THE ISOPARAMETRIC STORY

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This is a slightly revised and updated version of the notes for the summer mini-course on isoparametric hypersurfaces I gave at National Taiwan University, June 25-July 6, 2012.

Due to the request of the participants of the summer course, I placed the emphasis on the homogeneous isoparametric hypersurfaces in the sphere. For this purpose, I outlined in Section 2, in considerable details, the classification of symmetric spaces before looking at the isotropy representation of a symmetric space of rank two, whose orbits form a 1-parameter family of homogeneous isoparametric hypersurfaces and its two focal manifolds, in the unit sphere of the tangent space of the symmetric space at the origin. Since the classification of symmetric spaces is so comprehensive that it can be overwhelming to a beginning student, the outline in these notes serves, I hope, as a clear overview of the ingredients, both algebraic and geometric with emphasis more on the latter, entailed in the classification.

As of this writing, the classification of isoparametric hypersurfaces has been completed [13]. The survey article [12] may be a good introduction to the background commutative algebra central to the classification.

1. **Early History of Isoparametric Hypersurfaces**

Wikipedia. In physics, a *wavefront* is the locus of points having the same phase: a line or curve in 2-d or a surface for a wave propagating in 3-d.

A typical example is the crests of ocean waves forming wave fronts. A skillful surfer, on the other hand, knows how to ride a wavefront below the crest.

Question 1. (**Laura**, 1918 [24])*: What are the wavefronts whose front speed remains constant on each front surface*?

The wave equation is

$$\Delta\phi = \frac{\partial^2\phi}{\partial t^2}$$

Wave fronts are *level surfaces* of ϕ, at each moment, which propagate along the normal directions of the level surfaces. That the front speed remains constant on each level surface means

$$|\nabla\phi| = \text{ change per unit length of } \phi \text{ along the normal} = a(\phi),$$
$$ds/dt = b(\phi),$$

for some smooth functions a and b, where s is the distance a front travels. Therefore,

$$\frac{\partial\phi}{\partial t} = \frac{\partial\phi}{\partial s}\frac{ds}{dt} = a(\phi)b(\phi) := c(\phi),$$
$$\Delta\phi = \frac{\partial^2\phi}{\partial t^2} = c'(\phi)c(\phi).$$

Definition 2. *A smooth function f over $\mathbb{R}^3$ is transnormal if*

$$|\nabla f| = A(f)$$

for some smooth function A. A transnormal function is isoparametric if

$$\Delta f = B(f).$$

Let c be a regular value of an isoparametric function f. The level surface $f^{-1}(c)$ (i.e., a regular level surface) is called an isoparametric surface.

Theorem 3. (**Somigliana**, 1918-1919 [40]) *A transnormal function f is isoparametric if and only if each regular level surface of f has constant mean curvature.*

Proof. For each regular level surface $M := f^{-1}(c)$ of a transnormal function f,

$$\mathbf{n} = \nabla f/|\nabla f| = \nabla f/A(f)$$

is a unit normal field to M. The *shape operator* S of the surface M is

$$S(X) := -d\mathbf{n}(X)$$

for a tangent vector X of M. However,

$$\begin{aligned}
&d(\nabla f)(X)\\
&= d(A(f)\mathbf{n})(X)\\
&= A'(f)df(X)\mathbf{n} + A(f)d\mathbf{n}(X), \quad \text{(Leibniz Rule)}\\
&= A(f)d\mathbf{n}(X), \quad (df = 0 \text{ over } M)\\
&= -A(f)S(X).
\end{aligned}$$

On the other hand, as a linear operator,

$$d(\nabla f) : X \mapsto \text{Hessian}(f)X,$$

where $\text{Hessian}(f) = (\partial_{ij}^2 f)$. Taking trace, we obtain

$$\begin{aligned}
&\Delta f \\
&= trace(d(\nabla f)) \\
&= -A(f)\, trace(S) + < d(\nabla f)(\mathbf{n}), \mathbf{n} > \\
&= -2A(f)H + A'(f)A(f), \quad H \text{ is the mean curvature of } M.
\end{aligned}$$

That is,

$$H = -(B(f) - A'(f)A(f))/2A(f)$$

is a constant along M if the transnormal f is also isoparametric. Conversely, if H is constant along regular level surfaces of the transnormal f, then H is a function of f and so Δf is a function of f so that f is isoparametric. □

Theorem 4. (**Somigliana**) *The regular level surfaces of a isoparametric function must be either all spheres, all cylinders or all planes.*

Remark 5. *This theorem was rediscovered later by Segre in* 1924 [38] *and Levi-Civita in* 1937 [25]. *The approach Levi-Civita gave is what we will look at next.*

Lemma 6. (**Levi-Civita**, 1937) *A transnormal f is isoparametric if and only if the two principal curvatures of each regular level surface are constant.*

Proof. Observe first that the integral curves of the unit normal field $\mathbf{n} = \nabla f/|\nabla f|$ are just line segments. In fact, an integral curve c of $\mathbf{n}$ from $f = a$ to $f = b$ assumes the length

$$\text{Length of } c = \int_a^b \frac{df}{|\nabla f|} = \int_a^b \frac{df}{A(f)}.$$

On the other hand, for any curve $\gamma(t), 0 \le t \le 1$, beginning and ending at the two end points of the given integral curve, we have

$$|\frac{df(\gamma(t))}{dt}| = |\langle \nabla f(\gamma(t)), \gamma'(t)\rangle| \le A(f(\gamma(t))|\gamma'(t)|,$$

so that

$$\text{Length of } \gamma = \int_0^1 |\gamma'(t)| dt \ge \int_0^1 \frac{1}{A(f)} \frac{df}{dt} dt = \int_a^b \frac{df}{A(f)}.$$

In other words, the given integral curve c assumes the shortest distance among all curves beginning and ending at its end points. That is, the integral curve is a line segment. In view of this observation, instead of using f to parametrize the level surfaces, we might as well use the arc length s of the normal lines of an initial level surface to parametrize other level surfaces:

$$M_s := M + s\mathbf{n}$$

is now the 1-parameter family of level surfaces of the transnormal f, where M is the initial level surface with unit normal field $\mathbf{n}$.

Let us calculate the mean curvature H_s of M_s by using the fact that $\mathbf{n}$ is still the unit normal to M_s. The upshot is [36, p. 209]

$$H_s = \frac{H - sK}{1 - 2sH + s^2K},$$

where k_1 and k_2 are the eigenvalues (the principal curvatures) of the shape operator S of M, so that $H = (k_1 + k_2)/2$, and $K = k_1k_2$ is the Gaussian curvature of M. Therefore, the mean curvature H_s is constant on M_s for all s, i.e., the transnormal f is isoparametric, if and only if H, K are constant on M, if and only if the principal curvatures k_1, k_2 are constant.
Case 1. $k_1 = k_2 \neq 0$. M is a sphere.
Case 2. $k_1 = k_2 = 0$. M is a plane.
Case 3. $k_1 \neq k_2$. One employs $dk_1 = dk_2 = 0$ and a bit more surface geometry to conclude $k_1k_2 = 0$ [36, p. 255], so that one of k_1, k_2 is zero. Then M is a cylinder. □

Theorem 7. (**Segre**, 1938 [39]) *The same conclusion holds on $\mathbb{R}^n$. That is, an isoparametric hypersurface, which is a regular level hypersurface of an isoparametric function f over $\mathbb{R}^n$ satisfying*

$$|\nabla f| = A(f), \quad \Delta f = B(f),$$

is either a hypersphere, a hyperplane, both are totally umbilic (one principal curvature), or a cylinder $S^k \times \mathbb{R}^{n-1-k}$.

Theorem 8. (**Cartan**, 1938 [5]) *The same conclusion holds on the hyperbolic space H^n of constant curvature -1. That is, an isoparametric hypersurface in H^n must be either a sphere, a hyperbolic H^{n-1}, a Euclidean $\mathbb{R}^{n-1}$ (called a horosphere), all three are totally umbilic, or a cylinder $S^k \times H^{n-k-1}$.*

Proof. (sketch) Show again that there are at most two (constant) principal values of the shape operator. Indeed, let $\lambda_1, \cdots, \lambda_{n-1}$ be the principal curvatures of an isoparametric hypersurface in a standard space form of dimension n with constant curvature C. Then we have

$$\sum_{j \neq k} m_j \frac{C + \lambda_k \lambda_j}{\lambda_k - \lambda_j} = 0, \quad \text{summed on } j, \tag{1}$$

referred to by Cartan as the "Fundamental Formula", which was proved by Segre in the Euclidean case and by Cartan in general [4, p. 84]. Here, m_j is the multiplicity of λ_j and $\lambda_i \neq \lambda_j$ if $i \neq j$.

$C = 0$. Let λ_k be the smallest positive principal curvature. Note that each term, if nontrivial, in the fundamental formula must be negative, which is a contradiction. Therefore, there are at most two principal curvatures, one of them is zero if there are two.

$C = -1$. It is easy to see that

$$\frac{C + \lambda_k \lambda_j}{\lambda_k - \lambda_j} < 0 \tag{2}$$

if $\lambda_j \leq 0$ and $\lambda_k > 0$. Consider those positive principal values. If there is a $0 < \lambda_l \leq 1$ such that $(\lambda_j)^{-1} \leq \lambda_l$ for all $\lambda_j > 1$, we let λ_k be the largest positive principal value ≤ 1. It follows that (2) is negative for all those $0 < \lambda_j < 1$ and for those $\lambda_j > 1$ not reciprocal to λ_k. We conclude that none of the positive λ_j other than the reciprocal of λ_k exist. Otherwise, there exists some $\lambda_j > 1$ such that its reciprocal is greater than the above λ_k, which we replace by the smallest principal value > 1. Once more, (2) is negative for all positive λ_j not reciprocal to λ_k. We arrive at the same conclusion as in the preceding case. So, we have at most two principal curvatures reciprocal to each other. In the case of two distinct principal curvatures λ and μ, the isoparametric hypersurface is the product of two simply connected space forms of constant curvatures $\lambda^2 - 1$ and $\mu^2 - 1$. □

Remark 9. *Say, $C = 1$. That is, the ambient space is the unit sphere in which the isoparametric hypersurface sits. Write*

$$\lambda_j = \cot(\theta_j).$$

Then the fundamental formula is nothing but

$$\sum_{j \neq k} \cot(\theta_k - \theta_j) = 0,$$

which carries a significant geometric meaning. Namely, in the spherical case, wavefronts, that is, the 1-parameter family of isoparametric hypersurfaces, eventually degenerate to two subspaces of smaller dimensions whose mean curvatures are zero.

The case $C = 1$ is remarkably deep! At this point, 94 years after Laura first investigated isoparametric surfaces, there remains the last case (out of infinitely many) to be classified:

Classify the isoparametric hypersurfaces in S^{31} with four principal curvatures of multiplicities $7, 7, 8, 8$.

Different fields of mathematics, such as differential geometry, algebraic geometry, algebraic topology, homotopy theory, K-theory, representation theory, etc., interplay in this arena.

Definition 10. *g is the number of principal curvatures of an isoparametric hypersurface in S^n.*

Theorem 11. (**Cartan**, 1939-1940 [6], [8])

$g = 1$. *This is the 1-parameter family of parallel hyperspheres degenerating to the North and South Poles, called the focal manifolds of the family.*

$g = 2$. *This is the 1-parameter family of generalized tori $S^k \times S^{n-k-1}$, whose points are*

$$(x_0, \cdots, x_k, x_{k+1}, \cdots, x_n),\ x_0^2 + \cdots + x_k^2 = r^2,\ x_{k+1}^2 + \cdots + x_n^2 = s^2,\ r^2 + s^2 = 1,$$

which degenerates to two focal manifolds S^k and S^{n-k-1} of radius 1 as r approaches 0 or 1.

$g = 3$.
- I. *The three principal values have equal multiplicity $m = 1, 2, 4$, or 8.*
- II. *In the ambient Euclidean space $\mathbb{R}^{n+1} \supset S^n$, there is a homogeneous polynomial F of degree 3, satisfying*

$$|\nabla F|^2 = 9r^2, \quad r \text{ is the Euclidean radial distance, } \quad \text{and,}$$
$$\Delta F = 0,$$

whose restriction to S^n is exactly the isoparametric function f. The range of f is $[-1, 1]$. Then ± 1 are the only critical values. Thus $f^{-1}(c), -1 < c < 1$, form a 1-parameter family of isoparametric hypersurfaces that degenerates to the two focal manifolds $f^{-1}(1)$ and $f^{-1}(-1)$.
- III. *The two focal manifolds are the real, complex, quaternionic, or octonion projective plane corresponding to the principal multiplicity $m = 1, 2, 4$, or 8. Each isoparametric hypersurface in the family is a tube around the projective plane.*
- IV. *Let $\mathbb{F}$ be one of the normed algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}$, and $\mathbb{O}$. Let $X, Y, Z \in \mathbb{F}$ and $a, b \in \mathbb{R}$. Then*

$$
F = a^3 - 3ab^2 + \frac{3a}{2}(X\overline{X} + Y\overline{Y} - 2Z\overline{Z}) + \frac{3\sqrt{3}b}{2}(X\overline{X} - Y\overline{Y}) + \frac{3\sqrt{3}}{2}((XY)Z + \overline{(XY)Z}) \tag{3}
$$

$g = 4$. *He assumed equal multiplicity m and classified the cases when $m = 1$ or 2.*

Question 12. (**Cartan**, 1940 [8])

(i) *What are the possible g?*
(ii) *Is equal multiplicity of principal values always true?*
(iii) *Are all isoparametric hypersurfaces homogeneous?*

2. Development in the early 1970s, the homogeneous case

Nomizu wrote two papers in the early 1970s [34], [35] that revived the interest in isoparametric hypersurfaces. At about the same time Takagi and Takahashi [42] classified homogeneous isoparametric hypersurfaces in spheres. We next report on Takagi and Takahashi's work, which is based on the comprehensive work of Cartan on the classification of symmetric spaces.

2.1. General structure theory of symmetric spaces.

Definition 13. *Let G be a topological group and X a topological space. G is called a topological transformation group on X if there is a continuous map*

$$\phi : G \times X \to X,$$

where we denote $\phi(g, x)$ by $g \cdot x$ for simplicity, such that

$$(gh) \cdot x = g \cdot (h \cdot x), \quad e \cdot x = x,$$

for all $g, h \in G$ and all $x \in X$; here e is the identity of G.

We say the action of G on X is effective if $g \cdot x = x$ for all $x \in X$ implies $g = e$, i.e., every nontrivial $g \in G$ moves at least some $x \in X$.

We say X is a homogeneous space of G if G acts transitively on X, i.e., for any two $x, y \in X$ there is a $g \in G$ such that $g \cdot x = y$.

Definition 14. *Let X be a homogeneous space of G. For $x \in X$ let*

$$K_x := \{g \in G : g \cdot x = x\}.$$

K_x is called the isotropy subgroup of G at x.

Example 15. *Let G be a topological group and let $K \subset G$ be a subgroup. The coset space $X := G/K$ equipped with the quotient topology with respect to the (open) projection map*

$$\pi : G \to G/K, \quad x \mapsto xK,$$

is a topological space. The transitive action of G on G/K is

$$g \cdot xK := (gx)K.$$

Furthermore, we know G/K is Hausdorff if and only if K is closed. Moreover, since $\pi : G \to G/K$ is an open map, G/K is locally compact if G is.

Note that the action is effective if and only if the only normal subgroup of G contained in K_x is the identity group.

Theorem 16. ([20, p. 121]) *Let X be a locally compact Hausdorff space which is a homogeneous space of a locally compact topological group with a countable base. Let K_x be the isotropy subgroup of G at $x \in X$. Then*

$$f : G/K_x \to X, \quad f : gK_x \mapsto g \cdot x, \tag{4}$$

is a homeomorphism from G/H onto X.

In our applications, G is a compact connected Lie group and X contained in a smooth manifold Y is a homogeneous space of G. It follows that G and X are both locally compact. Therefore, X can be homeomorphically identified with G/K_x.

Corollary 17. *Let G be a compact connected Lie group and let G be a smooth transformation group on Y. For each $x \in Y$, the orbit through x,*

$$O_x := \{g \cdot x\} \subset Y,$$

is diffeomorphic to the smooth manifold G/K_x as a regular submanifold of Y.

Proof. The isotropy group K_x is closed in G and so K_x is also a Lie group [20, p. 115]. Therefore, G/K_x can be given a standard smooth structure, which is the unique one that makes $\pi : G \to G/K_x$ a principal bundle; O_x then inherits a smooth structure through the homeomorphism f in (4). We need to establish that this smooth structure of O_x is the one induced from Y, i.e., that O_x is a regular submanifold of Y. To this end, consider

$$\pi : G \to G/K_x$$

as a principal bundle with fiber K_x, and let

$$F : G \to O_x, \quad F : g \mapsto g \cdot x.$$

Let $s : G/K_x \to G$ be a section of the principal bundle G. Then

$$f = f \circ id = f \circ \pi \circ s = F \circ s.$$

It follows that the homeomorphism f is in fact a smooth map into Y. Let us check that f_* is one-to-one at the origin $\circ := eK_x \in G/K_x$. Suppose this is not the case; let $v \neq 0$ be such that $f_*(v) = 0$ at $\circ$. Denote by L_g the left translation by g on both G/K_x and O_x. That is,

$$L_g(hK_x) = (gh)K_x, \quad L_g(h \cdot x) = (gh) \cdot x.$$

Then the fact that

$$f \circ L_g = L_g \circ f$$

implies that f_* annihilates the vector field $X := L_g * (v)$. Let $\gamma(t)$ be the integral curve of X with $\gamma(0) = \circ$. We obtain $f(\gamma(t)) \in Y$ is a constant and so $f(\gamma(t)) = x$, which implies that $f : G/K_x \to O_x$ is not one-to-one. This is a contradiction to f being a homeomorphism. Hence, f is an immersion. The compactness of G then implies that f is an embedding and O_x is a proper submanifold of Y. □

Remark 18. *O_x need not be a regular submanifold of Y if G is not compact. For instance, let T^2 be the torus $\mathbb{R}^2/\sim$ obtained by modding out the integral lattice. Denote a point in T^2 by $[x, y]$ to indicate it is the projection of $(x, y) \in \mathbb{R}^2$. Consider the action*

$$\mathbb{R} \times T^2 \to T^2, \quad (t, [x, y]) \mapsto [x + t, y + \tau t],$$

for some irrational number τ. Each orbit of the action is dense in T^2 and so cannot be a regular submanifold.

