1. At the extreme values of f, the level curves of f just touch the curve $g(x, y) = 8$ with a common tangent line. (See Figure 1 and the accompanying discussion.) We can observe several such occurrences on the contour map, but the level curve $f(x, y) = c$ with the largest value of c which still intersects the curve $g(x, y) = 8$ is approximately $c = 59$, and the smallest value of c corresponding to a level curve which intersects $g(x, y) = 8$ appears to be $c = 30$. Thus we estimate the maximum value of f subject to the constraint $g(x, y) = 8$ to be about 59 and the minimum to be 30.

4. $f(x, y) = 3x + y,\ g(x, y) = x^2 + y^2 = 10$, and $\nabla f = \lambda \nabla g \;\Rightarrow\; \langle 3, 1\rangle = \langle 2\lambda x, 2\lambda y\rangle$, so $3 = 2\lambda x$, $1 = 2\lambda y$, and $x^2 + y^2 = 10$. From the first two equations we have $\dfrac{3}{2x} = \lambda = \dfrac{1}{2y} \;\Rightarrow\; x = 3y$ (note that the first two equations imply $x \neq 0$ and $y \neq 0$) and substitution into the third equation gives $9y^2 + y^2 = 10 \;\Rightarrow\; y^2 = 1 \;\Rightarrow\; y = \pm 1$. Then f has possible extreme values at the points $(3, 1)$ and $(-3, -1)$. We compute $f(3, 1) = 10$ and $f(-3, -1) = -10$, so the maximum value of f on $x^2 + y^2 = 10$ is $f(3, 1) = 10$ and the minimum value is $f(-3, -1) = -10$.

6. $f(x, y) = xe^y,\ g(x, y) = x^2 + y^2 = 2$, and $\nabla f = \lambda \nabla g \;\Rightarrow\; \langle e^y, xe^y\rangle = \langle 2\lambda x, 2\lambda y\rangle$, so $e^y = 2\lambda x$, $xe^y = 2\lambda y$, and $x^2 + y^2 = 2$. First note that from the first equation $x \neq 0$. If $y = 0$, the second equation implies $x = 0$, so $y \neq 0$. Then from the first two equations we have $\dfrac{e^y}{2x} = \lambda = \dfrac{xe^y}{2y} \;\Rightarrow\; 2ye^y = 2x^2e^y \;\Rightarrow\; y = x^2$, and substituting into the third equation gives $x^2 + (x^2)^2 = 2 \;\Rightarrow\; x^4 + x^2 - 2 = 0 \;\Rightarrow\; (x^2 + 2)(x^2 - 1) = 0 \;\Rightarrow\; x = \pm 1$. From $y = x^2$ we have $y = 1$, so f has possible extreme values at $(\pm 1, 1)$. Evaluating f at these points, we see that the maximum value is $f(1, 1) = e$ and the minimum is $f(-1, 1) = -e$.

8. $f(x, y, z) = e^{xyz},\ g(x, y, z) = 2x^2 + y^2 + z^2 = 24$, and $\nabla f = \lambda \nabla g \;\Rightarrow\; \langle yze^{xyz}, xze^{xyz}, xye^{xyz}\rangle = \langle 4\lambda x, 2\lambda y, 2\lambda z\rangle$. Then $yze^{xyz} = 4\lambda x$, $xze^{xyz} = 2\lambda y$, $xye^{xyz} = 2\lambda z$, and $2x^2 + y^2 + z^2 = 24$. If any of x, y, z, or λ is zero, then the first three equations imply that two of the variables x, y, z must be zero. If $x = y = z = 0$ it contradicts the fourth equation, so exactly two are zero, and from the fourth equation the possibilities are $(\pm 2\sqrt{3}, 0, 0)$, $(0, \pm 2\sqrt{6}, 0)$, $(0, 0, \pm 2\sqrt{6})$, all with an f-value of $e^0 = 1$. If none of x, y, z, λ is zero then from the first three equations we have $\dfrac{4\lambda x}{yz} = e^{xyz} = \dfrac{2\lambda y}{xz} = \dfrac{2\lambda z}{xy} \;\Rightarrow\; \dfrac{2x}{yz} = \dfrac{y}{xz} = \dfrac{z}{xy}$. This gives $2x^2z = y^2z \;\Rightarrow\; 2x^2 = y^2$ and $xy^2 = xz^2 \;\Rightarrow\; y^2 = z^2$. Substituting into the fourth equation, we have $y^2 + y^2 + y^2 = 24 \;\Rightarrow\; y^2 = 8 \;\Rightarrow\; y = \pm 2\sqrt{2}$, so $x^2 = 4 \;\Rightarrow\; x = \pm 2$ and $z^2 = y^2 \;\Rightarrow\; z = \pm 2\sqrt{2}$, giving possible points $(\pm 2, \pm 2\sqrt{2}, \pm 2\sqrt{2})$ (all combinations). The value of f is e^{16} when all coordinates are positive or exactly two are negative, and the value is e^{-16} when all are negative or exactly one of the coordinates is negative. Thus the maximum of f subject to the constraint is e^{16} and the minimum is e^{-16}.

