Math 233 - Exam 3 - Spring 2013

April 10, 2013

NAME:

SOLUTIONS

STUDENT ID NUMBER:

General instructions: This exam has 16 questions, each worth the same amount. Check that no pages are missing and notify your proctor if you detect any problems with your copy of the exam. Mark your ID number on the six blank lines on the top of your answer card, using one line for each digit. Print your name on the top of the card. Choose the answer that is closest to the solution and mark your answer card with a PENCIL by shading in the correct box. You may use a $3\times$ 5 card with notes and any calculator that does not have graphing functions. GOOD LUCK!

1. Find the local maximum and minimum values of the function

$$f(x, y) = x^2 + xy + y^2 + 3y.$$

(A) local minimum = -3, no local maximum
(B) no local minimum, local maximum = -3
(C) local minimum = -2, local maximum = 1
(D) local minimum = -1, local maximum = 2
(E) local minimum = 3, no local maximum
(F) local minimum = -4, local maximum = 1
(G) no local minimum, no local maximum
(H) local minimum = -2, local maximum = 2

1. Locate critical points :

$$\nabla f = \langle 2x+y,\ x+2y+3 \rangle$$

$$\therefore \begin{cases} 2x+y=0 & (1) \\ x+2y+3=0 & (2) \end{cases}$$

$$(1) \Rightarrow y = -2x \xrightarrow{(2)} x + 2(-2x) + 3 = 0$$

$$-3x+3=0$$

$$x=1 \Rightarrow y=-2$$

$\therefore$ $(1,-2)$ is only critical point

2. Apply 2nd derivatives test :

At $(1,-2)$:

$$D = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3 > 0 \quad \Big\} \text{ local min at } (1,-2)$$

$$f_{xx} = 2 > 0$$

2

$$f(1,-2) = 1 - 2 + 4 - 6 = -3$$

2. Consider the critical points $(\pi/2, 0), (0, 1)$, and $(\pi, -1)$ of a function $f(x, y)$ whose domain is restricted to $D = \{(x, y) | -1 \leq x \leq 4\}$ and whose partial derivatives are

$$f_x = 2y sin\, x,\ \ f_y = 2y - 2cos\, x\ \ .$$

What does the Second Derivatives Test tell us about the behaviour of f near these critical points?

(A) Local maximum at both $(0, 1)$ and$(\pi, -1)$, local minimum at $(\pi/2, 0)$.

(B) Test fails at $(\pi/2, 0)$, local maximum at $(0, 1)$, local minimum at $(\pi, -1)$.

(C) Test fails at $(\pi/2, 0)$, local maximum at $(\pi, -1)$, local minimum at $(0, 1)$.

(D) Saddle point at $(\pi/2, 0)$, local maximum at $(0, 1)$, local minimum at $(\pi, -1)$.

(E) Saddle point at $(\pi/2, 0)$, local minimum at $(\pi, -1)$, local minimum at $(0, 1)$

(F) Saddle point at $(\pi, -1)$, local maximum at $(\pi/2, 0)$, local minimum at $(0, 1)$.

(G) Test fails at $(\pi, -1)$, local maximum at $(0, 1)$, local minimum at $(\pi/2, 0)$.

(H) Test fails at all three critical points.

$f_{xx} = 2y\cos x$

$f_{xy} = f_{yx} = 2\sin x$

$f_{yy} = 2$

- At $(\frac{\pi}{2}, 0)$: $D = \begin{vmatrix} 0 & 2 \\ 2 & 2 \end{vmatrix} = -4 < 0 \implies$ saddle point

- At $(0,1)$: $D = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 > 0 \implies$ local min

  $f_{xx} > 0$

- At $(\pi, -1)$: $D = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 > 0 \implies$ local min.

  $f_{xx} > 0$

3. Find the extreme values of $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraints $\underbrace{x - y}_{g(x,y,z)} = 1$ and $\underbrace{y^2 + z^2}_{h(x,y,z)} = 1$.