Definition 19. *A connected hypersurface M in a smooth manifold X is called* **homogeneous** *if $I(X, M)$, the group of isometries of X leaving M invariant, acts transitively on M.*

It is clear that for such a hypersurface, the principal curvatures of its shape operator are everywhere constant, counting multiplicities.

Definition 20. *A hypersurfaces in $\mathbb{R}^n, S^n$ or H^n, is called* **isoparametric** *if its principal curvatures are everywhere constant, counting multiplicities.*

Theorems 7 and 8 classify all isoparametric hypersurfaces in $\mathbb{R}^n$ and H^n to be exactly the homogeneous hypersurfaces in these space forms. What is interesting is then the spherical case.

Question 21. *Classify all isoparametric hypersurfaces in spheres.*

We start with understanding the homogeneous ones. Let $I(M)$ be the group of isometries of M and let $\iota : I(S^n, M) \to I(M)$ be the restriction map. Let $I_0(S^n, M)$ be the connected component of $I(S^n, M)$ and let $G := \iota(I_0(S^n, M))$.

Proposition 22. [37, II, p. 15] *$\iota : I_0(S^n, M) \to G$ is an isomorphism, so that $\iota^{-1} : G \hookrightarrow SO(n+1)$ is an effective representation (action) on $\mathbb{R}^{n+1}$ with M an orbit. Furthermore, M is compact, and so in particular G is compact and hence is a Lie group.*

Definition 23. *An effective representation $\rho : G \hookrightarrow SO(n+1)$ acting on $\mathbb{R}^{n+1}$ is said to be of* **cohomogeneity** *r if the smallest codimension of all orbits of ρ is r in $\mathbb{R}^{n+1}$.*

In particular, the representation ι above of a homogeneous hypersurface in S^n is of cohomogeneity 2.

Definition 24. *Given an effective representation $\rho : G \hookrightarrow SO(n+1)$ of cohomogeneity r, then ρ is called* **maximal** *if there is no effective representation $\rho_1 : G_1 :\hookrightarrow SO(n+1)$ of cohomogeneity r such that G is a proper subgroup of G_1 with $\rho(g) = \rho_1(g)$ for all $g \in G$.*

Proposition 25. [37, p. 16]

(1) *The effective representation $\iota : G \hookrightarrow SO(n+1)$ in Proposition 22 is a maximal effective representation of cohomogeneity 2.*
(2) *Let $\rho : G \hookrightarrow SO(n+1)$ be a maximal effective representation of cohomogeneity 2. Let M be a G-orbit of codimension 2 in $\mathbb{R}^{n+1}$. Then $\rho(G) = I_0(S^n, M)$.*
(3) *In particular, any maximal effective representation $\rho : G \hookrightarrow SO(n+1)$ is obtained as the representation of a homogeneous hypersurface in S^n, and*
(4) *Two homogeneous hypersurfaces M and N in S^n are equivalent, i.e., $N = f(M)$ for an $f \in O(n+1)$, if and only if $I_0(S^n, M) \simeq I_0(S^n, N)$ through the isomorphism $g \mapsto fgf^{-1}$.*

Therefore, the classification of homogeneous hypersurfaces in S^n is equivalent to first classifying maximal effective orthogonal representations $\rho : G \hookrightarrow SO(n+1)$ of cohomogeneity 2 and then classifying their orbits of codimension 2. Hsiang and Lawson classified all maximal orthogonal representations in [21]. They are closely tied with what are called the s-representations of symmetric spaces, which is what we will look at next. We will return to Hsiang and Lawson's work later.

Definition 26. *A Riemannian manifold M is called a symmetric space if for any $p \in M$, there is an isometry s_p of M that extends the local geodesic symmetry at p.*

Proposition 27. *A Riemannian symmetric space is homogeneous and complete.*

Proof. By successive "reflections" of the form s_p. □

Convention 28. *Let M be a Riemannian symmetric space. Let G be the identity component of the isometry group of M and let K be the isotropy subgroup of G at a point $\circ$ of M, fixed once and for all, to be called the origin of M. Then $M = G/K$. From now on we will assume implicitly this setup when we mention a Riemannian symmetric space.*

Note that K is compact [20, p. 204]. Moreover, if M is simply connected, then K is also connected by the homotopy exact sequence

$$\to \pi_1(M) \to \pi_0(K) \to \pi_0(G) \to \pi_0(M) \to 0.$$

Note also that since G is part of the isometry group, G must act on M effectively.

Definition 29. *Let G/K be a Riemannian symmetric space, we define the involution*

$$\sigma : G \to G, \quad \sigma(g) := s_\circ g s_\circ.$$

Proposition 30. *Let $M = G/K$ be a Riemannian symmetric space. Let G_σ be the element of G left fixed by σ. Then*

$$(G_\sigma)^0 \subset K \subset G_\sigma,$$

where $(G_\sigma)^0$ denotes the identity component of G_σ.

Proof. σ fixes K, because for $k \in K$, in a geodesic coordinate system, $s_\circ$ sends $k(x)$ to $-k(x) = k(-x)$, while k sends $s_\circ(x) = -x$ to $k(-x)$, so that $s_\circ k(x) = k s_\circ(x)$. Thus, $K \subset G_\sigma$.

On the other hand, for a 1-parameter group $\exp(tX) \in (G_\sigma)^0$, we have

$$s_\circ(\exp(tX)\cdot \circ) = \exp(tX)(s_\circ(\circ)) = \exp(tX)\cdot \circ,$$

so that $\exp(tX)\cdot \circ = \circ$, which gives $\exp(tX) \in K$. Hence $(G_\sigma)^0 \subset K$. □

Proposition 31. *Let G/K be a Riemannian symmetric space. Let $\mathcal{G}$ and $\mathcal{K}$ be their Lie algebras. Then there is a decomposition*

$$\mathcal{G} = \mathcal{K} \oplus \mathcal{M},$$

such that

$$[\mathcal{K}, \mathcal{M}] \subset \mathcal{M}, \quad [\mathcal{M}, \mathcal{M}] \subset \mathcal{K}, \quad [\mathcal{K}, \mathcal{K}] \subset \mathcal{K},$$

called the Cartan decomposition of $\mathcal{G}$.

Proof. By Proposition 30, the Lie algebra of G_σ is $\mathcal{K}$. Therefore,

$$d\sigma : k \in \mathcal{K} \mapsto k \in \mathcal{K},$$

where $d\sigma$ denotes the Jacobian map σ_* at $\circ$. So, $\mathcal{K}$ is contained in the eigenspace E_+ of $d\sigma$ with eigenvalue 1. Conversely, if $X \in E_+$, then $\sigma(\exp(tX))$ is a 1-parameter subgroup of G whose tangent vector at e is X. It follows that $\sigma(\exp(tX)) = \exp(tX)$ since $\exp(tX)$ is also a 1-parameter subgroup of G whose tangent vector at e is X, which means $\exp(tX) \in (G_\sigma)^0 \subset K$. So, $X \in \mathcal{K}$. That is, we have arrived at

$$E_+ = \mathcal{K}.$$

Since $(d\sigma)^2 = Id$, We have

$$\mathcal{G} = E_+ \oplus E_-,$$

where E_- denotes the eigenspace of $d\sigma$ with eigenvalue -1. Set

$$\mathcal{M} := E_-.$$

We thus have $\mathcal{G} = \mathcal{K} \oplus \mathcal{M}$.

Now,

$$d\sigma([\mathcal{K}, \mathcal{M}]) = [d\sigma(\mathcal{K}), d\sigma(\mathcal{M})] = [\mathcal{K}, -\mathcal{M}] = -[\mathcal{K}, \mathcal{M}],$$

so that $[\mathcal{K}, \mathcal{M}] \subset \mathcal{K}$. Likewise, we have the other two properties. (The last property is also a consequence of $\mathcal{K}$ being a Lie algebra.) □

Proposition 32. *Let G/K be a Riemannian symmetric space. Then there is an inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{G}$ such that*

$$\langle [X, Y], Z \rangle + \langle Y, [X, Z] \rangle = 0$$

for all $X \in \mathcal{K}$ and $Y, Z \in \mathcal{G}$.

Proof. Let us first recall the adjoint map. For any Lie group G, consider the map

$$\tau_g : x \in G \mapsto gxg^{-1} \in G.$$

The Jacobian map $(\tau_g)_*$ at e is denoted by $Ad(g) : \mathcal{G} \to \mathcal{G}$. The adjoint map

$$Ad : G \to GL(\mathcal{G})$$

defines an action of G on $\mathcal{G}$. We have the identity [20, p. 128]

$$Ad(\exp(X)) = e^{ad_X} \tag{5}$$

for $X \in \mathcal{G}$, where $ad_X := [X, \cdot]$. The intersection of the kernel of Ad and the identity component G^0 of G is the center of G^0.

Via the projection $\pi : G \to G/K, \pi(e) = \circ$, we identify $\mathcal{M}$ at e with the tangent space $T_\circ$ of G/K at $\circ$ by

$$(6) \qquad Y = \frac{d}{dt}|_{t=0} \exp(tY) \in \mathcal{M} \mapsto Y_\circ = \frac{d}{dt}|_{t=0} \exp(tY) \cdot p \in T_\circ,$$

and so $\mathcal{M}$ inherits the inner product $\langle \cdot, \cdot \rangle_{\mathcal{M}}$ from $T_\circ$. We claim that $\langle \cdot, \cdot \rangle_{\mathcal{M}}$ is $Ad(K)$-invariant. Indeed, for $k \in K$ and $Y \in \mathcal{M}$, we have

$$\tau_k(\exp(tY))(\circ) = k(\exp(tY) \cdot \circ),$$

so that

$$\frac{d}{dt}|_{t=0} \tau_k(exp(tY))(\circ) = dk(Y_\circ),$$

where dk denotes the Jacobian map $(\tau_k)_*$ at $\circ$, which is an orthogonal map as k is an isometry fixing $\circ$. However, by the definition of Ad

$$\frac{d}{dt}|_{t=0} \tau_k(\exp(tY)) = Ad(k)(Y).$$

That is, via the identification (6), we have that $Ad(k) : \mathcal{M} \to \mathcal{M}$ is orthogonal and so $\langle \cdot, \cdot \rangle_{\mathcal{M}}$ is $Ad(K)$-invariant.

Since $Ad(K) \in GL(\mathcal{K})$ is compact, we can define an $Ad(K)$-invariant inner product $\langle \cdot, \cdot \rangle_{\mathcal{K}}$ on $\mathcal{K}$. Set

$$\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_{\mathcal{K}} \oplus \langle \cdot, \cdot \rangle_{\mathcal{M}}.$$

$\langle \cdot, \cdot \rangle$ is $Ad(K)$-invariant. The proposition follows when we take the derivative of

$$\langle Ad(\exp(tX)(Y)), Ad(\exp(tX)(Z)) \rangle = \langle Y, Z \rangle$$

at $t = 0$. □

Definition 33. *Let $\mathcal{G}$ be a Lie algebra. The symmetric bilinear form*

$$B(X, Y) = tr(ad_X ad_Y)$$

is called the **Killing form** *of $\mathcal{G}$.*

Proposition 34.

$$B([X, Y], Z) + B(Y, [X, Z]) = 0$$

for all $X, Y, Z \in \mathcal{G}$.

Proof. $B(\sigma(X), \sigma(Y)) = B(X, Y)$ for an automorphism of $\mathcal{G}$. Observe that $Ad(\exp(tX))$ is an automorphism. □

Corollary 35. *Let G/K be a Riemannian symmetric space. Then the Killing form is negative-definitive on $\mathcal{K}$.*

Proof. Choose an inner product $< \cdot, \cdot >$ on $\mathcal{G}$ as given in Proposition 32. Then ad_X is skew-symmetric for $X \in \mathcal{K}$. Write ad_X as a matrix (a_{ij}). We have

$$B(X, X) = tr(ad_X ad_X) = -\sum_{ij}(a_{ij})^2 \leq 0$$

with equality if and only if $a_{ij} = 0$ for all i, j, if and only if $X \in \mathcal{Z}$, the center of $\mathcal{G}$, if and only if $\exp(X)$ lies in the center of G by (5), if and only if $\exp(X) = e$ since the action of G is effective, if and only if $X = 0$. □

We have derived enough motivation to give the following.

Definition 36. *Let G be a connected Lie group and K a closed subgroup of G. The pair (G, K) is called a* **Riemannian symmetric pair** *if there is an involutive automorphism σ of G with the following properties.*

(i) $(G_\sigma)^0 \subset K \subset G_\sigma$, *where G_σ is the set of fixed points of σ and $(G_\sigma)^0$ is its identity component.*
(ii) $Ad(K)$ *is compact, where Ad is the adjoint map of G.*

Furthermore, the symmetric pair is called effective if K contains no nontrivial normal subgroup of G.

The Lie algebra version of the above definition is the following.

Definition 37. *An* **orthogonal symmetric Lie algebra** *is a pair $(\mathcal{G}, \theta)$ satisfying the following three properties.*

(a) $\mathcal{G}$ *is a Lie algebra over $\mathbb{R}$.*
(b) θ *is an involutive automorphism of $\mathcal{G}$.*
(c) $\mathcal{K}$, *the set of fixed points of θ, is compactly embedded subalgebra of $\mathcal{G}$, i.e., $Int(\mathcal{K})$ (see the remark below) is compact in $GL(\mathcal{G})$.*

If furthermore $\mathcal{K} \cap \mathcal{Z} = 0$, where $\mathcal{Z}$ is the center of $\mathcal{G}$, then the orthogonal symmetric pair is called effective.

Remark 38. *Let $\mathcal{G}$ be a real Lie algebra. Then the Jacobi identity implies*

$$ad_X ad_Y - ad_Y ad_X = ad_{[X,Y]},$$

so that $ad(\mathcal{G})$ form a Lie subalgebra of $gl(\mathcal{G})$. The connected Lie subgroup of $GL(\mathcal{G})$ whose Lie algebra is $ad(\mathcal{G})$ is denoted by $Int(\mathcal{G})$, which is generated by elements of the form $e^{ad_X}, X \in \mathcal{G}$. Now $ad(\mathcal{K})$ is a Lie subalgebra of $ad(\mathcal{G})$ for a Lie subalgebra $\mathcal{K}$ of $\mathcal{G}$. The connected Lie subgroup of $Int(\mathcal{G})$ whose Lie algebra is $ad(\mathcal{K})$ is denoted by $Int(\mathcal{K})$, which is generated by elements of the form $e^{ad_X}, X \in \mathcal{K}$. This is compatible with (5).

A Riemannian symmetric space gives rise to an effective Riemannian symmetric pair (G, K) as detailed above. Its associated orthogonal symmetric algebra is $(\mathcal{G}, \theta)$, where $\theta := d\sigma$, the Jacobian map of σ at $\circ$. The converse is also true.

Theorem 39. *Given a Riemannian symmetric pair (G, K) with its associated symmetric algebra $(\mathcal{G}, \theta)$ with $\theta = d\sigma$. As in Proposition 32, let us choose an inner product $\langle\cdot,\cdot\rangle$ on $\mathcal{M}$ which is $Ad(K)$-invariant, where $\mathcal{M}$ is the eigenspace of θ with eigenvalue -1. Identify $\mathcal{M}$ at $e \in G$ with the tangent space $T_\circ$ at $\circ := eK \in G/K$, so that $\langle\cdot,\cdot\rangle$ is the inner product on $T_\circ$. Extend the inner product to a global one on G/K by*

$$\langle (L_g)_*(X_\circ), (L_g)_*(Y_\circ)\rangle = \langle X_\circ, Y_\circ\rangle;$$

here $L_g(xK) := (gx)K$ is the left translation by $g \in G$. The $Ad(K)$-invariance of $\langle\cdot,\cdot\rangle$ warrants that $\langle\cdot,\cdot\rangle$ is globally well-defined.

We define a "reflection" $s_\circ$ about $\circ$ on G/K by setting

$$s_\circ : gK \mapsto \sigma(g)K,$$

while for any $p := gK \in G/K$, we define a "reflection" about p by setting

$$s_p := L_g s_\circ (L_g)^{-1}.$$

Then s_p preserves the inner product $\langle\cdot,\cdot\rangle$ and turns $(G/K, \langle\cdot,\cdot\rangle)$ into a Riemannian symmetric space, so that it is an isometry.

