1. Let $E$ be the splitting field of $f(x) = x^4 - 2 \in \mathbb{Q}[x]$ and $G$ the Galois group of $E/\mathbb{Q}$. Describe the subfields of $E$ corresponding to the following subgroups of $G$: $< gh >$, $< gh^3 >$, and $< g, h^2 >$ (If $\rho = \sqrt{2}$, then $g$ is the automorphism which fixes $\rho$ and sends $i$ to $-i$, and $h$ is the automorphism which fixes $i$ and sends $\rho$ to $i\rho$.)

2. Find the Galois group of the following polynomials.
   (a) $x^3 + x^2 - 2x - 1$ over $\mathbb{Q}$
   (b) $x^3 - 10$ over $\mathbb{Q}(\sqrt{2})$

3. Suppose that $f(x) \in F[x]$ is an irreducible separable polynomial and $E$ is the splitting field of $f$. Then show that the Galois group of $E/F$ acts transitively on the roots of $f(x)$.

4. Let $E/F$ be a finite Galois extension and $G = \text{Gal}(E/F)$. If $H_1 \trianglelefteq H_2 \leq G$, then show that $H_1 \trianglelefteq H_2$ if and only if $E^{H_1}$ is a normal extension of $E^{H_2}$.

5. Let $f(x) = x^4 + ax^2 + b$ be an irreducible polynomial over $\mathbb{Q}$, with roots $\pm \alpha, \pm \beta$, and splitting field $E$.
   (a) Show that $\text{Gal}(E/\mathbb{Q})$ is isomorphic to a subgroup of the Dihedral group of order 8, $D_8$, and is therefore isomorphic to $\mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, or $D_8$.
   (b) Show that if $\alpha \beta \in \mathbb{Q}$, then $G = \mathbb{Z}_2 \times \mathbb{Z}_2$.
   (c) Show that $\frac{\alpha}{\beta} - \frac{\beta}{\alpha} \in \mathbb{Q}$ if and only if $G = \mathbb{Z}_4$.

6. Show that if the Galois group of an irreducible polynomial of degree 3 in $\mathbb{Q}[x]$ is $\mathbb{Z}_3$, then all the roots of the polynomial are real.