta = \left(100 - 50\sqrt{2}\right) 2\pi = \left(100 - 10\sqrt{50}\right) 2\pi.
\]

11. We use the parametrization $x$ and $z$.

\[ r^2(x, z) = \langle x, 4 - z^2, z \rangle \quad 0 \leq x \leq 6 \]
\[ 0 \leq z \leq 6 \]

\[
\vec{r}_x = \langle 1, 0, 0 \rangle \quad \vec{r}_z = \langle 0, -2z, 1 \rangle
\]

\[
|\vec{r}_x \times \vec{r}_z| = \sqrt{1 + 4z^2}
\]

\[
\int_{\frac{3}{2}} f \, d\tau = \int_0^{6} \frac{1}{6} \int_0^{6} \frac{1}{\sqrt{1 + 4z^2}} \, dz \, dx = \int_0^6 \frac{1}{12} \left(1 + 4z^2\right)^{\frac{3}{2}} \bigg|_{z=0}^6 \, dx
\]

\[ = \int_0^6 \frac{1}{12} \left(145 \frac{3}{2} - 1 \frac{3}{2}\right) \, dx = \frac{1}{2} \left(145 \frac{3}{2} - 1 \frac{3}{2}\right) \]