Proof. We show s_p is an isometry. To see this, it suffices to look at $s_\circ$. It is readily checked that

$$ds_\circ = -Id,$$

where $ds_\circ$ denotes the Jacobian map of $s_\circ$ at $\circ$. Let $p := gK$. It is straightforward to see

$$s_\circ L_g = L_{\sigma(g)} s_\circ.$$

Hence, for $X = (L_g)_*(X_\circ), Y = (L_g)_*(Y_\circ)$ at T_p, we have

$$\begin{aligned}
&\langle (s_\circ)_*(X), (s_\circ)_*(Y)\rangle \\
&= \langle (L_{\sigma(g)})_*(s_\circ)_*(X_\circ), (L_{\sigma(g)})_*(s_\circ)_*(Y_\circ)\rangle \\
&= \langle (s_\circ)_*(X_\circ), (s_\circ)_*(Y_\circ)\rangle = \langle -X_\circ, -Y_\circ\rangle \\
&= \langle X_\circ, Y_\circ\rangle.
\end{aligned}$$

Therefore, G/K is a Riemannian symmetric space. □

Remark 40. *Note that G/K in Theorem 39 need not be an effective Riemannian symmetric pair for the above construction to go through. However, let N be the largest normal subgroup of G contained in K.*

Then $(G/N, K/N)$ is an effective symmetric pair. The involutive automorphism σ of G induces an automorphism on G/N in a natural way since $\sigma(N) = N$.

Theorem 39 is the recipe by which we construct examples of Riemannian symmetric spaces. Before we do that let us understand the curvature of a Riemannian symmetric space.

Theorem 41. *Let G/K be a Riemanian symmetric space. The curvature at $\circ$ is given by*

$$R(X,Y)Z = -[[X,Y],Z] \tag{7}$$

for $X, Y, Z \in \mathcal{M}$. Moreover, $\nabla R = 0$.

Proof. We claim that if we let $\gamma(t)$ be a geodesic emanating from $\circ$, then $f_t := s_{\gamma(t/2)} s_\circ$ has the property that $Y(t) := (f_t)_*(Y_\circ)$ is a parallel vector field along $\gamma(t)$. To see this, let Z be the parallel vector field along γ such that $Z = Y_\circ$ at $t = 0$. Since $s_{\gamma(t/2)}$ is an isometry, $W := (s_{\gamma(t/2)})_*(Z)$ is a parallel vector field along γ, which is equal to $-Z$ at $t/2$. It follows that $W = -Z$ everywhere. In particular, $(s_{\gamma(t/2)})_*(Y_\circ) = -Z(t)$. That is, $(f_t)_*(Y_\circ) = Z(t)$, proving the claim.

Now it is easy to see that $f_{t+s} = f_t \circ f_s$. So f_t is a 1-parameter group of isometries; $f_t = exp(tX)$ for some $X \in \mathcal{G}$. Consider the automorphism $\sigma : g \mapsto s_\circ g s_\circ$ of G. We find $\sigma(f_t) = f_{-t} = (f_t)^{-1}$. Taking derivative at $t = 0$, we see $\theta(X) = d\sigma(X) = -X$, so that we conclude $X \in \mathcal{M}$.

In conclusion, we have shown that the orbit of the 1-parameter group $\exp(tX)$ through $\circ$, for $X \in \mathcal{M}$, is a geodesic and $(\exp(tX))_*$ parallel translates any initial vector along the geodesic to form a parallel vector field on it.

$X \in \mathcal{M}$ generates the right-invariant vector field $\tilde{X}$ on G which projects to a Killing vector field X^* on G/K. We claim that

$$A_{X^*}(V) := (\nabla_V X^*)(\circ) = 0$$

for all $V \in \mathcal{M}$. To see this, consider the 2-parameter family of curves

$$c(t,s) = \exp(tX)\exp(sV)\cdot \circ.$$

$c(0,s)$ is a geodesic by the preceding paragraph. X^* restricted to $c(0,s)$ is $\partial c/\partial t(0,s)$. We have

$$(\nabla_V X^*)(\circ) = \frac{D}{ds}\frac{\partial c}{\partial t}(0,0) = \frac{D}{dt}\frac{\partial c}{\partial s}(0,0) = 0$$

because $c(t,0)$ is a geodesic through $\circ$ and $\partial c/\partial s(t,0)$ is exactly the parallel transport of V along $c(t,0)$ by the preceding paragraph.

As a consequence, for $X, Y \in \mathcal{M}$, the formula [23, I, p. 236]

$$R(X,Y)|_\circ = [A_{X^*}, A_{Y^*}] - A_{[X^*,Y^*]}$$

reduces the calculation of the curvature to

$$(\nabla_V[X^*,Y^*])(\circ) = \nabla_V[X,Y]^*(\circ)$$

for $V \in \mathcal{M}$.

Note that $[X,Y] \in \mathcal{K}$. Since K fixes $\circ$, we see $[X,Y]^*$ vanishes at $\circ$. We replace X in $c(t,s)$ by $[X,Y]$ so that now $c(t,0) = \circ$ for all t. A similar analysis as above shows that, in view of (5),

$$(\nabla_V[X,Y]^*)(\circ) = \frac{d}{dt}|_{t=0}\exp(t[X,Y])_*(V) = ad_{[X,Y]}(V) = [[X,Y],V].$$

Therefore,

$$R(X,Y)V = -[[X,Y],V].$$

Since ∇R is a tensor of type $(1,4)$ of odd degree, which is preserved by $ds_\circ = -Id$, it follows that $\nabla R = 0$. □

Example 42. *Let $\mathbb{R}^{n+1}$ be equipped with the Lorentzian form*

$$\langle X,Y\rangle = -x_0y_0 + x_1y_1 + \cdots + x_ny_n.$$

Let $O(1,n)$ be the Lorentzian group that preserves the Lorentzian form

$$O(1,n) = \{A \in GL(n+1,\mathbb{R}) : A^{tr}SA = S, \quad S := \begin{pmatrix} -1 & 0 \\ 0 & I_n \end{pmatrix}\},$$

where I_n is the n by n identity matrix. Its Lie algebra is

$$o(1,n) := \{M \in gl(n+1,\mathbb{R}) : M^{tr}S + SM = 0\}.$$

$\langle p,p\rangle = -1$ is a hyperboloid of two sheets. We choose the branch $x_0 \geq 1$ and call it H^n. The tangent space at $p \in H^n$ is

$$T_p = \{x \in \mathbb{R}^{n+1} : \langle p,x\rangle = 0\}.$$

H^n with the restriction of $\langle\cdot,\cdot\rangle$ on it, which is positive-definitive, is the n-dimensional non-Euclidean (hyperbolic) space.

Let G be the identity component of $O(1,n)$, which acts transitively on H^n. We find the isotropy subgroup of G at e_0 to be

$$K = \{\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}, \quad A \in SO(n)\}.$$

Then $H^n = G/K$. We define the involutive automorphism σ on G by

$$\sigma : Y \mapsto SYS^{-1}.$$

It is then seen that the fixed points of σ is exactly K. Therefore, Theorem 39 *gives that H^n is a symmetric space, whose Cartan decomposition is*

$$o(1,n) = \mathcal{K} \oplus \mathcal{M}, \tag{8}$$

where

$$\mathcal{M} := \{\begin{pmatrix} 0 & v^{tr} \\ v & 0 \end{pmatrix}, \quad v \text{ is a column vector } \in \mathbb{R}^n\}.$$

The action of K on $\mathcal{M}$ is the adjoint representation

$$\begin{pmatrix} 0 & v^{tr} \\ v & 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} 0 & v^{tr} \\ v & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}^{-1} = \begin{pmatrix} 0 & (Av)^{tr} \\ Av & 0 \end{pmatrix},$$

which is the standard orthogonal representation on $\mathbb{R}^n$. The Killing form is

$$B(X,Y) = -(n-1)tr(XY).$$

The curvature tensor is, by (7)*,*

$$R(X,Y) = -X \wedge Y : Z \mapsto \langle X, Z\rangle Y - \langle Y, Z\rangle X,$$

so that the sectional curvatures are all $= -1$.

Remark 43. *With a slight modification, the setup in the preceding example works for S^n as well. The only change is that we now have a positive-definite inner product on $\mathbb{R}^{n+1}$, $S = Id$ in the definition of $O(1+n)$, and $-v^{tr}$ is in place of v^{tr} in the definition of $\mathcal{M}$. The involution σ is identical with the one for H^n. The Killing form and the curvature are negative of those of H^n.*

This is not accidental, as we will see a duality later.

See more examples in [23, II, pp. 264-289]*, especially, for the complex Grassmann manifolds and the bounded symmetric domains, which are dual to each other.*

Definition 44. *Let $\mathcal{G}$ be a real* (*or complex*) *Lie algebra and let $B(X,Y)$ be its Killing form. $\mathcal{G}$ is called* **semisimple** *if $B(X,Y)$ is nondegenerate. A semisimple symmetric Lie algebra $\mathcal{G}$ is called* **compact** (**noncompact**) *type if $B|_{\mathcal{M}}$ is negative-definite* (*positive-definite*) *in the Cartan decomposition $\mathcal{G} = \mathcal{K} \oplus \mathcal{M}$.*

For instance, the symmetric Lie algebra in (8) for H^n is of noncompact type and is of compact type for S^n.

Theorem 45. [20, p. 231] *Let $(\mathcal{G}, \mathcal{K}, \theta)$ be an effective orthogonal symmetric Lie algebra. Then it is decomposed into the direct sum of three effective orthogonal symmetric Lie algebras*

$$(\mathcal{G}, \mathcal{K}, \theta) = (\mathcal{G}_0, \mathcal{K}_0, \theta_0) \oplus (\mathcal{G}_+, \mathcal{K}_+, \theta_+) \oplus (\mathcal{G}_-, \mathcal{K}_-, \theta_-)$$

such that

1. (1) $[\mathcal{M}_0, \mathcal{M}_0] = 0$ *in the Cartan decomposition* $\mathcal{G}_0 = \mathcal{K}_0 \oplus \mathcal{M}_0$ (*the Euclidean type*), *and*
2. (2) $\mathcal{G}_+$ (resp., $\mathcal{G}_-$) *is semisimple of compact* (resp., *noncompact*) *type, and* $[\mathcal{M}_+, \mathcal{M}_+] = \mathcal{K}_+$ (resp., $[\mathcal{M}_-, \mathcal{M}_-] = \mathcal{K}_-$) *in the Cartan decomposition of* $\mathcal{G}_+$ (resp., $\mathcal{G}_-$).

Proof. (Idea) Choose an inner product on $\mathcal{G}$ as given in Proposition 32, with respect to which we diagonalize the Killing form restricted to $\mathcal{M}$,

$$B = \alpha_1 (x_1)^2 + \cdots + \alpha_n (x_n)^2,$$

relative to an orthonormal basis $E_1, \cdots, E_n$. Set

$$\mathcal{M}_0 = \mathrm{span}_{\alpha_j = 0} < E_j >, \quad \mathcal{M}_+ = \mathrm{span}_{\alpha_j < 0} < E_j >, \quad \mathcal{M}_- = \mathrm{span}_{\alpha_j > 0} < E_j >,$$

and set

$$\mathcal{K}_+ = [\mathcal{M}_+, \mathcal{M}_+], \quad \mathcal{K}_- = [\mathcal{M}_-, \mathcal{M}_-], \quad \mathcal{K}_0 = (\mathcal{K}_+ \oplus \mathcal{K}_-)^{\perp}.$$

□

Theorem 46. *Let* $(\mathcal{G}, \mathcal{K}, \theta)$ *be an orthogonal semisimple symmetric Lie algebra. Then it can be decomposed into the direct sum*

$$\oplus_{j=1}^{k} (\mathcal{G}_j \oplus \mathcal{G}_j', \mathcal{K}_j, \theta_j) \bigoplus \oplus_{j=k+1}^{r} (\mathcal{G}_j, \mathcal{K}_j, \theta_j),$$

where

$$\mathcal{G}_j' = \theta(\mathcal{G}_j), \quad \theta_i = \theta|_{\mathcal{G}_j \oplus \mathcal{G}_j'}, \quad \mathcal{K}_j = \{(X, \theta(X)) \in \mathcal{G}_j \oplus \mathcal{G}_j'\}, \quad 1 \le j \le k,$$

and

$$\theta_j = \theta|_{\mathcal{G}_j}, \quad \mathcal{K}_j = \mathcal{G}_j \cap \mathcal{K}, \quad j \ge k+1.$$

Proof. (Idea) $\mathcal{G}$ is decomposed into simple ideals and θ permutes them. Either an ideal is mapped to another one, then we group them into one of the first k orthogonal symmetric Lie algebra, or the ideal is mapped into itself, which belongs to the remaining symmetric Lie algebras in the decomposition. □

Corollary 47. *A fundamental building block of orthogonal semisimple Lie algebras belongs to one the following two types.*

1. (1) $(\mathcal{G}, \mathcal{K}, \theta)$, *where* $\mathcal{G}$ *is simple and* $\mathcal{K}$ *is a compactly embedded subalgebra of* $\mathcal{G}$ (*see Remark* 38). *The Killing form can be either positive or negative definite on* $\mathcal{M}$.
2. (2) $(\mathcal{G} \oplus \mathcal{G}, \Delta\mathcal{G}, \theta)$, *where* $\mathcal{G}$ *is simple of compact type,*

$$\mathcal{K} = \Delta\mathcal{G} = \{(X, X) : X \in \mathcal{G}\}, \quad \theta(X, Y) = (Y, X).$$

In both cases $ad(\mathcal{K})$ *acts irreducibly on* $\mathcal{M}$.

Proof. For Item (1), choose an inner product $\langle , \rangle$ as in Proposition 32. Define T by

$$B(X,Y) = \langle TX, Y\rangle.$$

Then T has only one eigenvalue, because if $\mathcal{M}_i$ is the i-th eigenspace of T, then $[\mathcal{M}_i, \mathcal{M}_i] + \mathcal{M}_i$ is an ideal of $\mathcal{G}$, whose simplicity implies $\mathcal{M}_i = \mathcal{M}$.

For Item (2),

$$B((X,Y),(X,Y)) = B(X,X) + B(Y,Y)$$

for $(X,Y) \in \mathcal{G} \oplus \mathcal{G}$. However, since $\mathcal{K} = \Delta(\mathcal{G} \oplus \mathcal{G})$ is a compactly embedded subalgebra of $\mathcal{G} \oplus \mathcal{G}$, Corollary 35 says that B is negative-definite on $\mathcal{K} = \Delta(\mathcal{G} \oplus \mathcal{G})$. That is

$$0 > B((X,X),(X,X)) = 2B(X,X).$$

Therefore, the Killing forms of $\mathcal{G}$ and $\mathcal{G} \oplus \mathcal{G}$ are both negative-definite. □

As a consequence, we obtain

Theorem 48. *The irreducible orthogonal symmetric Lie algebras of compact type are divided into the following two types.*

Type I. $(\mathcal{G}, \mathcal{K}, \theta)$*, where* $\mathcal{G}$ *is simple of compact type.*

Type II. $(\mathcal{G} \oplus \mathcal{G}, \Delta\mathcal{G}, \theta)$*, where* $\mathcal{G}$ *is simple of compact type.*

Definition 49. *Let* $(\mathcal{G}, \mathcal{K}, \theta)$ *be a symmetric Lie algebra with the Cartan decomposition* $\mathcal{G} = \mathcal{K} \oplus \mathcal{M}$*. Let* $\mathcal{G}^{\mathbb{C}}$ *be the complexification of* $\mathcal{G}$*:*

$$\mathcal{G}^{\mathbb{C}} := \{X + \sqrt{-1}Y : X, Y \in \mathcal{G}\}.$$

Set

$$\mathcal{G}^* := \mathcal{K} \oplus \sqrt{-1}\mathcal{M},$$

and

$$\theta^* = \theta^{\mathbb{C}}|_{\mathcal{G}^*},$$

where

$$\theta^{\mathbb{C}}(X + \sqrt{-1}Y) := \theta(X) + \sqrt{-1}\theta(Y).$$

$(\mathcal{G}^*, \mathcal{K}, \theta^*)$ *is called the dual of* $(\mathcal{G}, \mathcal{K}, \theta)$*.*

Example 50. H^n *is dual to* S^n*. In fact, the identification*

$$\begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix} \in o(n) \mapsto \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix} \in o(n) = \mathcal{K}$$

$$\sqrt{-1}\begin{pmatrix} 0 & -v^{tr} \\ v & 0 \end{pmatrix} \in \sqrt{-1}\mathcal{M} \mapsto \begin{pmatrix} 0 & v^{tr} \\ v & 0 \end{pmatrix} \in \mathcal{M}^*$$

from S^n *to* H^n *is an symmetric Lie algebra isomorphism.*

The duality sets up a one-to-one correspondence between compact and noncompact types. Therefore,

Theorem 51. *The irreducible orthogonal symmetric Lie algebras of noncompact type are divided into the following two types.*

Type IV. $(\mathcal{G}^{\mathbb{C}}, \mathcal{G}, \overline{\theta})$, *where $\mathcal{G}$ is simple of compact type.*

$$\overline{\theta} : X + \sqrt{-1}Y \mapsto X - \sqrt{-1}Y.$$

It is dual to Type II.

Type III. $(\mathcal{G}, \mathcal{K}, \theta)$, *where $\mathcal{G}$ is simple of noncompact type and does not admit any compatible complex structure, i.e., any $J : \mathcal{G} \to \mathcal{G}$, $J^2 = -Id$, such that*

$$ad_{JX} = J \circ ad_X = ad_X \circ J.$$

It is dual to Type I.