(A) max = 12, min = 3

(B) max = 27, min = 1

(C) max = 25, no min

(D) max = 12, no min

(E) max = 12, min = 1

(F) max = 5, min = 2

(G) max = 5, min = 1

(H) max = 2, min = 0

$\nabla f = \langle 2x, 2y, 2z \rangle$

$\nabla g = \langle 1, -1, 0 \rangle$

$\nabla h = \langle 0, 2y, 2z \rangle$

Inspection says extrema exist. $\therefore$ apply method of Lagrange multipliers.

$$\begin{cases} 2x = \lambda & (1) \\ 2y = -\lambda + 2\mu y & (2) \\ 2z = 2\mu z & (3) \\ x - y = 1 & (4) \\ y^2 + z^2 = 1 & (5) \end{cases}$$

$(3) \Longrightarrow 2z(1-\mu) = 0 \Rightarrow z = 0$ or $\mu = 1$

$z = 0 \xRightarrow{(5)} y^2 = 1 \Rightarrow y = 1 \overset{(4)}{\Downarrow} x = 2$ or $y = -1 \overset{(4)}{\Downarrow} x = 0$ $\therefore (2,1,0)$ + $(0,-1,0)$

$\mu = 1 \xRightarrow{(2)} 2y = -\lambda + 2y \Rightarrow \lambda = 0 \xRightarrow{(1)} x = 0 \xRightarrow{(4)} y = -1 \xRightarrow{(5)} z = 0$ $\therefore (0,-1,0)$

$f(2,1,0) = 4 + 1 = 5$

$f(0,-1,0) = 1$

[Alternately: $(5) \Rightarrow f(x,y,z) = x^2 + 1$ and $x = 1 + y$ with $-1 \le y \le 1 \Rightarrow 0 \le x \le 2$

So inspection says abs. max $= 2^2 + 1 = 5$]

4. Find the dimensions of the rectangle of greatest area that can be inscribed in the ellipse

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

with sides parallel to the coordinate axes. (The base is parallel to the x-axis and the height is parallel to the y-axis.)

(A) base $= 1$ and height $= 2$

(B) base $= 2$ and height $= 1$

(C) base $= 2\sqrt{2}$ and height $= 3\sqrt{2}$

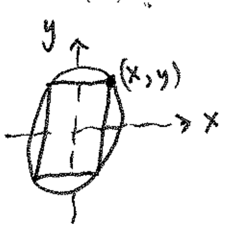
(D) base $= 2\sqrt{3}$ and height $= 3\sqrt{2}$

(E) base $= \sqrt{2}$ and height $= 3$

(F) base $= 2$ and height $= 3$

(G) base $= 3$ and height $= 2$

(H) base $= 1$ and height $= 3$



$x > 0, y > 0$

maximize $f(x,y) = xy$

subject to $g(x,y) = \frac{x^2}{4} + \frac{y^2}{9} = 1$

note: maximize $\Rightarrow$ $x, y \neq 0$ $\Downarrow$ $\lambda \neq 0$

$\nabla f = \langle y, x \rangle$

$\nabla g = \langle \frac{x}{2}, \frac{2y}{9} \rangle$

$$\therefore \begin{cases} y = \lambda\left(\frac{x}{2}\right) & (1) \quad \Rightarrow \quad xy = \lambda\left(\frac{x^2}{2}\right) \\ x = \lambda\left(\frac{2y}{9}\right) & (2) \quad \Rightarrow \quad xy = \lambda\left(\frac{2y^2}{9}\right) \\ \frac{x^2}{4} + \frac{y^2}{9} = 1 & (3) \end{cases}$$

$\lambda\left(\frac{x^2}{2} - \frac{2y^2}{9}\right) = 0$

$\Rightarrow \frac{x^2}{4} - \frac{y^2}{9} = 0$

$\frac{x^2}{4} + \frac{y^2}{9} = 1$

$\frac{x^2}{2} = 1$

$x = \sqrt{2}$

$\therefore \frac{y^2}{9} = \frac{1}{2}$

$y^2 = \frac{9 \cdot 2}{2}$

$y = \frac{1}{2} \cdot 3\sqrt{2}$

$\therefore x = \sqrt{2}, \; y = \frac{1}{2} \cdot 3\sqrt{2}$

corresponding to dimensions base $= 2\sqrt{2}$, height $= 3\sqrt{2}$

5. Compute

$$\iint_R (y + xe^y)\, dA.$$

when $R = [-2, 2] \times [0, 1]$.

(A) e

(B) e-1

(C) 1/2(e-1)

(D) 1/2

(E) 2

(F) 2+2e

(G) 2e

(H) -2e

$$\int_0^1 \int_{-2}^{2} (y + xe^y)\, dx\, dy$$

$$= \int_0^1 \left[x\Big|_{-2}^{2} y + \frac{x^2}{2}\Big|_{-2}^{2} e^y \right] dy$$

$$= \int_0^1 (4y + 0)\, dy$$

$$= 2y^2\Big|_0^1$$

$$= 2$$

6. Calculate

$$\iint\limits_R \frac{xy^2}{x^2+1}\, dA$$

when $R = [0,1] \times [-2,2]$.

