Complex Analysis, Spring 2018

Problem Set 1

Due: January 30 in class

1. Let K be a compact subset of $\mathbf{C}$ and S the set of continuous functions on U which can be approximated uniformly on K by polynomials.

(a) Show that if $f, g \in S$, then $f + g \in S$ and $gf \in S$.

(b) Show that if $f_n \in S$ and f is a function on U such that f_n converges to f uniformly on K, then $f \in S$.

2. Let $U \subset \mathbf{C}$ be a non-empty open set. Show that there is a sequence of compact subsets K_n of U such that $U = \cup_n K_n$ and K_n is contained in the interior of K_{n+1} for every $n \geq 1$. (Let K_n be the set of point z in U such that $|z| \leq n$ and the distance between z and the boundary of U is $\geq \frac{1}{n}$.)

3. Use Problem 2 to show that if $U \subset \mathbf{C}$ is an open subset, $\{a_n\}_{n=1}^{\infty}$ is a sequence of distinct points of U without a limit point in U, and p_n polynomials whose constant coefficients are zero, then there is a meromorphic function f on U such that the poles of f are exactly the a_n and the principal part of f at a_n is $p_n(\frac{1}{z-a_n})$.

4. Show that if f and g are entire functions with no common zeros, then there are entire functions r and s such that $fr + gs = 1$. (Such f and g are called *relatively prime.*) Hint: Write $\frac{1}{fg} = F + G$ where F only has poles at zeros of f and G has poles only at zeros of g.

5. Let $U \subset \mathbf{C}$ be an open subset, and let $K_n, n \geq 1$, be as in Problem 2. For continuous functions f, g on U, we define $d(f, g)$ as follows: For $n \geq 1$, let

$$\delta_n(f, g) = \max_{z \in K_n} \frac{|f(z) - g(z)|}{1 + |f(z) - g(z)|},$$

and

$$d(f,g) = \sum_{n=1}^{\infty} 2^{-n}\delta_n(f,g).$$

(a) Show that d is a metric on $\mathcal{C}(U)$, the space of continuos functions on U.

(b) Show that $f_n \to f$ in $(\mathcal{C}(U), d)$ if and only if f_n converges uniformly to f on all compact subsets of U.