12. $f(x, y, z) = x^4 + y^4 + z^4,\ g(x, y, z) = x^2 + y^2 + z^2 = 1 \;\Rightarrow\; \nabla f = \langle 4x^3, 4y^3, 4z^3\rangle$, $\lambda \nabla g = \langle 2\lambda x, 2\lambda y, 2\lambda z\rangle$.

Case 1: If $x \neq 0$, $y \neq 0$, and $z \neq 0$ then $\nabla f = \lambda \nabla g$ implies $\lambda = 2x^2 = 2y^2 = 2z^2$ or $x^2 = y^2 = z^2 = \frac{1}{3}$ giving 8 points each with an f-value of $\frac{1}{3}$.

Case 2: If one of the variables is 0 and the other two are not, then the squares of the two nonzero coordinates are equal with common value $\frac{1}{2}$ and the corresponding f-value is $\frac{1}{2}$.

Case 3: If exactly two of the variables are 0, then the third variable has value ± 1 with corresponding f-value of 1.

Thus on $x^2 + y^2 + z^2 = 1$, the maximum value of f is 1 and the minimum value is $\frac{1}{3}$.

16. $f(x, y, z) = x^2 + 2y^2 + 3z^2$, $g(x, y) = x + 2y + 3z = 10$, and $\nabla f = \lambda \nabla g \Rightarrow \langle 2x, 4y, 6z \rangle = \langle \lambda, 2\lambda, 3\lambda \rangle$, so $2x = \lambda$, $4y = 2\lambda$, $6z = 3\lambda$, and $x + 2y + 3z = 10$. From the first three equations we have $2x = \lambda = 2y = 2z \Rightarrow x = y = z$, and substituting into the fourth equation gives $x + 2x + 3x = 10 \Rightarrow x = \frac{5}{3} = y = z$. Thus the only possible point for an extreme value of f is $\left(\frac{5}{3}, \frac{5}{3}, \frac{5}{3}\right)$. Notice here that the constraint $x + 2y + 3z = 10$ allows any of $|x|$, $|y|$, or $|z|$ to be arbitrarily large, and hence $f(x, y, z) = x^2 + 2y^2 + 3z^2$ can be made arbitrarily large. So f has no maximum value subject to the constraint. The minimum value is $f\left(\frac{5}{3}, \frac{5}{3}, \frac{5}{3}\right) = 6\left(\frac{5}{3}\right)^2 = \frac{50}{3}$.

22. $f(x, y) = 2x^2 + 3y^2 - 4x - 5 \Rightarrow \nabla f = \langle 4x - 4, 6y \rangle = \langle 0, 0 \rangle \Rightarrow x = 1, y = 0$. Thus $(1, 0)$ is the only critical point of f, and it lies in the region $x^2 + y^2 < 16$. On the boundary, $g(x, y) = x^2 + y^2 = 16 \Rightarrow \lambda \nabla g = \langle 2\lambda x, 2\lambda y \rangle$, so $6y = 2\lambda y \Rightarrow$ either $y = 0$ or $\lambda = 3$. If $y = 0$, then $x = \pm 4$; if $\lambda = 3$, then $4x - 4 = 2\lambda x \Rightarrow x = -2$ and $y = \pm 2\sqrt{3}$. Now $f(1, 0) = -7$, $f(4, 0) = 11$, $f(-4, 0) = 43$, and $f\left(-2, \pm 2\sqrt{3}\right) = 47$. Thus the maximum value of $f(x, y)$ on the disk $x^2 + y^2 \leq 16$ is $f\left(-2, \pm 2\sqrt{3}\right) = 47$, and the minimum value is $f(1, 0) = -7$.