Proof. We display the duality between Type II and Type IV:

$$(\mathcal{G} \oplus \mathcal{G})^* = \mathcal{K} \oplus \sqrt{-1}\mathcal{M} \to \mathcal{G}^{\mathbb{C}}$$

given by

$$(X, X) \oplus \sqrt{-1}(Y, -Y) \mapsto X + \sqrt{-1}Y$$

is an isomorphism. □

Theorem 52. *Let (G, K, σ) be a Riemannian symmetric pair such that $(\mathcal{G}, \mathcal{K}, \theta)$ is irreducible.*

(1) *If $(\mathcal{G}, \mathcal{K}, \theta)$ is of Type I or II, then G/K is compact, Einstein with nonnegative sectional curvatures and positive Ricci tensor.*
(2) *If $(\mathcal{G}, \mathcal{K}, \theta)$ is of Type III or IV, then G/K is diffeomorphic to a Euclidean space, Einstein with nonpositive sectional curvature and negative Ricci tensor.*

Proof. The irreducibility of K on $\mathcal{M}$ implies by Schur's lemma that the Riemannian inner product is a constant multiple of the Killing form

$$\langle \cdot, \cdot \rangle = aB(\cdot, \cdot),$$

where $a < 0$ for compact type and $a > 0$ for noncompact type. By Proposition 34 and (7), we conclude that the sectional curvatures are given by

$$\langle R(X, Y)Y, X \rangle = aB([X, Y], [X, Y]). \tag{9}$$

Note that $[X, Y] \in \mathcal{K}$ on which B is negative-definite by Proposition 35. It follows that the sectional curvatures are ≥ 0 for compact type and ≤ 0 for noncompact type.

Schur's lemma once more implies that the Ricci tensor is a multiple of $\langle \cdot, \cdot \rangle$, so that the manifold is Einstein. If the Einstein constant is $= 0$, then the sectional curvatures will be identically zero, which implies that the manifold is of Euclidean type (i.e., $[\mathcal{M}, \mathcal{M}] = 0$ by (9)), a contradiction. So the Einstein constant is > 0 for compact type and < 0 for noncompact type.

Lastly, that the noncompact type is diffeomorphic to Euclidean space follows from a general fact [23, II, p. 105] that a connected homogeneous Riemannian manifold M with nonpositive sectional curvature and negative-definite Ricci tensor is simply connected. □

Corollary 53. *Let (G, K, σ) be an effective Riemannian symmetric pair such that $(\mathcal{G}, \mathcal{K}, \theta)$ is irreducible of noncompact type. Then G has trivial center and K is connected and is a maximal compact subgroup of G. Any two maximal compact subgroups of G are conjugate.*

Proof. (Sketch) The effectiveness of G/K means that G is contained in the isometry group of G/K. By the preceding theorem G/K is homogeneous, has nonpositive sectional curvature and negative-definite Ricci tensor and is simply connected, then the result follows by a general theorem on transformation groups [23, II, p. 107, p. 112]. □

2.2. **The classification of symmetric spaces.** The classification of irreducible symmetric spaces is made simpler by looking at the noncompact type, because of Corollary 53 and the following two theorems.

Theorem 54. *Simply connected irreducible Riemannian symmetric spaces are in one-to-one correspondence with (isomorphic) effective irreducible orthogonal symmetric Lie algebras $(\mathcal{G}, \mathcal{K}, \theta)$.*

Proof. For the effective orthogonal symmetric Lie algebra, Let $\tilde{G}$ be the simply connected Lie group whose Lie algebra is $\mathcal{G}$. The involution θ extends to a unique involutive automorphism $\tilde{\sigma}$ on $\tilde{G}$. Any subgroup $\tilde{K}$ satisfying $(\tilde{G})^0_{\tilde{\sigma}} \subset \tilde{K} \subset \tilde{G}_{\tilde{\sigma}}$ makes $M := \tilde{G}/\tilde{K}$ a Riemannian symmetric space. We mod out the maximal normal subgroup N of $\tilde{G}$ contained in $\tilde{K}$. Then $M := G/K$, where $G := \tilde{G}/N$ and $K := \tilde{K}/N$, is effective. G is contained in the identity component G_1 of the group of isometries of M.

We claim that $G = G_1$. The symmetry of M at the origin $\circ$ induces an involutive automorphism of G_1, which extends $\sigma := \tilde{\sigma}|_G$ of G. Let K_1 be the isotropy subgroup of G_1 at $\circ$. Then $M = G_1/K_1$ and we have two Cartan decompositions

$$\mathcal{G} = \mathcal{K} \oplus \mathcal{M}, \quad \mathcal{G}_1 = \mathcal{K}_1 \oplus \mathcal{M}_1$$

associated with $M := G/K$ and $M := G_1/K_1$. But then $\mathcal{M} = \mathcal{M}_1$ and so

$$\mathcal{K} = [\mathcal{M}, \mathcal{M}] = [\mathcal{M}_1, \mathcal{M}_1] = \mathcal{K}_1.$$

It follows that $\mathcal{G} = \mathcal{G}_1$, and so $G = G_1$. □

Theorem 55. *Irreducible effective orthogonal symmetric Lie algebras* $(\mathcal{G}, \mathcal{K}, \theta)$ *of noncompact type are in one-to-one correspondence with the real simple Lie algebras of noncompact type.*

Proof. (Sketch) Corollary 53 implies that there is at most one effective orthogonal symmetric Lie algebra $(\mathcal{G}, \mathcal{K}, \theta)$ for each $\mathcal{G}$.

To show the existence of such an orthogonal symmetric Lie algebra, we consider $\mathcal{G}^{\mathbb{C}}$, which is semisimple. Now we know $\mathcal{G}^{\mathbb{C}}$ has a compact real form $\mathcal{U}$ [20, p. 181]. That is, $\mathcal{U}$ is a real Lie algebra such that $\mathcal{U}^{\mathbb{C}} = \mathcal{G}^{\mathbb{C}}$ and the Killing form on $\mathcal{U}$ is negative-definite. This immediately takes care of type IV, because $(\mathcal{G}^{\mathbb{C}}, \mathcal{U}, \theta)$, where θ is the complex conjugation with respect to $\mathcal{U}$ is an irreducible effective orthogonal symmetric Lie algebra.

For type III, again we are given $(\mathcal{G}^{\mathbb{C}}, \mathcal{U}, \theta)$; we denote by (G_1, U, σ_1) an effective orthogonal symmetric pair assuming the symmetirc Lie algebra. We denote by τ the complex conjugation of $\mathcal{G}^{\mathbb{C}}$ with respect to $\mathcal{G}$. τ is an involutive automorphism of $\mathcal{G}^{\mathbb{C}}$ (as a real Lie algebra), which induces an involutive automorphism σ on the universal covering group $\tilde{G}_1$ of G_1 and in turn it induces an involutive automorphism σ on G_1 (because G_1 is the quotient group of $\tilde{G}_1$ by its center). We know there is a maximal compact subgroup K_1 of G_1 left invariant by σ by a general theorem about the isometry group of a connected, simply connected homogeneous space of nonpositive curvature [23, II, p. 112, Theorem 9.4]; by Corollary 53, U is conjugate to K_1 in G_1, so that the Lie algebra $\mathcal{K}_1$ of K_1 is also a compact real form of $\mathcal{G}^{\mathbb{C}}$. Let μ be the complex conjugation of $\mathcal{G}^{\mathbb{C}}$ with respect to $\mathcal{K}_1$. Then $(\mathcal{G}^{\mathbb{C}}, \mathcal{K}_1, \mu)$ is an irreducible orthogonal symmetric Lie algebra of noncompact type. Since $\mathcal{K}_1$ is left invariant by σ and $\mathcal{G}$ is left fixed by σ, we obtain

$$\mathcal{G} = \mathcal{G} \cap \mathcal{K}_1 \oplus \mathcal{G} \cap \sqrt{-1}\mathcal{K}_1.$$

Let $\mathcal{K} := \mathcal{G} \cap \mathcal{K}_1$. Then $(\mathcal{G}, \mathcal{K}, \mu)$ is an effective orthogonal symmetric Lie algebra of noncompact type. □

Remark 56. [20, pp. 182-186] *contains an intrinsic algebraic description of the geometric one given in the preceding theorem. It is shown there that two compact forms* $\mathcal{U}_1$ *and* $\mathcal{U}_2$ *in a complex semisimple Lie algebras* $\mathcal{G}^{\mathbb{C}}$ *are identical via an automorphism of* $\mathcal{G}^{\mathbb{C}}$*; therefore, Theorem* 54 *says that the simply connected Riemannian symmetric spaces of Type I are in one-to-one correspondence with* $(\mathcal{U}, \theta)$*, where* $\mathcal{U}$ *is a simple Lie algebra of compact type and* θ *is (up to isomorphism) an involutive automorphism of* $\mathcal{U}$*. This will become clear by the classification tables given below.*

Definition 57. *Let $(\mathcal{G}, \mathcal{K}, \theta)$ be an effective orthogonal symmetric Lie algebra. Then the* **rank** *of it is the maximum dimension of linear abelian spaces of $\mathcal{M}$, i.e., linear spaces $V \subset \mathcal{M}$ for which $[V, V] = 0$.*

Theorem 58. (The Classification)

(1) [20, p. 439] *The Riemannian symmetric spaces of Type II are exactly the compact, connected simple Lie groups G with a bi-invariant metric. They are also identified with $G \times G / \Delta(G \times G)$. The following are all the simply connected ones:*

Type	G	Center	Rank	Dimension
$A_n, n \geq 1$	$SU(n+1)$	$\mathbb{Z}_{n+1}$	n	$n(n+2)$
$B_n, n \geq 2$	$Spin(2n+1)$	$\mathbb{Z}_2$	n	$n(2n+1)$
$C_n, n \geq 3$	$Sp(n)$	$\mathbb{Z}_2$	n	$n(2n+1)$
$D_n, n \geq 4$	$Spin(2n)$	$\mathbb{Z}_4$, for n odd, or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, for n even	n	$n(2n-1)$
	G_2	$\mathbb{Z}_1$	2	14
	F_4	$\mathbb{Z}_1$	4	52
	E_6	$\mathbb{Z}_3$	6	78
	E_7	$\mathbb{Z}_2$	7	133
	E_8	$\mathbb{Z}_1$	8	248

Modding out any subgroup of the center results in a simple Lie group of Type II.

(2) [20, p. 516] *The Riemannian symmetric spaces of Type IV are G/U, where G is a connected group whose Lie algebra is a complex simple Lie algebra $\mathcal{G}$ over $\mathbb{C}$ considered as a real Lie algebra, and U is a maximal compact subgroup of G. The following is the list:*

G	U	Rank	Dimension
$SL(n+1, \mathbb{C})$	$SU(n+1)$	n	$n(n+2)$
$SO(2n+1, \mathbb{C})$	$SO(2n+1)$	n	$n(2n+1)$
$Sp(n, \mathbb{C})$	$Sp(n)$	n	$n(2n+1)$
$SO(2n, \mathbb{C})$	$SO(2n)$	n	$n(2n-1)$
$G_2^{\mathbb{C}}$	G_2	2	14
$F_4^{\mathbb{C}}$	F_4	4	52
$E_6^{\mathbb{C}}$	E_6	6	78
$E_7^{\mathbb{C}}$	E_7	7	133
$E_8^{\mathbb{C}}$	E_8	8	248

(3) *The simply connected Riemannian symmetric spaces of Type I are:*

G	U	Rank	Dimension
$SU(n)$	$SO(n)$	$n-1$	$(n-1)(n+2)/2$
$SU(2n)$	$Sp(n)$	$n-1$	$(n-1)(2n+1)$
$SU(p+q)$	$S(U_p \times U_q)$	$\min(p,q)$	$2pq$
$SO(p+q)$	$SO(p) \times SO(q)$	$\min(p,q)$	pq
$SO(2n)$	$U(n)$	$[n/2]$	$n(n-1)$
$Sp(n)$	$U(n)$	n	$n(n+1)$
$Sp(p+q)$	$Sp(p) \times Sp(q)$	$\min(p,q)$	$4pq$
G_2	$SO(4)$	2	8
F_4	$(Sp(3) \times Sp(1))/\mathbb{Z}_2$	4	28
F_4	$Spin(9)$	1	16
E_6	$Sp(4)/\mathbb{Z}_2$	6	42
E_6	$(SU(6) \times SU(2))/\mathbb{Z}_2$	4	40
E_6	$(Spin(10) \times SO(2))/\mathbb{Z}_4$	2	32
E_6	F_4	2	26
E_7	$(E_6 \times SO(2))/\mathbb{Z}_3$	3	54
E_7	$SU(8)/\mathbb{Z}_2$	7	70
E_7	$(Spin(12) \times SU(2))/\mathbb{Z}_2$	4	64
E_8	$SO(16)$	8	128
E_8	$(E_7 \times SU(2))/\mathbb{Z}_2$	4	112

All G/K in the table are effective, and finitely cover all compact symmetric spaces of Type I by the scheme given in [20, p. 327, Corollary 9.3].

(4) *The Riemannian symmetric spaces of Type III are:*

G	U	Rank	Dimension
$SL(n,\mathbb{R})$	$SO(n)$	$n-1$	$(n-1)(n+2)/2$
$SU^*(2n)$	$Sp(n)$	$n-1$	$(n-1)(2n+1)$
$SU(p,q)$	$S(U_p \times U_q)$	$\min(p,q)$	$2pq$
$SO_0(p,q)$	$SO(p) \times SO(q)$	$\min(p,q)$	pq
$SO^*(2n)$	$U(n)$	$[n/2]$	$n(n-1)$
$Sp(n,\mathbb{R})$	$U(n)$	n	$n(n+1)$
$Sp(p,q)$	$Sp(p) \times Sp(q)$	$\min(p,q)$	$4pq$
$G_2(2)$	$SO(4)$	2	8
$F_4(4)$	$(Sp(3) \times Sp(1))/\mathbb{Z}_2$	4	28
$F_4(-20)$	$Spin(9)$	1	16
$E_6(6)$	$Sp(4)/\mathbb{Z}_2$	6	42
$E_6(2)$	$(SU(6) \times SU(2))/\mathbb{Z}_2$	4	40
$E_6(-14)$	$(Spin(10) \times SO(2))/\mathbb{Z}_4$	2	32
$E_6(-26)$	F_4	2	26
$E_7(-25)$	$(E_6 \times SO(2))/\mathbb{Z}_3$	3	54
$E_7(7)$	$SU(8)/\mathbb{Z}_2$	7	70
$E_7(-5)$	$(Spin(12) \times SU(2))/\mathbb{Z}_2$	4	64
$E_8(8)$	$SO(16)$	8	128
$E_8(-24)$	$(E_7 \times SU(2))/\mathbb{Z}_2$	4	112

All G/K in the table are effective. Here, the -14 in $E_6(-14)$ means

$$\dim(\mathcal{M}) - \dim(\mathcal{K}) = -14$$

in the Cartan decomposition, etc.

Article [47] gives an explicit construction of Type II and III exceptional symmetric spaces in the tables.

Remark 59. *In view of Remark 56, one way to go about the classification is to first of all classify all compact simple Lie groups, or equivalently, to classify all complex simple Lie algebras and find their compact real forms, which are of Type II, A_n through E_8, given in the table. Then classify, up to isomorphism, all involutive automorphisms of the compact real forms to come up with Type I given in the table. The other two types are obtained by duality.*

2.3. **Root systems.** The classification of (real or complex) simple Lie algebras rests on understanding root systems of semisimple Lie algebras. We will have a quick look at this concept.

Let $(\mathcal{G}, \mathcal{K}, \theta)$ be an irreducible symmetric Lie algebra of noncompact type. Let $\mathcal{A} \subset \mathcal{M}$ be a maximum abelian subspace, i.e., a linear subspace of maximum dimension in $\mathcal{M}$ satisfying $[\mathcal{A}, \mathcal{A}] = 0$; recall the dimension of $\mathcal{A}$ is called the rank of the symmetric Lie algebra. It is easy to see by the Jacobi identity that ad_{H_1} and ad_{H_2} commute, for $H_1, H_2 \in \mathcal{A}$, as operators of $\mathcal{G}$. Since the Killing form is negative-definite on $\mathcal{K}$ and positive-definite on $\mathcal{M}$, the form

$$B_\theta(Y, Z) := -B(Y, \theta(Z))$$

is positive-definite and $ad_H, H \in \mathcal{A}$, is symmetric with respect to B_θ, so that $ad_H, H \in \mathcal{A}$, can be simultaneously diagonalized.

