1. Do the following exercises from the book.
   - Section 27: 25, 30
   - Section 29: 5, 31 (mimic the construction of the field with 4 elements: $\mathbb{Z}_2[X]/<x^2 + x + 1>$).

2. The content of a polynomial $f(x) = a_n x^n + \cdots + a_1 x + a_0 \in \mathbb{Z}[x]$ is the greatest common divisor of $a_0, \ldots, a_n$. The polynomial $f(x)$ is called primitive if its content is 1. For example, $x^2 + 4x + 2$ is a primitive polynomial since the greatest common divisor of 1, 4, 2 is 1.

   Given any polynomial $f(x)$, we can write $f(x) = c g(x)$ where $c$ is the content of $f(x)$ and $g(x)$ is primitive. Also, if the content of $f(x)$ is $a$, and if $h(x) = bf(x)$, then the content of $h(x)$ is $ab$.

   (a) Show that if $f(x)$ and $g(x)$ are primitive polynomials, then the product $f(x)g(x)$ is also primitive by showing that there cannot be any prime $p$ dividing all coefficients of $fg$.

   (b) Read the following argument which shows that if a primitive polynomial $f(x)$ can be factored as the product of two polynomials having rational coefficients, then it can be factored as the product of two polynomials having integer coefficients: suppose
   
   $$f(x) = p(x)q(x), \quad p(x), q(x) \in \mathbb{Q}[x].$$

   By clearing denominators and taking out common factors, we can write
   
   $$f(x) = \frac{a}{b} g(x)h(x) \quad g, h \in \mathbb{Z}[x], \quad a, b \in \mathbb{Z}$$

   such that $g(x)$ and $h(x)$ are both primitive. Then
   
   $$bf(x) = ag(x)h(x).$$

   But the content of the left hand side is $b$ since $f(x)$ is primitive, and the content of the right hand side is $a$ since $g(x)h(x)$ is primitive by part $a$. So $a = b$ and
   
   $$f(x) = g(x)h(x), \quad g, h \in \mathbb{Z}[x].$$
This is what we wanted to prove.

(c) Show that if \( f(x) \in \mathbf{Z}[x] \) is a polynomial which factors as the product of two polynomials with rational coefficients, then it factors as the product of two polynomials with integer coefficients.