29. Let the sides of the rectangle be x and y. Then $f(x, y) = xy$, $g(x, y) = 2x + 2y = p \Rightarrow \nabla f(x, y) = \langle y, x \rangle$, $\lambda \nabla g = \langle 2\lambda, 2\lambda \rangle$. Then $\lambda = \frac{1}{2}y = \frac{1}{2}x$ implies $x = y$ and the rectangle with maximum area is a square with side length $\frac{1}{4}p$.

43. If the dimensions of the box are given by x, y, and z, then we need to find the maximum value of $f(x, y, z) = xyz$ $[x, y, z > 0]$ subject to the constraint $L = \sqrt{x^2 + y^2 + z^2}$ or $g(x, y, z) = x^2 + y^2 + z^2 = L^2$. $\nabla f = \lambda \nabla g \Rightarrow$ $\langle yz, xz, xy \rangle = \lambda \langle 2x, 2y, 2z \rangle$, so $yz = 2\lambda x \Rightarrow \lambda = \dfrac{yz}{2x}$, $xz = 2\lambda y \Rightarrow \lambda = \dfrac{xz}{2y}$, and $xy = 2\lambda z \Rightarrow \lambda = \dfrac{xy}{2z}$.

Thus $\lambda = \dfrac{yz}{2x} = \dfrac{xz}{2y} \Rightarrow x^2 = y^2$ [since $z \neq 0$] $\Rightarrow x = y$ and $\lambda = \dfrac{yz}{2x} = \dfrac{xy}{2z} \Rightarrow x = z$ [since $y \neq 0$].

Substituting into the constraint equation gives $x^2 + x^2 + x^2 = L^2 \Rightarrow x^2 = L^2/3 \Rightarrow x = L/\sqrt{3} = y = z$ and the maximum volume is $\left(L/\sqrt{3}\right)^3 = L^3/\left(3\sqrt{3}\right)$.

44. Let the dimensions of the box be x, y, and z, so its volume is $f(x, y, z) = xyz$, its surface area is $2xy + 2yz + 2xz = 1500$ and its total edge length is $4x + 4y + 4z = 200$. We find the extreme values of $f(x, y, z)$ subject to the constraints $g(x, y, z) = xy + yz + xz = 750$ and $h(x, y, z) = x + y + z = 50$. Then $\nabla f = \langle yz, xz, xy \rangle = \lambda \nabla g + \mu \nabla h = \langle \lambda(y + z), \lambda(x + z), \lambda(x + y) \rangle + \langle \mu, \mu, \mu \rangle$. So $yz = \lambda(y + z) + \mu$ **(1)**, $xz = \lambda(x + z) + \mu$ **(2)**, and $xy = \lambda(x + y) + \mu$ **(3)**. Notice that the box can't be a cube or else $x = y = z = \frac{50}{3}$ but then $xy + yz + xz = \frac{2500}{3} \neq 750$. Assume x is the distinct side, that is, $x \neq y$, $x \neq z$. Then **(1)** minus **(2)** implies $z(y - x) = \lambda(y - x)$ or $\lambda = z$, and **(1)** minus **(3)** implies $y(z - x) = \lambda(z - x)$ or $\lambda = y$. So $y = z = \lambda$ and $x + y + z = 50$ implies $x = 50 - 2\lambda$; also $xy + yz + xz = 750$ implies $x(2\lambda) + \lambda^2 = 750$. Hence $50 - 2\lambda = \dfrac{750 - \lambda^2}{2\lambda}$ or

$3\lambda^2 - 100\lambda + 750 = 0$ and $\lambda = \dfrac{50 \pm 5\sqrt{10}}{3}$, giving the points $\left(\frac{1}{3}\left(50 \mp 10\sqrt{10}\right), \frac{1}{3}\left(50 \pm 5\sqrt{10}\right), \frac{1}{3}\left(50 \pm 5\sqrt{10}\right)\right)$.

Thus the minimum of f is $f\left(\frac{1}{3}\left(50 - 10\sqrt{3}\right), \frac{1}{3}\left(50 + 5\sqrt{10}\right), \frac{1}{3}\left(50 + 5\sqrt{10}\right)\right) = \frac{1}{27}\left(87{,}500 - 2500\sqrt{10}\right)$, and its maximum is $f\left(\frac{1}{3}\left(50 + 10\sqrt{10}\right), \frac{1}{3}\left(50 - 5\sqrt{10}\right), \frac{1}{3}\left(50 - 5\sqrt{10}\right)\right) = \frac{1}{27}\left(87{,}500 + 2500\sqrt{10}\right)$.