Remark 60. *We can also consider the compact type, where the inner product is $-B(X, Y)$. But then ad_H is skew-symmetric with purely imaginary eigenvalues, which, when multiplied by $\sqrt{-1}$, give us the same setup as in the nonocompact type via duality.*

For $\lambda \in \mathcal{A}^*$, set

$$\mathcal{G}_\lambda = \{X \in \mathcal{G} : [H, X] = \lambda(H)X, \ \forall H \in \mathcal{A}\}.$$

We have

$$[\mathcal{G}_\lambda, \mathcal{G}_\mu] \subset \mathcal{G}_{\lambda+\mu}.$$

Definition 61. *If $\lambda \neq 0$ and $\mathcal{G}_\lambda \neq 0$, then λ is called a* **root** *of $\mathcal{G}$ with respect to $\mathcal{A}$, and $\mathcal{G}_\lambda$ is called a root space. The set of all roots, denoted by Σ, is called the root system of $\mathcal{G}$ with respect to $\mathcal{A}$. The dimension of a root space is called its* **multiplicity***.*

Remark 62. *If we conduct the root space decomposition with respect to the compact type, then the decomposition is in $\mathcal{G}^{\mathbb{C}}$. In fact, a symmetric space of Type II, when regarded as a simple Lie group G with Lie algebra $\mathcal{G}$, has the property that a maximal abelian space in $\mathcal{M}$ is isomorphic to a maximal abelian Lie subalgebra $\mathcal{H}$ in $\mathcal{G}$, whose complexification is called a Cartan subalgebra in $\mathcal{G}^{\mathbb{C}}$.*

The root space decomposition of $\mathcal{G}^{\mathbb{C}}$ with respect to $\mathcal{H}$ will be specifically denoted by Δ instead of Σ. The root space decomposition of $\mathcal{G}^{\mathbb{C}}$ with respect to Δ has particularly nice structures [20, p. 165-178]*. For instance, all roots have multiplicity* 1*.*

On the other hand, when we regard a group as a symmetric space of Type II, all roots of Σ have multiplicity 2*, and vice versa* [26, p. 78-85]*.*

Remark 63. *The inner product $B_\sigma(Y, Z)$ naturally induces an inner product $< Y, Z >$ on $\mathcal{A}^*$ by setting*

$$\lambda(v) = B_\theta(t_\lambda, v)$$

and set

$$< \lambda, \tau >:= B_\theta(t_\lambda, t_\tau).$$

We have

$$\mathcal{G} = \mathcal{G}_0 \oplus \sum_{\lambda \in \Sigma} \mathcal{G}_\lambda.$$

Definition 64. *Fix a $v \neq 0$ in $\mathcal{A}$ so that $\lambda(v) \neq 0$ for all $\lambda \in \Sigma$. We say a root $\lambda \in \Sigma$ is positive (negative) if $\lambda(v) > 0$ (< 0).*

Theorem 65. '[20, p. 290] Σ *satisfies*

(1) *Σ generates $\mathcal{A}^*$,*
(2) *for each $\alpha \in \mathcal{A}^*$, the reflection S_α along α leaves Σ invariant, where*
$$S_\alpha(\beta) = \beta - 2\frac{< \beta, \alpha >}{< \alpha, \alpha >}\alpha,$$
(3) *$a_{\beta\alpha} := 2\frac{<\beta,\alpha>}{<\alpha,\alpha>} \in \mathbb{Z}$, and*
(4) *in fact the α-chain through β, i.e., $\beta + n\alpha, p \leq n \leq q$, is unbroken, and we have*
$$a_{\beta\alpha} = -(p + q).$$

Corollary 66. *If $\beta = m\alpha$ for $\alpha, \beta \in \Sigma$, then $m = \pm\frac{1}{2}, \pm 1, \pm 2$.*

Proof. $a_{\beta\alpha} = 2m \in \mathbb{Z}$ and $a_{\alpha\beta} = 2/m \in \mathbb{Z}$. □

Definition 67. *A finite set $\Sigma \subset \mathbb{R}^n$ is called a* **root system** *if it satisfies items* (1), (2), (3) *in the preceding theorem. The root system is called* **reduced** *if $\beta = m\alpha$ for $\alpha, \beta \in \Sigma$ implies $m = \pm 1$.*

Remark 68. *Referring to Remark* 62*, the root system Δ of a simple Lie algebra of compact type, is reduced. On the other hand, the root system of a symmetric Lie algebra $\mathcal{G}$ of Type III need not have a reduced root system.*

Corollary 69. *Let Σ be a root system. Let*

$$\Sigma_1 := \{\alpha \in \Sigma : \alpha/2 \notin \Sigma\}, \quad \Sigma_2 := \{\alpha \in \Sigma : 2\alpha \notin \Sigma\}.$$

Then Σ_1 and Σ_2 are reduced.

Definition 70. *Let $\Sigma \subset \mathbb{R}^n$ be a root system. A subset $B \subset \Sigma$ is called a* **basis** *of Σ if*

(1) *B is a basis of $\mathbb{R}^n$, and*
(2) *for $\beta \in \Sigma$, if $\beta = \sum_{\alpha \in B} n_\alpha \alpha$, then $n_\alpha \geq 0$ or $n_\alpha \leq 0$ for all $\alpha \in B$.*

A positive root is called a **simple root** *if it cannot be written as the sum of two positive roots.*

Theorem 71. [20, p. 178, p.458]

(1) *Every root system has a basis consisting of simple roots, and*
(2) $a_{\beta\alpha} \leq 0$ *for any two simple roots.*

Definition 72. *A root system* Σ *is called irreducible if it cannot be decomposed into two disjoint nonempty orthogonal subsets.*
Proposition 73. [20, p. 458] *A root system decomposes uniquely as the union of irreducible root systems.*

Theorem 74. (Classification of irreducible root systems) [20, p. 462] Reduced case. (Here, e_i are standard basis elements of a Euclidean space.)

(1) $\mathbf{A_n}, n \geq 1 : \Sigma = \{e_i - e_j, 1 \leq i \neq j \leq n+1\}$,
Simple roots*:* $\alpha_i = e_i - e_{i+1}, 1 \leq i \leq n$.
(2) $\mathbf{B_n}, n \geq 1 : \Sigma = \{\pm e_i, 1 \leq i \leq n, \pm e_i \pm e_j, 1 \leq i \neq j \leq n\}$
Simple roots: $\alpha_i = e_i - e_{i+1}, 1 \leq i \leq n-1, \alpha_n = e_n$.
(3) $\mathbf{C_n}, n \geq 1 : \Sigma = \{\pm 2e_i, 1 \leq i \leq n, \pm e_i \pm e_j, 1 \leq i \neq j \leq n\}$,
Simple roots: $\alpha_i = e_i - e_{i+1}, 1 \leq i \leq n-1, \alpha_n = 2e_n$.
(4) $\mathbf{D_n}, n \geq 2 : \Sigma = \{\pm e_i \pm e_j, 1 \leq i \neq j \leq n\}$,
Simple roots: $\alpha_i = e_i - e_{i+1}, 1 \leq i \leq n-1, \alpha_n = e_{n-1} + e_n$.
(5) $\mathbf{G_2} : \Sigma = \{\pm(e_2 - e_3), \pm(e_3 - e_1), \pm(e_1 - e_2), \pm(2e_1 - e_2 - e_3), \pm(2e_2 - e_1 - e_3), \pm(2e_3 - e_1 - e_2)\}$,
Simple roots: $\alpha_1 = e_1 - e_2, \alpha_2 = -2e_1 + e_2 + e_3$.
(6) $\mathbf{F_4} : \Sigma = \{\pm e_i, \pm e_i \pm e_j, i < j,\ (\pm e_1 \pm e_2 \pm e_3 \pm e_4)/2\}$,
Simple roots: $\alpha_1 = e_2 - e_3, \alpha_2 = e_3 - e_4, \alpha_3 = e_4, \alpha_4 = (e_1 - e_2 - e_3 - e_4)/2$.
(7) $\mathbf{E_8} : \Sigma = \{\pm e_i \pm e_j, i < j,\ (\sum_{i=1}^{8} (-1)^{n_i} e_i)/2,\ \sum_{i=1}^{8} n_i \textit{ even}\}$,
Simple roots: $\alpha_1 = e_1 + e_2$, $\alpha_2 = (e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 + e_8)/2$, $\alpha_{i+1} = e_i - e_{i-1}$, $2 \leq i \leq 7$.
(8)
$$\mathbf{E_7} : \Sigma = \{\pm e_i \pm e_j, 1 \leq i < j \leq 6, \pm(e_7 - e_8), \pm (e_7 - e_8 + \sum_{i=1}^{6} (-1)^{n_i} e_i)/2,\ \sum_{i=1}^{6} n_i \textit{ even}\},$$
Simple roots: $\alpha_1 = e_1 + e_2$, $\alpha_2 = (e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 + e_8)/2$, $\alpha_{i+1} = e_i - e_{i-1}$, $2 \leq i \leq 6$.
(9) $\mathbf{E_6} : \Sigma = \{\pm e_i \pm e_j, 1 \leq i < j \leq 5, \pm(e_8 - e_7 - e_6 + \sum_{i=1}^{5} (-1)^{n_i} e_i)/2,\ \sum_{i=1}^{5} n_i \textit{ even}\}$,
Simple roots: $\alpha_1 = e_1 + e_2$, $\alpha_2 = (e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 + e_8)/2$, $\alpha_{i+1} = e_i - e_{i-1}$, $2 \leq i \leq 5$.

Nonreduced case.

$\mathbf{BC_n}, n \geq 1 : \Sigma = \{\pm e_i \pm e_j, 1 \leq i < j \leq n, \pm e_i, 1 \leq i \leq n, \pm 2e_i, 1 \leq i \leq n\}$,
Simple roots: $\alpha_i = e_i - e_{i+1}, 1 \leq i \leq n-1, \alpha_n = e_n$

Note that E_8 restricts to E_7 and E_8. In fact, the $\mathbb{R}^7$ (vs. $\mathbb{R}^6$) spanned by the first seven (vs. six) simple roots of E_8 intersecting the root system of E_8 is the root system of E_7 (vs. E_6).

Example 75. (The root systems Δ of the classical simple Lie algebras)
$\mathbf{A_n} = \mathbf{sl}(\mathbf{n+1}, \mathbb{C})$. *Cartan subalgebra (maximal abelian subalgebra) is*

$$\mathcal{H} := \{x_1 e_{11} + \cdots + x_{n+1} e_{n+1\, n+1} : x_1 + \cdots + x_{n+1} = 0,$$

where e_{ij} *is the matrix whose* (i,j)*-entry is* 1 *and is* 0 *elsewhere. Set* $h_i = e_{ii}$. *Let* $\omega_1, \cdots, \omega_{n+1}$ *be the dual basis to* $h_1, \cdots, h_{n+1}$. *The Killing form is the standard inner product on* $\mathcal{H}$. *For any* $h \in \mathcal{H}$,

$$ad_h : e_{ij} \mapsto he_{ij} - e_{ij}h = (\omega_i - \omega_j)(h) e_{ij}, \quad i \neq j.$$

Therefore, the root system of A_{n+1} *is*

$$\omega_i - \omega_j, \quad 1 \leq i \neq j \leq n.$$

Let

$$v := a_1 h_1 + \cdots + a_{n+1} h_{n+1}, \quad a_1 > a_2 > \cdots > a_{n+1} > 0.$$

λ *is a positive root if* $\lambda(v) > 0$. *So,* $\omega_i - \omega_j, i > j$, *are positive roots, with the simple roots given above.*
$\mathbf{B_n} = \mathbf{so}(\mathbf{2n+1}, \mathbb{C}), \mathbf{n} \geq \mathbf{1}$. *It consists of* $(2n+1)$ *by* $(2n+1)$ *matrices of the form*

$$\begin{pmatrix} A & B & E \\ C & D & F \\ G & H & 0 \end{pmatrix},$$

where A, B, C, D *are of size* n *by* n, B, C *are skew-symmetric,* $(A, D), (E, H), (F, G)$ *are pairs whose components are negative transposes of each other. The Cartan subalgebra* $\mathcal{H}$ *consists of matrices of the form*

$$\begin{pmatrix} X & 0 & 0 \\ 0 & -X & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X = \mathrm{diag}(x_1, \cdots, x_n).$$

With $h_i := \mathrm{diag}(e_{ii}, -e_{n+i\, n+i})$ *and* ω_i *its dual, we calculate to see that the eigenvectors of* $ad_h, h \in \mathcal{H}$, *are*

$$e_{ij} - e_{n+j\, n+i},\ e_{i\, n+j} - e_{j\, n+i},\ e_{i\, 2n+1} - e_{2n+1\, n+1}, \quad i \neq j,$$

and the roots are

$$\pm\omega_i, \quad 1 \leq n, \quad \pm\omega_i \pm \omega_j, \quad i \neq j.$$

Setting

$$v = a_1 h_1 + \cdots a_n h_n, \quad a_1 > \cdots > a_n > 0,$$

the positive roots are

$$\omega_i + \omega_j,\ \omega_i - \omega_j,\ \omega_i, \quad i < j,$$

with the simple roots given above.

$\mathbf{C_n = sp(n, \mathbb{C}), n \geq 1}$. *It consists of matrices of the form*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A, D *are negative transposes of each other and* B, C *are symmetric; all of them are of size* n *by* n*. The Cartan subalgebra* $\mathcal{H}$ *consists of matrices of the form*

$$\begin{pmatrix} X & 0 \\ 0 & -X \end{pmatrix}, \quad X = \mathrm{diag}(x_1, \cdots, x_n).$$

With $h_i := \mathrm{diag}(e_{ii}, -e_{n+i\, n+i})$ *and* ω_i *its dual, we calculate to see that the eigenvectors are*

$$e_{i\, n+j} + e_{j\, n+i},\ e_{n+i\, j} - e_{n+j\ i},\ e_{ij} - e_{n+j\ n+i}, \quad i \neq j,$$

with roots $\pm\omega_i \pm \omega_j$*. Setting* $v = a_1 h_1 + \cdots + a_n h_n, \quad a_1 > \cdots > a_n > 0$*, the positive roots are*

$$\omega_i + \omega_j,\ \omega_i - \omega_j, \quad i < j,$$

with the simple roots given above.

$\mathbf{D_n = so(2n, \mathbb{C}), n \geq 2}$. *It consists of matrices of the form*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A, D *are negative transposes of each other and* B, C *are skew-symmetric; all of them are of size* n *by* n*. The Cartan subalgebra consists of matrices of the form*

$$\begin{pmatrix} X & 0 \\ 0 & -X \end{pmatrix}, \quad X = \mathrm{diag}(x_1, \cdots, x_n).$$

With $h_i := \mathrm{diag}(e_{ii}, -e_{n+i\, n+i})$ *and* ω_i *its dual, we calculate to see that the eigenvectors are*

$$e_{ij} - e_{n+j\ n+i},\ e_{i\, n+j} - e_{j\, n+i}, \quad i \neq j,$$

with roots $\pm\omega_i \pm \omega_j, i \neq j$*. Setting* $v = a_1 h_1 + \cdots + a_n h_n, a_1 > \cdots > a_n > 0$*, the positive roots are*

$$\omega_i + \omega_j,\ \omega_i - \omega_j, \quad i < j,$$

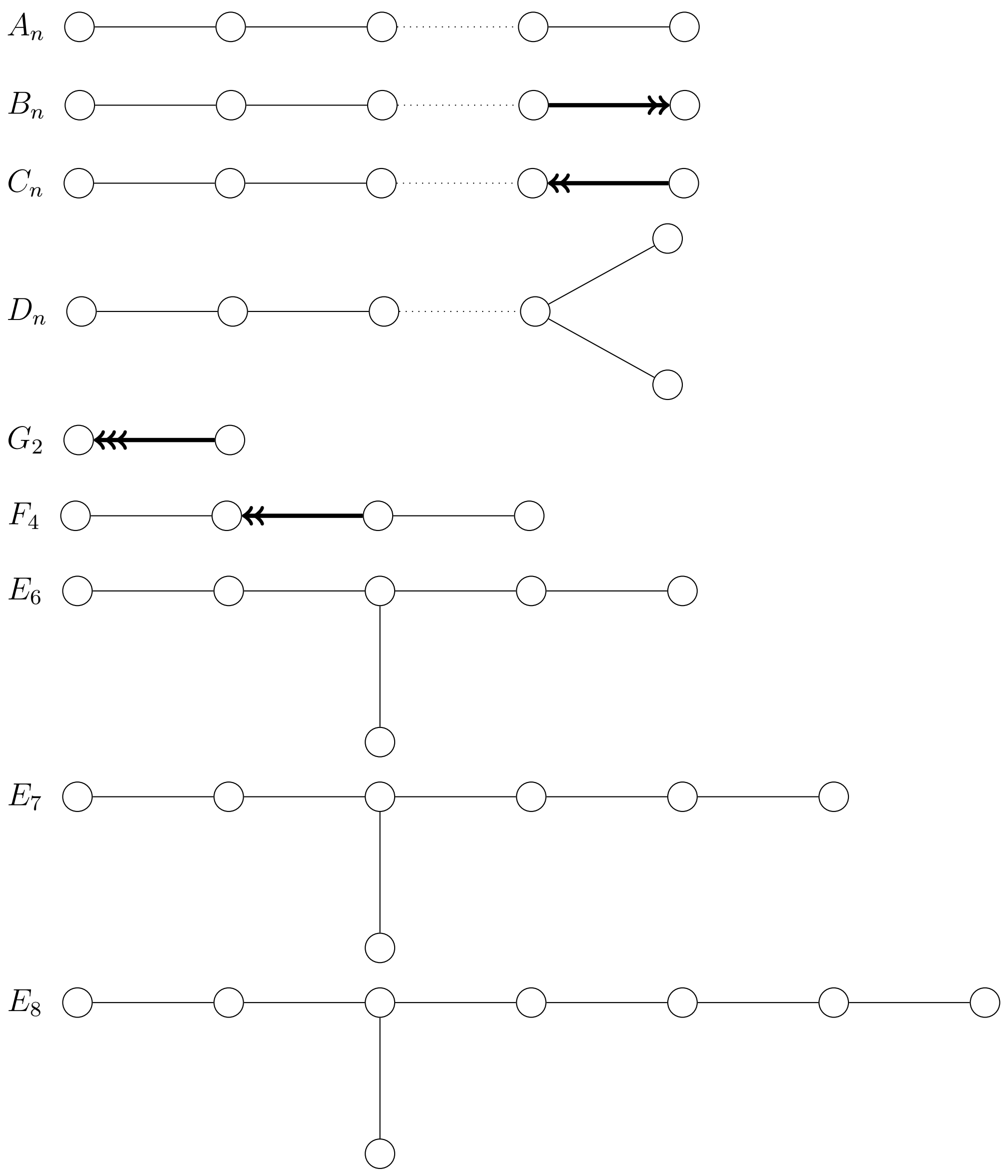
with the simple roots given above.

