The Kadison-Singer Problem and the Uncertainty Principle

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Abstract: We endeavor to tell a story which weaves together the Kadison-Singer problem, the exponential functions, and the “Heisenberg” Uncertainty Principle. The goal is to compare and contrast what Kadison-Singer wants to tell us, and what Uncertainty does tell us.

Problem 1 (Kadison-Singer)**.** Can every pure state on the algebra of diagonal operators on $\ell^2(\mathbb{Z})$ be uniquely extended to all of $B(\ell^2(\mathbb{Z}))$?

Note 1. In their paper, Kadison and Singer suspect the answer is no.

Reformulation 1:

Problem 2 (Anderson-Akemann)**.** Can every bounded operator be paved? That is, given $T \in B(\ell^2(\mathbb{Z}))$ and $\epsilon > 0$, is there a finite partition $\{A_j\}_{j=1}^N$ of $\mathbb{Z}$ such that

$$\|Q_{A_j}(T - D(T))Q_{A_j}\| < \epsilon?$$

Here $D(T)$ is the diagonal of T, and Q_{A_j} is the canonical projection of $\ell^2(\mathbb{Z})$ onto $\ell^2(A_j)$.

(Well-known/Folklore?) Theorem

Theorem 1 (Casazza-Fickus-Tremain-Weber, GPOTS05)**.** *It suffices to show any of the following classes of operators are pavable:*

1. *Unitary operators*
2. *Orthogonal projections*
3. *Positive operators*
4. *Self-adjoint operators*
5. *Invertible operators*
6. *Gram matrices for unit norm Bessel sequences (defined below)*

Reformulation: Frames.

Let $\mathbb{J}$ be a countable index set. We say $\mathbb{X} := \{x_j\}_{j\in\mathbb{J}} \subset H$ is *Bessel* if the following (formally densely defined) operator is well-defined and bounded:

$$\Theta_{\mathbb{X}}^* : \ell^2(\mathbb{J}) \to H : (c_j)_j \mapsto \sum_{j\in\mathbb{J}} c_j x_j.$$

This is called the synthesis operator.

If $\mathbb{X}$ is Bessel, then

$$\Theta_{\mathbb{X}} : H \to \ell^2(\mathbb{J}) : v \mapsto (\langle v, x_j\rangle)_j.$$

This is called the analysis operator.

The set $\mathbb{X} = \{x_j\}_{j \in \mathbb{J}}$ is a *Riesz basic sequence* if

$$\Theta_{\mathbb{X}} \Theta_{\mathbb{X}}^* : \ell^2(\mathbb{J}) \to \ell^2(\mathbb{J})$$

is invertible. If so, there are constants $B_1, B_2 > 0$, called the Riesz basis bounds, such that

$$B_1 \|(c_j)\|_{\ell^2(\mathbb{J})} \leq \|\sum_j c_j x_j\|_H \leq B_2 \|(c_j)\|_{\ell^2(\mathbb{J})} \quad \forall (c_j) \in \ell^2(\mathbb{J}).$$

The set $\mathbb{X}$ is a *frame* if

$$\Theta_{\mathbb{X}}^* \Theta_{\mathbb{X}} : H \to H$$

is invertible. If so, there are constants $C_1, C_2 > 0$, called the frame bounds, such that

$$C_1 \|v\|_H^2 \leq \sum_j |\langle v, x_j \rangle|^2 \leq C_2 \|v\|_H^2 \quad \forall v \in H.$$

Note 2. Observe the squared norms in the frame bounds.

A Riesz basis for H is both a Riesz basic sequence and a frame, or equivalently, a complete Riesz basic sequence.

Definition 1. For $0 \leq \epsilon < 1$ (fixed), we say $\mathbb{X} = \{x_j\}_{j \in \mathbb{J}}$ is a $1+\epsilon$ Riesz basic sequence if the Riesz basis bounds B_1, B_2 of $\mathbb{X}$ satisfy $1 - \epsilon < B_1 \leq B_2 < 1 + \epsilon$.

The operator $\Theta_{\mathbb{X}} \Theta_{\mathbb{X}}^*$ is called the Grammian, or Gram matrix, for $\mathbb{X}$; the entries are given by $\langle x_j, x_k \rangle$.

The operator

$$\Theta_{\mathbb{X}}^* \Theta_{\mathbb{X}} = \sum_{j \in \mathbb{J}} \langle \cdot, x_j \rangle x_j .$$

is called the frame operator.

