Complex Analysis, Spring 2018

Problem Set 6

Due: April 17 in class

1. Use Picard's theorem to show that a non-constant meromorphic function on $\mathbf{C}$ takes every value in $\mathbf{C}$ except for possibly 2 values. (Hint: every meromorphic function is of the from f/g where f and g are entire functions.)

2. Let $n \geq 3$. Show that if f and g are entire functions satisfying Fermat's equation

$$f^n + g^n = 1,$$

then f and g are constant. (Hint: consider f/g and use the previous question.)

3. Show that the family of injective holomorphic functions on $\mathbf{D}$ which omit one fixed value is normal in the generalized sense. (Hint: assume the omitted value is 0, and for every function f in the family, write $f = g^2$ for a holomorphic function g on $\mathbf{D}$. What values is g omitting?)

4. Let γ be a smooth non-intersecting curve. Then Green's theorem for area implies that the area of the region enclosed by γ is given by

$$\frac{1}{2i}\int_\gamma \bar{z}\, dz.$$

(to see this note that

$$\int_\gamma \bar{z}dz = \int_\gamma (x-iy)(dx+idy) = \int_\gamma xdx + ydy + i\int_\gamma xdy - ydx = 0 + i(2 * \text{area}).$$

where the last equality comes from Green's theorem.)

Let now f be an injective holomorphic map on an annulus $A_{1,R}$ (see Homework 5, Question 3) given by the Laurent series

$$f(z) = \sum_{n\in\mathbf{Z}} b_n z^n.$$

Let C_r be the circle of radius r, $1 < r < R$, around the origin.

(i) Use polar coordinates and Green's formula for the area to show that the area $A(r)$ of the region enclosed by $f(C_r)$ is

$$\frac{1}{2i} \int_{C_r} \bar{f}(z) f'(z)\, dz = \pi \sum_{n \in \mathbf{Z}} n |b_n|^2 r^{2n}.$$

(ii) Assuming parts (a)-(c) of Question 3 on homework 5, show that $\sum_n n|b_n|^2 = 1$.

(iii) Give another proof of the statement of the question by showing that $A(r) \geq \pi r^2$ for every r and hence $S \geq R$.