Remark 76. *The subscripts in A_n through E_8 refer to the rank, or the dimension of the maximal abelian space, of the symmetric space.*

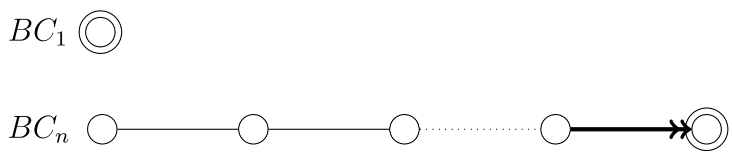
Dynkin Diagrams of Irreducible Root Systems

Let $\alpha_1, \cdots, \alpha_n$ be the simple roots marked by n circles. Connect α_i and α_j by a line if $a_{ij}a_{ji} \neq 0$, where $a_{\beta\alpha}$ is given in Theorem 65. Assign $a_{ij}a_{ji}$-many arrowheads, pointing to the shorter root, to a line if $a_{ij}a_{ji} > 1$, where the line is thickened for the purpose of distinction. The Dykin diagrams for the irreducible root systems are:

Reduced Root Systems



Nonreduced Root Systems



($n \geq 2$. The two concentric circles mark the simple root α for which 2α is also a root.)

We can calculate by the defining relations in Theorem 74 to match the Dynkin diagrams. Note that for the root systems E_6, E_7, and E_8, the singleton circle beneath the horizontal ones is α_1 in Theorem 74.

Theorem 77. [20, p. 482] *BC_n is excluded from the root systems Δ of compact simple Lie groups. Conversely, Δ recovers its corresponding simple Lie algebra.*

In particular, a look at the Dynkin diagrams shows that there are Lie algebra isomorphisms $A_1 \simeq B_1 \simeq C_1$, i.e., $sl(2,\mathbb{C}) \simeq so(3,\mathbb{C}) \simeq sp(1,\mathbb{C})$, $B_2 \simeq C_2$, i.e., $so(5,\mathbb{C}) \simeq sp(2,\mathbb{C})$, $D_2 \simeq A_1 \oplus A_1$, i.e., $so(4,\mathbb{C}) \simeq so(3,\mathbb{C}) \oplus so(3,\mathbb{C})$, and $D_3 \simeq A_3$, i.e., $so(6,\mathbb{C}) \simeq sl(4,\mathbb{C})$. On the Lie group level, there exist the well known corresponding isomorphisms $Spin(3) \simeq Sp(1)$, $Spin(5) \simeq Sp(2)$, $Spin(4) \simeq Spin(3) \times Spin(3)$, and $Spin(6) \simeq SU(4)$ [1].

Definition 78. *Let $\Sigma \subset \mathbb{R}^n$ be a root system. The complement of the hyperplanes fixed by the reflections $s_\lambda, \lambda \in \Sigma$, are called* **Weyl chambers***, while the hyperplanes are called the* **chamber walls***. Any vector in a Weyl chamber* (chamber wall) *is called* **regular** (singular)*. The group $W(\Sigma)$ generated by the reflections $s_\lambda, \lambda \in \Sigma$, is called the* **Weyl group** *associated with Σ.*

Theorem 79. [20, p.288] *The Weyl group acts simply transitively on the set of Weyl chambers.*

Now that we have the simple complex Lie algebras in hand, we can, in view of Remark 59, find their compact real forms $\mathcal{U}$, which are given in the table for type II. So now we are given such a $\mathcal{U}$ with the corresponding simply connected compact group U in the table. We want to find all the involutive automorphisms σ of U up to equivalence. These automorphisms will then give rise to all the Type I symmetric spaces. There are two types of automorphisms: the inner and outer ones. The inner automorphisms are $Ad(g)$ for $g \in U$. The set $Int(U)$ of all inner automorphisms on U is a group isomorphic to U/Z, where

Z is the center of U. The set of all automorphisms will be denoted by $Aut(G)$. Recall from Remark 62 the root systems are Δ.

Theorem 80. [26, II, p. 44, p. 90]

(1) *An automorphism on U is inner if and only if it leaves a maximal torus pointwise fixed.*

(2)
$$Aut(U)/Int(U) \simeq Aut(\Delta)/W(\Delta),$$
where $Aut(\Delta)$ is group of isometries of the Euclidean space, spanned by Δ, which leave Δ invariant.

(3) *$Aut(\Delta)/W(\Delta)$ is isomorphic to the group of symmetries of the Dynkin diagram. Therefore, we have the following table of the order r of $Aut(U)/Int(U)$.*

(4)

U	A_1	A_n	B_n	C_n	D_4	D_n	G_2	F_4	E_6	E_7	E_8	BC_n
r	1	2	1	1	6	2	1	1	2	1	1	1

With this theorem we can classify all the automorphisms for the classical groups U. If the automorphism is inner, say, $Ad(g)$ for some $g \in U$, then g^2 is in the center of U. We can then solve for all the g. Except for D_4, all the outer automorphisms are of the form $\sigma Ad(g)$, for some σ to be found, since $r = 2$. For details see [26, p. 101]. We have the following.

Theorem 81. (1) *There are, up to equivalence, three types of involutive automorphisms of $SU(n)$:*

(a) *$Ad(I_{pq}), p + q = n, 1 \leq q \leq [n/2]$, where*
$$I_{pq} := \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix}.$$
The automorphism is inner. The fixed point set is $S(U_p \times U_q)$.

(b) *τ, the complex conjugation and an outer automorphism. The fixed point set is $SO(n)$.*

(c) *$\tau \circ J_{q+1}$, where*
$$J_k = \begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix},$$
and $n = 2(q + 1)$. It is an outer automorphism. The fixed point set is $Sp(n)$.

(2) *There is only one type of involutive automorphism of $SO(2n + 1)$, which is $Ad(I_{pq}), p + q = 2n + 1$. It is inner. The fixed point set is $SO(p) \times SO(q)$.*

(3) *There are two types of involutive automorphisms, which are*

$$\tau, \quad Ad(I_{pq}), p+q=n, 1 \leq q \leq [n/2],$$

of $Sp(n)$. Both are inner. The fixed point set of the former is $U(n)$ and is $SO(p) \times SO(q)$ of the latter.

(4) *There are two types of involutive automorphisms of $SO(2n)$, which are*

$$Ad(I_{pq}), p+q=2n, \quad Ad(J_n).$$

The former is inner, with fixed point set $SO(p) \times SO(q)$, and the latter is outer, with fixed point set $U(n)$.

As a consequence, we obtain symmetric spaces of Type I associated with the classical simple groups.

Remark 82. *To handle the exceptional cases to come up with the remaining spaces of Type I, one needs to study more about the root systems Δ and their* **Satake** *diagrams* [26, p. 132]*, which determines the involutions of Δ* [26, p. 135]*, and hence the involutive automorphisms, of the simply connected compact U. One can read off from the Satake diagrams the root space decomposition with respect to the root system Σ of a symmetric space of Type I or III* [26, p. 119, p. 145-146]*. Specifically, a simply connected symmetric space of Type I is determined uniquely by its root system Σ and the root multiplicities. The nonreduced root systems BC_n occur in both the classical and exceptional categories.*

Lastly, the center of a compact simply connected simple Lie group, given in the table for Type II, can be read off from the root system Δ, or as a consequence, from the extended Dynkin diagram [26, p. 15, p. 96]*. They can be calculated in a straightforward manner for the classical groups.*

To conclude, we record the important theorem.

Theorem 83. [20, p. 284, p. 289] *Let (G, K, σ) be an irreducible Riemannian symmetric space of noncompact type. Let*

$$M := \{k \in K : Ad(k) \cdot v = v, \forall v \in \mathcal{A}\}, \quad M' := \{k \in K : Ad(k) \cdot \mathcal{A} \subset \mathcal{A}\}.$$

Then M'/M is the Weyl group.

Definition 84. *An s-***representation** *of rank r is the isotropy representation of a connected, simply connected semisimple Riemannian symmetric space of rank r. Here, if the symmetric space is decomposed into its irreducible components, the rank is the sum of the ranks of the components.*

An s-representation of rank 2 is either the isotropy representation of two irreducible symmetric spaces of rank 1, or of an irreducible symmetric space of rank 2. Note that $\mathbb{R} \times M$, where M is irreducible symmetric of rank 1, is also of rank 2, although its isotropy representation is not an s-representation.

In connection with classifying homogeneous hypersurfaces in S^n, we are particularly interested in the isotropy representations of simply connected symmetric spaces of rank 2, because of the theorem of Hsiang and Lawson [21] on the classification of all maximal effective orthogonal representations $\rho : G \hookrightarrow SO(n+1)$ of cohomogeneity 2:

Theorem 85. *Up to equivalence, the maximal effective orthogonal representations $\rho : G \hookrightarrow SO(n+1)$ of cohomogeneity 2 are exactly the isotropy representations of the simply connected noncompact symmetric spaces of rank 2, i.e., the isotropy representations of*

(1) $\mathbb{R} \times H^n$, *where the principal orbits are spheres* $S^{n-1} \subset S^n \subset \mathbb{R}^{n+1}$,
(2) $H^p \times H^q$, *where the principal orbits are* $S^{p-1} \times S^{q-1} \subset S^{p+q-1} \subset \mathbb{R}^{p+q}$, *and*
(3) *the noncompact irreducible symmetric spaces of rank 2, where principal orbits are those of s-representations.*

The principal orbits of the first two items are easy to visualize. In Example 50 we have seen that the isotropy, or s-, representation of $O(1,n)$ is the standard orthogonal representation $SO(n)$ on $\mathbb{R}^n$, whose typical principal orbit is the sphere S^{n-1}. The Euclidean factor in item (1) acts trivially, so that a principal orbit of codimension 2 of the isotropy representation is $S^{n-1} \subset S^n \subset \mathbb{R}^{n+1}$. In the same vein, a typical principal orbit of the isotropy representation in item (2) is $S^{p-1} \times S^{q-1} \subset S^{p+q-1} \subset \mathbb{R}^{p+q}$.

Definition 86. *Let M be an isoparametric hypersurface in S^n. The number of principal curvatures of M is denoted by g.*

In particular, we have $g = 1$ or 2 for the homogeneous spaces in the first two items of Theorem 85. The isoparametric hypersurfaces with $g = 1$ or 2, classified by Cartan, are exactly the ones in the first two items.

Let us study item (3) in Theorem 85, where the principal orbits give rise to all homogeneous isoparametric hypersurfaces with $g \geq 3$ in the sphere.

Let G/K be a noncompact irreducible symmetric space of rank 2 with the Cartan decomposition $\mathcal{G} = \mathcal{K} \oplus \mathcal{M}$. Fix a $v \neq 0 \in \mathcal{M}$. We know [20, p. 247] there is a $k \in K$ such that $Ad(k) \cdot v \in \mathcal{A}$, where

$\mathcal{A}$ is the maximal abelian subspace of $\mathcal{M}$. Therefore, we may assume without loss of generality that $v \in \mathcal{A}$.

Proposition 87. *With the setup above, an orbit $Ad(K) \cdot v$, where $v \in \mathcal{A}$, is principal, of codimension 2, if and only if v lies in a Weyl chamber.*

Proof. The isotropy subgroup L of $Ad(K)$ leaving v fixed assumes the Lie algebra

$$\mathcal{L} := \{X \in \mathcal{K} : [X, v] = 0\}.$$

We have the root space decomposition

$$0 = ad_h(Z) = \sum_{\lambda \in \Sigma} \lambda(h) Z_\lambda \tag{10}$$

with $h \in \mathcal{A}$ and $Z_\lambda \in \mathcal{G}_\lambda$, and so

$$\mathcal{G} = \mathcal{N}_0 \oplus \sum_{\lambda \in \Sigma} \mathcal{G}_\lambda,$$

where $\mathcal{N}_0$ is the centralizer of $\mathcal{A}$ in $\mathcal{G}$.

If v belongs to a Weyl chamber, then $\lambda(v) \neq 0$ for all $\lambda \in \Sigma$, so that by (10) $X_\lambda = 0$ for all $\lambda \in \Sigma$. That is, $X \in \mathcal{L}$ if and only if $X \in \mathcal{M}_0 := \mathcal{N}_0 \cap \mathcal{K}$, the centralizer of $\mathcal{A}$ in $\mathcal{K}$.

By [20, Lemma 3.6, p. 261], we know $\mathcal{M}_0$ has the same codimension in $\mathcal{K}$ as $\mathcal{A}$ in $\mathcal{M}$. That is,

$$\dim(Ad(K)/L) = \dim(\mathcal{K}) - \dim(\mathcal{L}) = \dim(\mathcal{M}) - \dim(\mathcal{A}).$$

In other words, the isotropy orbit is of codimension $\dim(\mathcal{A}) = 2$.

If v lies in a chamber wall, then by (10) $X \in \mathcal{L}$ if and only if

$$X \in \mathcal{M}_0 \oplus \sum_{\lambda, \lambda(v) \neq 0} (\mathcal{G}_\lambda \cap \mathcal{K}).$$

Therefore, the codimension of the orbit of v is larger than 2. $\square$

Corollary 88. *The isotropy representation of an irreducible noncompact symmetric space of rank 2 has only two singular orbits and a 1-parameter family of diffeomorphic principal orbits of codimension 2 degenerating to the two singular orbits.*

Proof. In the rank 2 case, a Weyl chamber is a sector of the plane of angle measure $\pi/3$ for A_2, $\pi/4$ for B_2 and $\pi/6$ for G_2. Let us say $\theta_0 < \theta < \theta_0 + \pi/l, l = 3, 4, 6$, defines the chamber. Then the preceding proposition says that for any unit v assuming angle θ in the chamber, its isotropic orbit is homogeneous (and hence isoparametric) of codimension 2 and is diffeomorphic to $Ad(K)/L$. So we have a 1-parameter family of diffeomorphic homogeneous isoparametric hypersurfaces. At the two chamber walls, that is, when v assumes the angle θ_0 or $\theta_0+\pi/l$, the dimension of the orbit drops. Meanwhile, since the normalizer of $\mathcal{A}$ serves as the Weyl group by Theorem 83, we see that the isotropic orbit of v intersecting the chamber plane at some points $v_1 = v, v_2, \cdots, v_{2l}$, one in each chamber. So the isotropic representation has only two singular orbits, even though there are $2g$ Weyl chambers. All other orbits are principal of codimension 2. □

Proposition 89. *With the same setup, $Ad(k)(\mathcal{A})$ is the normal plane to the principal orbit $Ad(K)(v)$ at $Ad(k)(v)$ for $v \in \mathcal{A}$.*

Proof. It suffices to check this at v, where the tangent space of the orbit is

$$T_v = \{[h, v] : h \in \mathcal{K}\}.$$

But then for $w \in \mathcal{A}$, we have, since the inner product is proportional to the Killing form,

$$\langle w, [h, v]\rangle = \langle [v, w], h\rangle = 0.$$

□

Proposition 90. *With the same setup, let w be a unit vector perpendicular to v in $\mathcal{A}$, and extend it to a global normal field on the principal orbit $Ad(K) \cdot v, |v| = 1$, in the unit sphere of $\mathcal{M}$. The shape operator S_w of the orbit at v satisfies that the eigenvalues are*

$$-\lambda(w)/\lambda(v),$$

where λ are reduced positive roots such that $\lambda/2 \notin \Sigma$. The eigenspace associated with the above eigenvalue is

$$E_\lambda = \mathcal{G}_\lambda \oplus \mathcal{G}_{-\lambda} \oplus \mathcal{G}_{2\lambda} \oplus \mathcal{G}_{-2\lambda}.$$

In particular, g, the number of principal curvatures of the shape operator, is 3, 4, or 6. If we label the principal curvatures by $\lambda_1 > \cdots > \lambda_g$ and their multiplicities by $m_1, \cdots, m_g$, then $m_i = m_{i+2}$, where the subscripts are modulo g. In particular, the multiplicities are all equal when $g = 3$. Moreover, if we choose the angles

$$\theta_i = (2i - 1)\pi/2g, \quad i = 1, \cdots, g,$$

to coordinatize the positive roots, then the principal curvatures are

$$\lambda_i = \tan(\theta - \theta_i), \quad -\pi/g < \theta < \pi/g,$$

when v assumes the angle θ and w the angle $\theta + \pi/2$.

Proof. As mentioned in the preceding proposition, a vector X tangent to the orbit is of the form

$$X = [k, v] = -\sum_{\lambda\in\Sigma} \lambda(v) X_\lambda$$

for $k \in \mathcal{K}$. Since

$$n = Ad(K) \cdot w$$

is a normal vector field to the orbit, the shape operator is

$$S(X) := -dn(X) = -[X, w] = \sum_{\lambda\in\Sigma} \lambda(w) X_\lambda.$$

Therefore, we obtain

$$-\lambda(v) S(X_\lambda) = \lambda(w) X_\lambda.$$

Since v is regular we have $\lambda(v) \neq 0$ for all $\lambda \in \Sigma$. It follows that

$$S(X_\lambda) = -\lambda(w)/\lambda(v) X_\lambda.$$

The principal curvatures of S are thus $-\lambda(w)/\lambda(v)$, which is assumed by $\pm\lambda, \pm 2\lambda$. It follows that the eigenspace E_λ with the principal curvature $-\lambda(w)/\lambda(v)$ is the desired one, where λ need only go through the positive roots λ for which $\lambda/2 \notin \Sigma$, which form a reduced root system. The number of positive roots in the A_2, B_2, or G_2 root system is 3, 4, or 6, respectively, which is g.