Reformulation 2:

Problem 3 (R_ϵ-conjecture; Casazza?)**.** If $\{x_j\}_{j \in \mathbb{J}}$ is a unit norm ($\|x_j\| = 1$) Bessel sequence, given any $\epsilon > 0$, is there a finite partition $\{A_k\}_{k=1}^N$ of $\mathbb{J}$ such that each subset $\{x_j\}_{j \in A_k}$ is a $1+\epsilon$ Riesz basic sequence?

Proof of Equivalence. If every operator is pavable, then so is the Gram matrix for $\mathbb{X}$; this matrix has 1's on the diagonal. After paving the Gram matrix for $\mathbb{X}$ and using the spectral mapping theorem, the submatrices given by the paving of the Gram matrix identify $1+\epsilon$ Riesz basic subsets of $\mathbb{X}$.

The converse is a combination of a dilation type argument with a bootstrapping argument. $\square$

Reformulation 3:

Problem 4 (Feichtinger)**.** If $\{x_j\}_{j\in\mathbb{J}}$ is a unit norm Bessel sequence, is there a finite partition $\{A_k\}_{k=1}^{N}$ such that the subsets $\{x_j\}_{j\in A_k}$ are Riesz basic sequences?

Note 3. This is equivalent to R_ϵ (and Paving), despite the fact that there are no Riesz basis constants! The proof (Casazza-Tremain, PNAS) of this equivalence winds through the strong and weak Bourgain-Tzafriri conjectures (hence are equivalent to them as well), and also uses important results of Weaver.

Special case: Discrete exponential frames

We let f_j denote the jth Fourier basis element of $\ell^2(\mathbb{Z}_N)$. If $E \subset \{0, \ldots, N-1\}$, χ_E is the indicator function of E, so $f_j\chi_E$ is f_j on E and 0 outside E.

We consider exponential frames of the form

$$\{f_j\chi_{E_N} : j \in F_N \subset \mathbb{Z}_N; E_N \subset \mathbb{Z}_N\}.$$

In particular, we consider a sequence of frames of this form.

What Kadison-Singer claims:

Theorem 2. *Suppose Kadison-Singer is true. For the pair of sequences* $\{E_N\}$, $\{F_N\}$, *with* $E_N, F_N \subset \{0, \ldots, N-1\}$ *and* $|E_N| = O(N)$, *then there exist constants* K, L *independent of* N *such that* F_N *can be partitioned into at most* K *subsets* $\{A_N^j\}$ *where* $\{f_l \chi_{E_N} | l \in A_N^j\}$ *is a Riesz basic sequence with lower bound greater than* L.

The classical uncertainty principle says there is a lower bound on how localized a function and its Fourier transform are.

Theorem 3 (Donoho-Stark)**.** *If* $v \in \ell^2(\mathbb{Z}_N)$ *(non-zero), then*

$$|\{k = 1, \ldots, N : v(k) \neq 0\}| \cdot |\{k = 1, \ldots, N : \hat{v}(k) \neq 0\}| \geq N.$$

Corollary 1. *If* $E, F \subset \{0, \ldots, N-1\}$ *and* $|F| < \dfrac{N}{N - |E|}$, *then* $\{f_j \chi_E : j \in F\}$ *is linearly independent, and therefore a Riesz basic sequence.*

What the Uncertainty Principle provides:

Theorem 4. *Let* $\{M_N\}$ *be a sequence of positive integers such that* $\alpha := \sup_N N - M_N < \infty$. *For the pair of sequences* $\{E_N\}$, $\{F_N\}$, *with* $E_N, F_N \subset \{0, \ldots, N-1\}$ *and* $|E_N| \geq M_N$, *then there exists a constant* $K := K(\alpha)$ *independent of* N *such that there is a partition of* $\{0, \ldots, N-1\}$ *into at most* K *subsets* $\{A_N^j\}$ *where* $\{f_l \chi_{E_N} : l \in A_N^j \cap F_N\}$ *is linearly independent (i.e. Riesz basic sequence).*

Note 4. Of course, we can choose $F_N = \{0, \ldots, N-1\}$. So, if we restrict the Fourier basis to a subset of its domain, we can partition that frame into finitely many linearly independent sets, and the number depends on the size of the subset of the domain.

Comparison and Constrast

1. growth rates (Kadison-Singer)

2. basis constants (Kadison-Singer)

3. selection of partition before E_N (Uncertainty Principle)

4. estimate on size of partition (Uncertainty Principle)

Special case: exponential frames

Problem 5. If $\phi \in L^{\infty}[0,1]$ (non-zero), is there a finite partition $\{A_k\}_{k=1}^{N}$ of $\mathbb{Z}$ such that the following sets are Riesz basic sequences:

$$\{\phi(\xi)e^{2\pi i\xi n} : n \in A_k\}?$$

Note 5. Here $\{\phi(\xi)e^{2\pi i\xi n} : n \in \mathbb{Z}\}$ is a bounded Bessel sequence.

Halpern-Kaftal-Weiss: Yes, if ϕ is Riemann integrable.

Laurent operators:

If $\phi \in L^\infty[0,1]$, we denote by $T_\phi \in B(\ell^2(\mathbb{Z}))$ the Laurent operator determined by the symbol ϕ.

Theorem 5 (Halpern-Kaftal-Weiss)**.** *If the symbol ϕ of the Laurent operator T_ϕ is bounded and Riemann integrable, then T_ϕ is pavable (in fact uniformly pavable).*

Theorem 6. *Suppose that* T_ϕ *is a Laurent operator, with normalized symbol* ϕ*. If* T_ϕ *is pavable, then the Bessel sequence* $\{e^{2\pi i n\xi}\phi(\xi) : n \in \mathbb{Z}\}$ *satisfies the Feichtinger conjecture. That is to say, there is a finite partition* $\{A_k\}_{k=1}^{K}$ *such that each subset* $\{e^{2\pi i n\xi}\phi(\xi) : n \in A_k\}$ *is a Riesz basic sequence. Moreover, given* $1 > \epsilon > 0$ *the partition may be chosen so that each subset has lower Riesz basis bound greater than* $1 - \epsilon$*.*

Theorem (Smith)**.** *If* $f \in L^2[0,1]$ *is nonzero, supported on* $V \subset [0,1]$ *and* $\hat{f}$ *is supported on* $W \subset \mathbb{Z}$*, then* $|V||W| \geq 1$*, where* $|V|$ *is the Lebesgue measure of* V*, and* $|W|$ *is the cardinality.*

Corollary 2. *If* $E \subset [0,1]$ *(measurable) and* $W \subset \mathbb{Z}$ *such that* $(1 - |E|)|W| < 1$*, then*

$$\{e^{2\pi in\xi}\chi_E(\xi) : n \in W\}$$

is linearly independent, and therefore a Riesz basic sequence.

Difficulties 1: Weak inequality.
The uncertainty principle between $\mathbb{Z}$ and $[0,1]$ is much too coarse.

Theorem 7 (Smith)**.** *If* $f \in L^2[0,1]$ *and* $\hat{f} \in \ell^2(\mathbb{Z})$ *are "essentially" localized on* $E \subset [0,1]$ *and* $F \subset \mathbb{Z}$*, respectively, then*

$$|E||F| \geq (1-\epsilon)^2.$$

Difficulties 2: Robustness.
The uncertainty principle is robust to perturbations. Paving is not!

Example 1. Let $E = [0, \frac{1}{2}]$; the projection P_E on $L^2([0,1])$ can be paved with $2\mathbb{Z}$ and $2\mathbb{Z}+1$. However, shift any $F \subset [0, \frac{1}{6}]$ by $\frac{2}{3}$, i.e.

$$E' = (E \setminus F) \cup (F + \frac{2}{3})$$

and the projection $P_{E'}$ is no longer paved by the same subsets of $\mathbb{Z}$!

Difficulties 3: Quantitative estimates.

Uncertainty principle does not provide for basis constants, which are required for paving. Are they really needed? (Recall Feichtinger conjecture.) For extending results from $\mathbb{C}^N$ to $L^2[0,1]$, they are needed.

What the Rado-Horn theorem provides:

Theorem 8 (Casazza-Christensen-Vershynin)**.** *For the pair of sequences* $\{E_N\}$, $\{F_N\}$, *with* $E_N, F_N \subset \{0, \ldots, N-1\}$ *and* $|E_N| = O(N)$, *there exists a constant* K *independent of* N *such that* F_N *can be partitioned into at most* K *subsets* $\{A_N^j\}$ *where* $\{f_l \chi_{E_N} | l \in A_N^j\}$ *is linearly independent.*

Note 6. No lower Riesz basis bound, as claimed by Kadison-Singer.