We choose the angles $\theta_i = (2i - 1)\pi/2g$ to coordinatize the positive roots. We see the Weyl group is generated by

$$\theta \mapsto \pi/g - \theta, \quad \theta \mapsto \theta + 2\pi/g. \tag{11}$$

By Theorem 83, the Weyl group preserves the principal curvatures and their multiplicities. Hence, $m_i = m_{i+2}$ with index modulo g.

Lastly, since

$$v = (\cos(\theta), \sin(\theta)), \quad w = (-\sin(\theta), \cos(\theta)), \quad \lambda_i = (\cos(\theta_i), \sin(\theta_i)),$$

we calculate

$$-\lambda_i(w)/\lambda_i(v) = -\langle w, \lambda_i\rangle/\langle v, \lambda_i\rangle = \tan(\theta - \theta_i).$$

□

Let us now look at

$$F(\theta) := \sin(g\theta), \quad -\pi/2g < \theta < \pi/2g.$$

It is left invariant by the two generators of the Weyl group in (11). In fact, $F(\theta)$ is the restriction to the unit circle of the homogeneous polynomial of degree g

$$(12) \qquad F_{\mathcal{A}} := \sum_{0}^{[(g-1)/2]} \binom{g}{2i+1} (-1)^i x^{g-(2i+1)} y^{2i+1}$$

defined by the maximal abelian space $\mathcal{A}$. $F_{\mathcal{A}}$ is left invariant by the Weyl group.

Theorem 91. [23, p. 299] *The space of homogeneous polynomials on $\mathcal{M}$ left invariant by $Ad(K)$ is isomorphic to the space of homogeneous polynomials on $\mathcal{A}$ left invariant by the Weyl group.*

This theorem is called Chevalley Restriction Theorem. In [23], the proof is given for a compact Lie group, or for a symmetric space of Type II. But the proof there can be modified easily to arrive at the preceding theorem in view of Theorem 83.

Theorem 92. [14] *The space of homogeneous polynomials on a maximal abelain space $\mathcal{A}$ of dimension r left invariant by the Weyl group is generated by r algebraically independent polynomials.*

Since $r = 2$ in our case and we have found two generators, namely, x^2+y^2 and $F_{\mathcal{A}}$ on $\mathcal{A}$, the space of homogeneous polynomials left invariant by $Ad(K)$ on $\mathcal{M}$ of dimension n is thus generated by $(x_1)^2 + \cdots + (x_n)^2$ and a homogeneous polynomial F of degree g whose restriction to the circle is $F_{\mathcal{A}}$.

F, homogeneous of degree g, thus leaves each isotopic orbit invariant. Therefore, we conclude the following.

Theorem 93. *There is a homogeneous polynomial F of degree g, called Cartan polynomial, for $g = 3, 4, 6$, on $\mathcal{M}$, whose restriction f to the unit sphere of $\mathcal{M}$ satisfies the property that its range is $[-1, 1]$. For each $c \in (-1, 1)$, $f^{-1}(c)$ is a homogeneous (isoparametric) hypersurface degenerating to two singular submanifolds $f^{-1}(\pm 1)$. The statement is clearly true when $g = 1$ or 2. All homogeneous hypersurfaces in spheres are constructed this way.*

We remark that for $g = 1$ in the preceding theorem, the polynomial is $F = x_{n+1}$ over $\mathbb{R}^{n+1}$, while for $g = 2$ the polynomial is $F = (x_1)^2 + \cdots + (x_r)^2 - (x_{r+1})^2 - \cdots - (x_{r+s})^2$ over $\mathbb{R}^{r+s}$.

Example 94. *We find F in the case $g = 3$ when the symmetric space is $SU(3)/SO(3)$ of Type I and rank 2.*

Let $\mathcal{M}$ be the space of 5-dimensional 3 by 3 real traceless symmetric matrices. The Cartan decomposition is

$$su(3) = so(3) \oplus \sqrt{-1}\mathcal{M}, \quad \mathcal{K} = so(3).$$

$\mathcal{M}$ is equipped with the inner product

$$< Y, Y > := tr(YY) = \alpha^2 + \beta^2 + \gamma^2 + x^2 + y^2 + z^2,$$

which is a multiple of the Killing form of $su(3)$ restricted to $\mathcal{M}$, where we write

$$Y := \begin{pmatrix} \alpha & x/\sqrt{2} & y/\sqrt{2} \\ x/\sqrt{2} & \beta & z/\sqrt{2} \\ y/\sqrt{2} & z/\sqrt{2} & \gamma \end{pmatrix}, \quad \alpha + \beta + \gamma = 0.$$

The isotropic action is the adjoint action

$$Ad(T) : V \in \mathcal{M} \mapsto TVT^{-1} \in \mathcal{M}, \quad T \in SO(3).$$

The diagonal block of $\mathcal{M}$ is the maximal abelian subspace $\mathcal{A}$ of $\mathcal{M}$. The three positive roots are

$$\alpha_1 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} /\sqrt{2}, \quad \alpha_2 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} /\sqrt{2}, \quad \alpha_3 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} /\sqrt{2},$$

where α_1 and α_2 are simple roots. We choose the unit angle bisector as the standard basis element

$$e_1 := (2\alpha_1 + \alpha_2)/\sqrt{6},$$

and

$$e_2 := \alpha_2.$$

Then e_1, e_2 form an orthonormal basis of $\mathcal{A}$, so that an element in $\mathcal{M}$ relative to e_1, e_2 is

$$X := ae_1 + be_2 = \begin{pmatrix} 2a/\sqrt{6} & 0 & 0 \\ 0 & b/\sqrt{2} - a/\sqrt{6} & 0 \\ 0 & 0 & -b/\sqrt{2} - a/\sqrt{6} \end{pmatrix},$$

and the Y above is

$$Y := \begin{pmatrix} 2a/\sqrt{6} & x/\sqrt{2} & y/\sqrt{2} \\ x/\sqrt{2} & b/\sqrt{2} - a/\sqrt{6} & z/\sqrt{2} \\ y/\sqrt{2} & z/\sqrt{2} & -b/\sqrt{2} - a/\sqrt{6} \end{pmatrix}.$$

Now it is clear that $\det(Y)$ *is $Ad(SO(3))$-invariant. We calculate*

$$\det(X) = \frac{1}{3\sqrt{6}}(a^3 - 3ab^2) = \frac{1}{3\sqrt{6}} F_{\mathcal{A}},$$

where $F_{\mathcal{A}}$ is given in (12)*, when we set*

$$a = \cos(\pi/6 - \theta), \quad b = \sin(\pi/6 - \theta), \quad -\pi/6 < \theta < \pi/6.$$

It follows that

$$F := 3\sqrt{6}\det(Y)$$

restricts to $F_{\mathcal{A}}$ and so F is the Cartan polynomial given in Theorem 93. A calculation shows F is exactly the polynomial given in (3) by Cartan in the case when $\mathbb{F}$ is $\mathbb{R}$.

Note that f, the restriction of F to the unit sphere, has range $[-1,1]$ and $f^{-1}(\pm 1)$ are the two singular submanifolds, both being the projective plane. To see this, we set $\theta = \pm\pi/6$. Then, respectively,

$$X = \begin{pmatrix} 2/\sqrt{6} & 0 & 0 \\ 0 & -1/\sqrt{6} & 0 \\ 0 & 0 & -1/\sqrt{6} \end{pmatrix}, \quad \begin{pmatrix} 1/\sqrt{6} & 0 & 0 \\ 0 & 1/\sqrt{6} & 0 \\ 0 & 0 & -2/\sqrt{6} \end{pmatrix}.$$

Let us find the isotropy group L of the isotropy action on X, where L consists of all $T \in SO(3)$ commuting with X. We see L is in diagonal block form. Hence,

$$L \simeq S(O(1) \times O(2)),$$

so that the singular orbits are

$$Ad(SO(3))/L = SO(3)/S(O(1) \times O(2)) = \mathbb{R}P^2.$$

Remark 95. *A look at the tables for the symmetric spaces of rank 2 of Types I and II shows that there are four such spaces with $g = 3$, which are*

$$SU(3)/SO(3), \quad SU(3)\times SU(3)/\Delta(SU(3)\times SU(3)), \quad SU(6)/Sp(3), \quad E_6/F_4,$$

whose Cartan polynomials of their isotropic orbits are the ones given in (3).

As in the $SU(3)/SO(3)$ case, the singular orbits of the other three examples are, respectively, the complex, quaternionic and octonion projective planes. The principal orbits are tubes around the projective planes.

The following table is the collection of all symmetric spaces G/K of Types I and II whose isotropy representations give homogeneous (isoparametric) hypersurfacs M. There are at most two multiplicities $(m_1, m_2), m_1 \leq m_2$, for the g principal curvatures.

G	K	$\dim M$	g	(m_1, m_2)
$S^1 \times SO(n+1)$	$SO(n)$	n	1	$(1,1)$
$SO(p+1) \times SO(n+1-p)$	$SO(p) \times SO(n-p)$	n	2	$(p, n-p)$
$SU(3)$	$SO(3)$	3	3	$(1,1)$
$SU(3) \times SU(3)$	$SU(3)$	6	3	$(2,2)$
$SU(6)$	$Sp(3)$	12	3	$(4,4)$
E_6	F_4	24	3	$(8,8)$
$SO(5) \times SO(5)$	$SO(5)$	8	4	$(2,2)$
$SO(10)$	$U(5)$	18	4	$(4,5)$
$SO(m+2), m \geq 3$	$SO(m) \times SO(2)$	$2m-2$	4	$(1, m-2)$
$SU(m+2), m \geq 2$	$S(U(m) \times U(2))$	$4m-2$	4	$(2, 2m-2)$
$Sp(m+2), m \geq 2$	$Sp(m) \times Sp(2)$	$8m-2$	4	$(4, 4m-5)$
E_6	$(Spin(10) \times SO(2))/\mathbb{Z}_4$	30	4	$(6,9)$
G_2	$SO(4)$	6	6	$(1,1)$
$G_2 \times G_2$	G_2	12	6	$(2,2)$

3. Development in the early 1970s, the general case

Münzner [33] in 1973 proved a remarkable result that extended Cartan's investigation, recorded in Theorem 11, in a far-reaching manner:

Theorem 96. (Münzner's structure theory on isoparametric hypersurfaces) *Let M be any isoparametric hypersurfaces with g principal curvatures in S^n. Then we have the following.*

(1) *There is a homogeneous polynomial F, called Cartan-Münzner polynomial, of degree g over $\mathbb{R}^n$ satisfying*
$$|\nabla F|^2 = g^2 r^{2g-2}, \quad \Delta F = \frac{m_- - m_+}{2} g^2 r^{g-2},$$
where r is the radial function over $\mathbb{R}^{n+1}$.

(2) *Let $f := F|_{S^n}$. Then the range of f is $[-1,1]$. The only critical values of f are ± 1. Moreover, $M_\pm := f^{-1}(\pm 1)$ are connected submanifolds of codimension $m_\pm + 1$ in S^n, called focal submanifolds, whose principal curvatures are* $\cot(k\pi/g), 1 \leq k \leq g-1$.

(3) *For any $c \in (-1,1)$, $f^{-1}(c)$ is an isoparametric hypersurface with at most two multiplicities $m_\pm$ associated with the principal curvatures. In fact, if we order the principal curvatures $\lambda_1 > \cdots > \lambda_g$ with multiplicities $m_1, \cdots, m_g$, then $m_i = m_{i+2}$ with index modulo g; in particular, all multiplicities are equal when g*

is odd, and when g is even, there are at most two multiplicities precisely equal to $m_{\pm}$.

(4) *Each of the* 1*-parameter isoparametric hypersurfaces is a tube around the two focal submanifolds, so that* S^n *is obtained by gluing two disk bundles over* $M_{\pm}$ *along the isoparametric hypersurface* $M_0 := f^{-1}(0)$. *As a consequence, algebraic topology implies that the only possible values of* g *are* $1, 2, 3, 4,$ *or* 6.

Indeed, starting from an isoparametric hypersurface

$$x : M \hookrightarrow S^n$$

whose principal curvatures are set to be

$$\lambda_j = \cot(\theta_j), \quad 0 < \theta_1 < \cdots < \theta_g < \pi,$$

with respect to the outward normal field n. Let us consider the parallel transport of M

$$x_t := \cos(t)x + \sin(t)n, \tag{13}$$

which is the counterpart to the Euclidean parallel transport along the normal direction. A priori, $M_t := x_t(M)$ is an embedding for small t. Since

$$n_t := -\sin(t)x + \cos(t)n$$

is normal to M_t, a straightforward calculation derives that the principal curvatures of M_t, with respect to the chosen normal field n_t, are

$$\lambda_j(t) = \cot(\theta_j - t) \tag{14}$$

with the same eigenspace and multiplicity as λ_j. On the other hand, for a fixed l, the eigenspace of λ_l from point to point defines an integrable distribution, called the l-th curvature distribution, on M with spheres of radius $|\sin(\theta_l)|$ as leaves. This can be directly checked by differentiating

$$f_l(x) := x + v_l(x)/|v_l(x)|^2, \quad v_l(x) := -x + \cot(\theta_l)n,$$

to see that $f_l(x)$ is a constant c_l on the l-th curvature leaf through x; we have

$$c_l = \cos(\theta_l)(\cos(\theta_l)x + \sin(\theta_l)n), \tag{15}$$

i.e., the unit vector along c_l assumes the angle θ_l on the unit circle spanned by x and n. Now that the curvature leaf through x is a sphere of radius $|\sin(\theta_l)|$ centered at c_l, the antipodal point to x on this leaf is the reflection map ϕ_l about c_l:

$$\phi_l(x) := x + 2v_l(x)/|v_l(x)|^2 = \cos(2\theta_l)x + \sin(2\theta_l)n;$$

that is, $\phi(x)$ is the point of reflection of x about the line spanned by c_l on the (x, n)-plane. Therefore, by (14), the principal values of M at $\phi_l(x)$ are

$$-\cot(\theta_j - 2\theta_l), \tag{16}$$

where j varies, with the same eigenspaces and multiplicities as x. Note that the sign in (16) differs from that in (14), because the circle x_t leaves M at x and enters M at $\phi(x)$, so that $n_{2\theta}$ at $\phi(x)$ is negative of the chosen outward normal field n of M at $\phi(x)$. Since M has constant principal values, counting multiplicities, we conclude that the following sets

$$\{\cot(\theta_j)\}, \quad \{\cot(2\theta_l - \theta_j)\} \tag{17}$$

are identical for all j, l, and two numbers, one from each set, having the same index j have the same principal multiplicity m_j, regardless of what l is.

Now (17) means that the lines L_j spanned by c_j on the (x, n)-plane, all through the origin, satisfies the property that the reflection of L_j about any L_l is another L_k. It follows that these lines $L_j, 1 \leq j \leq g$, are equally spaced in the (x, n)-plane so that

$$\theta_j = (j - 1)\pi/g + \theta_1.$$

Thus the reflections about the lines L_j results in $m_i = m_{i+2}$ with index modulo g. Accordingly, we denote m_1 and m_2 by m_+ and m_-, respectively. (This is reminiscent of a root system and its Weyl chambers.)

Having done so, Münzner went on to construct a local isoparametric function, which is nothing but an appropriate distance function, in a neighborhood of M as follows. Any p in a tubular neighborhood U of M can be written uniquely as

$$p = \cos(t)x + \sin(t)n$$

for some small t. Define

$$\tau(p) := \theta_1 - t, \quad V(p) = \cos(g\tau(p)).$$

Extend $V(p)$ to a neighborhood of M in the ambient Euclidean space by defining

$$F(rp) = r^g V(p),$$

where r is the Euclidean radial function. Münzner then verified the following.

Theorem 97. *F is in fact a homogeneous polynomial of degree g satisfying*

$$|\nabla F|^2 = g^2 r^{2g-2}, \quad \Delta F = g^2 \frac{m_- - m_+}{2}.$$

Proof. (Sketch) Define

$$G := F - ar^g,$$

where

$$a := \frac{g}{g+n-1}\frac{m_- - m_+}{2}.$$

Then verify that

$$\Delta G = 0.$$

In general, it is true that for a harmonic function G over $\mathbb{R}^{n+1}$, we have

$$\Delta^g |\nabla G|^2 = \sum (\partial^{g+1} G/\partial x_{i_1} \cdots \partial x_{i_{g+1}})^2. \tag{18}$$

On the other hand, for the G involved a calculation gives

$$|\nabla G|^2 = g^2 r^{2g-2}(1+a^2) - 2ag^2 r^{g-2} F.$$

We therefore find

$$\Delta^{g-1}|\nabla G|^2 = c$$

with c an appropriate constant. F is thus a homogeneous polynomial by (18). □

Now that F is globally analytic over $\mathbb{R}^{n+1}$, we set $f := F|_{S^n}$. A calculation by the formulae

$$|\nabla F|^2 = (\frac{\partial F}{\partial r})^2 + |\nabla f|^2, \quad \Delta f = \Delta F - \frac{\partial^2 F}{\partial r^2} - n\frac{\partial F}{\partial r}$$

derives, by Theorem 97, that

$$|\nabla f|^2 = A(f), \quad \Delta f = B(f),$$

where

$$A(f) = g^2(1-f^2), \quad B(f) = -g(n+g-1)f + \frac{m_- - m_+}{2} g^2.$$

So, f is an isoparametric function on S^n. Note that $A(f) = 0$ only at $f = \pm 1$, so that the range of f is $[-1,1]$ and ± 1 are the only critical values. Let $M_\pm := f^{-1}(\pm 1)$ be the singular set. $S^n \setminus (M_+ \cup M_-)$ is open and dense and is diffeomorphic to $M_c \times (-1,1)$ for any fixed c, where $M_c := f^{-1}(c)$ for $c \in (-1,1)$.

A priori, M_c need not be connected. We claim that this is not the case. Define

$$d : M \times (0, \pi/g) \to S^n, \quad d(x,\tau) = \cos(\theta_1 - \tau)x + \sin(\theta_1 - \tau)n.$$

Then

$$f(d(x,\tau)) = \cos(g\tau)$$

by the analytic nature of f, since $f|_U = V$ and the identity is true on U. That is, M is contained in M_c, where $c = \cos(g\theta_1)$. But then the map

(19) $d_c : M_c \times (0, \pi/g) \to S^n, \quad d_c(x, \tau) = \cos(\theta_1 - \tau)x + \sin(\theta_1 - \tau)n$

also satisfies $f(d_c(x, \tau)) = \cos(g\tau)$ and

$$d_c : M_c \times (0, \pi/g) \to S^n \setminus (M_+ \cup M_-)$$

is a diffeomorphism. From this we see that the map

$$g := M_c \to S^n, \quad x \mapsto \cos(\theta_1)x + \sin(\theta_1)n$$

maps M_c to M_+. Observe that $g(x) = c_1$ for any curvature leaf through x whose tangent space is the eigenspace with principal value $\cot(\theta_1)$, where c_l is defined in (15), remarking that c_1 is the center of the spherical leaf, and vice versa. We see that $g : M_c \to M_+$ is a sphere bundle whose fiber is a curvature leaf diffeomorphic to S^{m_+}. Meanwhile, it is easy to check that dg has kernel dimension m_+; at x, the derivative dg preserves eigenspaces of all principal values other than that of $\cot(\theta_1)$. Therefor, M_+ is a manifold of dimension $\dim(M) - m_+$, which is of codimension at least 2 in S^n. Likewise, the codimension of M_- is at least 2 in S^n.

Returning to the map (19), we see now $S^n \setminus (M_+ \cup M_-)$ is connected as M_+ and M_- are of codimension at least 2 in S^n. Therefore, that d_c is a diffeomorphism ensures that M_c is connected, for all c. As a consequence, $M_\pm$ are also connected via the map g.

Lastly, since $g = x_{\theta_1}$ defined in (13), we see by (14) that the principal values of M_+, in any normal direction, are

$$\cot(\theta_j - \theta_1) = \cot((j-1)\pi/g), \quad 2 \le j \le g.$$

This also holds true for M_-.

Corollary 98. *$M_\pm$ are minimal submanifolds of S^n. The minimality condition is exactly equation* (1)*, the fundamental formula of Segre and Cartan.*

Proof. By the preceding formula, the mean curvature of M_+ in any normal direction is

$$\sum_{j=1}^{g-1} \cot(j\pi/g) = 0,$$

which is exactly the fundamental formula by Remark 9. ☐

Corollary 99. *There is a unique minimal isoparametric hypersurface in the* 1*-parameter family M_t.*

Proof. By (14), the mean curvature of M_t is

$$H := \sum_{j=1}^{g} \cot(\theta_j - t)$$

for $t \in (0, \pi/g)$. H is strictly increasing as the derivative is > 0. Near $t = \theta_1 < \pi/g$ the function is > 0 whereas near $\theta = \pi/g - \theta_1 > 0$ the function is < 0. Therefore, there is a unique $t \in (0, \pi/g)$ at which $H = 0$. □

Now that S^n is obtained by gluing two disk bundles over the focal submanifolds $M_\pm$ along an isoparametric hypersurface M, Münzner used algebraic topology to express the cohomology ring of M, with $\mathbb{Z}_2$ coefficients, as modules of those of $M_\pm$, whose module structures then give $g = 1, 2, 3, 4$, or 6.

Based on Münzner's work, Ozeki and Takeuchi [37, I] constructed two classes, each with infinitely many members, of inhomogeneous isoparametric hypersurfaces with $g = 4$. They also classified all isoparametric hypersurfaces with $g = 4$ when one of the multiplicities $m_\pm$ is 2, which are all homogeneous [37, II].

An important ingredient in their work is their expansion formula of the Cartan-Münzner polynomial:

$$\begin{aligned} F(tx + y + w) &= t^4 + (2|y|^2 - 6|w|^2)t^2 + 8(\sum_{a=0}^{m_+} p_a w_a)t \\ &+ |y|^4 - 6|y|^2|w|^2 + |w|^4 - 2\sum_{a=0}^{m_+}(p_a)^2 + 8\sum_{a=0}^{m_+} q_a w_a \\ &+ 2\sum_{a,b=0}^{m_+} \langle \nabla p_a, \nabla p_b \rangle w_a w_b. \end{aligned}$$

Here, x is a point on M_+, y is tangent to M_+ at x, and w is normal to M_+ with coordinates w_i with respect to the chosen orthonormal normal basis $\mathbf{n}_0, \mathbf{n}_1, \cdots, \mathbf{n}_{m_+}$ at x. Moreover, $p_a(y)$ (resp., $q_a(y)$) is the a-th component of the 2nd (resp., 3rd) fundamental form of M_+ at x. Furthermore, p_a and q_a are subject to ten convoluted equations [37, I, pp 529-530], of which the first three assert that, since $S_\mathbf{n}$, the 2nd fundamental matrix of M_+ in any unit normal direction $\mathbf{n}$, has eigenvalues $1, -1, 0$ with fixed multiplicities, it must be that $(S_\mathbf{n})^3 = S_\mathbf{n}$.

The expansion formula coupled with the ten identities are fundamentally important for the classification of isoparametric hypersurfaces with $g = 4$ [9], [10], [11].

4. Development in the 1980s

The multiplicities of the principal values for $g = 4$ and $g = 6$ had remained undetermined until Arbresch [2] extended Münzner's work to show by algebraic topology that for $g = 6$ we have $m_+ = m_- = 1$ or 2. This is in agreement with the multiplicities of the homogeneous examples. Although he derived some constraints in the case $g = 4$, among which we have, for instance, $m_+ = m_-$ implies $m_+ = m_- = 1$ or 2, etc., the case remained open.

Meanwhile, Ferus, Karcher and Münzner generalized the inhomogeneous examples of Ozeki and Takeuchi to construct infinitely many classes, each with infinitely many members, of inhomogeneous isoparametric hypersurfaces with $g = 4$. Their construction can be best motivated by the example in Nomizu's paper [34] mentioned earlier:

Consider $\mathbb{C}^k = \mathbb{R}^k \oplus \mathbb{R}^k$ and write $z \in \mathbb{C}^k$ as $z = x + \sqrt{-1}y$ accordingly. Define an homogeneous polynomial of degree 4 on $\mathbb{C}^k$ by

$$\tilde{F} = (|x|^2 - |y|^2) + 4(\langle x, y\rangle)^2.$$

Then F is an isoparametric function with multiplicities $\{1, k-2\}$. In fact, the isoparametric hypersurfaces are the principal isotropy orbits of the symmetric spaces $SO(k+2)/S(2) \times SO(k)$.

Note that $\tilde{f} = \tilde{F}|_{S^{2k-1}}$ has range $[0, 1]$. So we normalize it by defining $f := 1 - 2\tilde{f}$, or rather, by setting

$$F := (|x|^2 + |y|^2)^2 - 2\tilde{F}.$$

F is an isoparametric function such that f has range $[-1, 1]$. Let us set

$$P_0 := \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad P_1 := \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad u := (x, y)^{tr},$$

where I is the k by k identity matrix. Then F can be rewritten as

$$F = |u|^4 - 2\sum_{i=0}^{1} \langle P_i u, u\rangle^2, \quad P_i P_j + P_j P_i = 2\delta_{ij} I.$$

Ferus, Karcher and Münzner's construction is a generalization of this.

Definition 100. *The skew-symmetric (resp., symmetric) Clifford algebra C_n (resp., C'_n) over $\mathbb{R}$ is the algebra generated by the standard basis $e_1, \cdots, e_n$ of $\mathbb{R}^n$ subject to the only constraint*

$$e_i e_j + e_j e_i = -2\delta_{ij} I \quad (\mathit{resp.,} \;\; e_i e_j + e_j e_i = 2\delta_{ij} I).$$

The classification of the Clifford algebras are known [22]:

n	1	2	3	4	5	6	7	8
C_n	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}\oplus\mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8)\oplus\mathbb{R}(8)$	$\mathbb{R}(16)$
δ_n	1	2	4	4	8	8	8	8
C'_n	$\mathbb{R}\oplus\mathbb{R}$	$\mathbb{R}(2)$	$\mathbb{C}(2)$	$\mathbb{H}(2)$	$\mathbb{H}(2)\oplus\mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$
θ_n	2	4	8	8	16	16	16	16

Here, δ_n is the dimension of an irreducible module of C_{n-1}, and θ_n is the dimension of an irreducible module of C'_{n+1}. Moreover, C_n (resp., C'_n) is subject to the periodicity condition $C_{n+8} = C_n \otimes \mathbb{R}(16)$ (resp., $C'_{n+8} = C'_n \otimes \mathbb{R}(16)$). The generators $e_1, \cdots, e_n$ acting on each irreducible module of either C_n or C'_n in the table give rise to n skew-symmetric or symmetric orthogonal matrices $T_1, \cdots, T_n$ satisfying

$$T_iT_j + T_jT_i = \pm 2\delta_{ij} I,$$

a representation of C_n or C'_n on the irreducible module. Note that we have

$$\theta_n = 2\delta_n.$$

This is not fortuitous. It says that we can construct symmetric representations of C'_{m+1} from skew-symmetric representations of C_{m-1}, and vice versa. Indeed, let us be given k irreducible representations $V_1, \cdots, V_k$ of C_{m-1}. Set

$$V := V_1 \oplus + \cdots \oplus V_k \simeq \mathbb{R}^l, \quad l = k\delta_m.$$

The representations of $e_1, \cdots, e_{m-1}$ on $V_1, \cdots, V_k$ give rise to $m-1$ skew-symmetric orthogonal matrices $E_1, \cdots, E_{m-1}$ on V. Set

$$P_0 := \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad P_1 := \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad P_{1+i} = \begin{pmatrix} 0 & E_i \\ -E_i & 0 \end{pmatrix}, \ 1 \leq i \leq m-1.$$

Then

$$P_iP_j + P_jP_i = 2\delta_{ij} I.$$

$P_0, \cdots, P_m$ give a representation of C'_{m+1} on $\mathbb{R}^{2l}$.

Ferus, Karcher and Münzner's examples, referred to as of OT-FKM type, are

$$F := 2|u|^4 - 2\sum_{i=0}^{m} (\langle P_i u, u\rangle)^2, \quad u \in \mathbb{R}^{2l}, \quad l = k\delta_m.$$

Note that we recover Nomizu's example when $m = 1$.

By a straightforward calculation, we conclude [17]

Proposition 101. *The two multiplicities $m_\pm$ of the associated isoparametric hypersurface are*

$$(m, k\delta_m - m - 1),$$

where $m, k \in \mathbb{N}$ to make the second entry positive.

We have the following table for the multiplicity pair $(m, k\delta_m - m - 1)$.

$\delta_m =$	1	2	4	4	8	8	8	8	16	$\cdots$
$k = 1$	–	–	–	–	$(5,2)$	$(6,1)$	–	–	$(9,6)$	$\cdots$
$k = 2$	–	$(2,1)$	$(3,4)$	$(4,3)$	$(5,10)$	$(6,9)$	$(7,8)$	$(8,7)$	$(9,22)$	$\cdots$
$k = 3$	$(1,1)$	$(2,3)$	$(3,8)$	$(4,7)$	$(5,18)$	$(6,17)$	$(7,16)$	$(8,15)$	$(9,38)$	$\cdots$
$k = 4$	$(1,2)$	$(2,5)$	$(3,12)$	$(4,11)$	$(5,26)$	$(6,25)$	$(7,24)$	$(8,23)$	$(9,54)$	$\cdots$
$k = 5$	$(1,3)$	$(2,7)$	$(3,16)$	$(4,15)$	$(5,34)$	$(6,33)$	$(7,32)$	$(8,31)$	$(9,70)$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Among other things, Ferus, Karcher and Münzner established

Theorem 102. (1) *OT-FKM type with multiplicities on the first, second, fourth columns, $(4,3)$ and $(9,6)$ are exactly the homogeneous examples, except for the two with multiplicities $\{2,2\}$ and $\{4,5\}$ not on the list.*

(2) *OT-FKM type with multiplicities on the third and seventh columns are exactly the inhomogeneous examples constructed by Ozeki and Takeuchi.*

So, except for the first, second and fourth columns, we have infinitely many families, each with infinitely many members, of inhomogeneous isoparametric hypersurfaces with four principal curvatures. Note that we also have the fact that OT-FKM type with multiplicities $(1, l)$ or $(2, l)$ is congruent to the one with multiplicities $(l, 1)$ or $(l, 2)$ [17, 6.5]. Note also that Cartan classified the cases when the multiplicities are $\{1,1\}$ and $\{2,2\}$, both being homogeneous as mentioned in Theorem 11.

Wang [46] investigated the topology of OT-FKM type by K-theory and showed that there are many pairs of minimal isoparametric hypersurfaces in spheres, of identical constant scalar curvature, which are diffeomorphic but noncongruent to each other. Also, in [45] he proved that on a compact manifold N, a transnormal function f alone, where $|\nabla f|^2 = A(f)$ for some smooth A, already warrants that f has only two critical values a and b, where $[a, b]$ is the range of f. Moreover, the two singular sets $M_- := f^{-1}(a)$ and $M_+ := f^{-1}(b)$ are smooth manifolds, and each level hypersurface $f^{-1}(c), c \in (a,b)$, is a tube over $M_\pm$. In particular, he obtained

Theorem 103. *An transnormal function f on S^n is already isoparametric so long as $M_\pm$ are of codimension at least 2. That is, with the assumption, $|\nabla f|^2 = A(f)$ implies $\Delta f = B(f)$ for some smooth B.*

The theorem is in fact also true for $\mathbb{R}^n$ as his analysis showed in the noncompact case. See [19] and [30] for a follow-up study.

Dorfmeister and Neher [16] settled one of the two cases when $g = 6$.

Theorem 104. *An isoparametric hypersurface with $g = 6$ and multiplicities $m_\pm = 1$ is homogeneous.*

5. Development in the 1990s

Another remarkable result, via homotopy theory, in the late 1990s was achieved by Stolz [41], who classified all the possible multiplicity pairs $(m_1, m_2), m_1 \leq m_2$, of isoparametric hypersurfaces with $g = 4$.

Theorem 105. *The multiplicity pairs $(m_1, m_2), m_1 \leq m_2$, of isoparametric hypersurfaces with four principal curvatures are exactly those in the above table for the OT-FKM type, barring the pairs $(2,2)$ and $(4,5)$ not in the table.*

He established that if $(m_1, m_2), m_1 \leq m_2$, is neither $(2,2)$ nor $(4,5)$, then m_1+m_2+1 is a multiple of $2^{\phi(m_1-1)}$, where $\phi(n)$ denotes the number of natural numbers $s, 1 \leq s \leq n$, such that $s \equiv 0,2,2,4 \pmod 8$. One can see easily that such pairs (m_1, m_2) are exactly those for the OT-FKM type in the above table.

His approach is reminiscent of the theorem of Adams:

Theorem 106. *If there are k independent vector fields on S^n, then $n+1$ is a multiple of $2^{\phi(k)}$.*

The core technique Adams developed for proving the above theorem on vector fields was to what Stolz reduced his proof.

6. Development in the 2000s

Theorem 107. [9], [10], [11] *When $g = 4$, except possibly for the case with multiplicities $\{7, 8\}$, an isoparametric hypersurface is, up to congruence, one of the hypersurfaces of OT-FKM type.*

The proof utilizes commutative algebra, algebraic geometry, and Stolz's multiplicity result.

Miyaoka recently settled the other case when $g = 6$.

Theorem 108. [31], [32] *An isoparamtric hypersurface with $g = 6$ and multiplicities $m_\pm = 2$ is homogeneous.*

She also gave a simpler and more geometric proof [29] of the theorem by Dorfmeister and Neher by the same technique.

So as it stands now, only the case with $g = 4$ and $\{m_+, m_-\} = \{7, 8\}$ remains open, for which we know three inhomogeneous examples of OT-FKM type.

Lastly, a recent application of the classification of isoparametric hypersurfaces is the result of Tang and Yan [44] on the first eigenvalue of isoparametric hypersurfaces in spheres, which verifies, in the case of isoparametric hypersurfaces, a conjecture of Yau that states that the first eigenvalue of every compact minimal hypersurface in S^{n+1} is n.

Another recent application of the classification of isoparametric hyperurfaces in spheres is the construction, by Tang, Xie and Yan [43], of new positive scalar curvature manifolds from the minimal isoparametric hypersurface given in Corollary 99.

As a final note, Ge and Tang [18], constructed isoparametric functions on exotic spheres, Ma and Ohnita determined Hamiltonian stability of the Gauss images of homogeneous isoparametric hypersurfaces in complex hyperquadrics as Lagrangian submanifolds [27], [28], and Dearricott [15] proved the existence of a contact CR structure of dimension 8 on the focal manifold of dimension 14 of the (homogeneous) isoparametric hypersurface with multiplicities $\{4, 5\}$ in S^{19}, giving rise to the notion of 13-dimensional 5-Sasakian manifolds fibered over $\mathbb{C}P^4$ that generalizes the 3-Sasakian ones. The 5-Sasakian manifold constructed from the focal manifold carries a metric of positive sectional curvature [3].

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