(A) $ln\, 2$

(B) $2\, ln\, 2$

(C) $\frac{8}{3}\, ln\, 2$

(D) $\frac{7}{2}\, ln\, 2$

(E) $ln\, 2 - 1$

(F) $2\, ln\, 2 - 1$

(G) $\frac{8}{3}\, ln\, 2 - 1$

(H) $\frac{7}{2}\, ln\, 2 - 1$

$$\left(\int_0^1 \frac{x}{x^2+1}\, dx \right) \left(\int_{-2}^{2} y^2\, dy \right)$$

$$= \left(\frac{1}{2} \ln(x^2+1)\Big|_0^1 \right) \left(\frac{1}{3} y^3 \right)\Big|_{-2}^{2}$$

$$= \left(\frac{1}{2} \ln 2 \right) \left(\frac{2}{3} \cdot 8 \right)$$

$$= \frac{8}{3} \ln 2$$

7. Calculate

$$\iint_D \frac{y}{x^5+1}\, dA$$

when $D = \{(x,y) | 0 \le x \le 1, 0 \le y \le x^2\}$.

(A) ln 2

(B) 1/2 ln 2

(C) 1/5 ln 2

(D) 1/10 ln 2

(E) ln 2 - 1/2

(F) 1/2 (1 - ln 2)

(G) 1/5 (1 - ln 2)

(H) 1/10 (1-ln 2)

$$\int_0^1 \int_0^{x^2} \frac{y}{x^5+1}\, dy\, dx = \int_0^1 \frac{y^2/2 \Big|_0^{x^2}}{x^5+1}\, dx$$

$$= \frac{1}{2} \int_0^1 \frac{x^4}{x^5+1}\, dx$$

$u = x^5 + 1$

$du = 5x^4 dx$

$x^4 dx = \frac{1}{5} du$

$$= \frac{1}{10} \int_1^2 \frac{du}{u}$$

$$= \frac{1}{10} \ln 2$$

8. Find the volume of the solid under the surface $z - xy = 1$ and above the bounded region in the upper half-plane enclosed by $x = y^2$, $y = 0$, and $x = 4$.

(A) 40/3

(B) 39/2

(C) 97/7

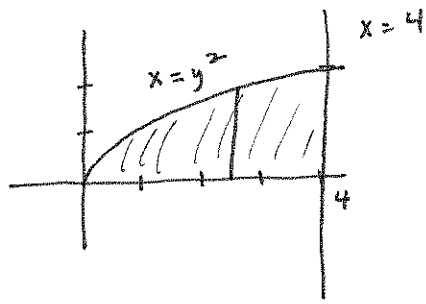
(D) $4\sqrt{2}$

(E) 15

(F) 16

(G) 17

(H) 18



$$\int_0^4 \int_0^{\sqrt{x}} (xy+1)\, dy\, dx$$

$$= \int_0^4 \left(\frac{1}{2} x y^2 \Big|_0^{\sqrt{x}} + y \Big|_0^{\sqrt{x}} \right) dx$$

$$= \int_0^4 \left(\frac{1}{2} x^2 + x^{1/2} \right) dx$$

$$= \frac{1}{6} x^3 + \frac{2}{3} x^{3/2} \Big|_0^4$$

$$= \frac{2 \cdot 2 \cdot 16}{2 \cdot 3} + \frac{2}{3} \cdot 8$$

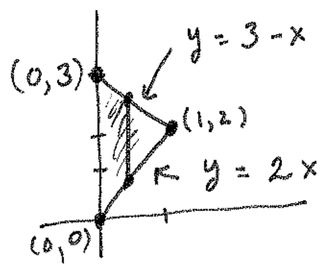
$$= \frac{32 + 16}{3} = \frac{48}{3} = 16$$

9. Evaluate

$$\iint_D 2xy\,dA$$

when D is the triangular region with vertices $(0,0),(1,2)$, and $(0,3)$.

(A) 7/4
(B) 8/3
(C) 9/5
(D) 11/7
(E) 2
(F) 3
(G) 4
(H) 5



$$\int_0^1 \int_{2x}^{3-x} 2xy\,dy\;dx$$

$$= \int_0^1 xy^2 \Big|_{2x}^{3-x} \; dx$$

$$= \int_0^1 x\left[(3-x)^2 - (2x)^2\right] dx$$

$$= \int_0^1 x\,(9 - 6x + x^2 - 4x^2)\,dx$$

$$= \int_0^1 (9x - 6x^2 - 3x^3)\;dx$$

$$= \frac{9}{2}x^2 - 2x^3 - \frac{3}{4}x^4 \Big|_0^1$$

$$= \frac{18 - 8 - \cancel{6}3}{4}$$

$$= 7/4$$

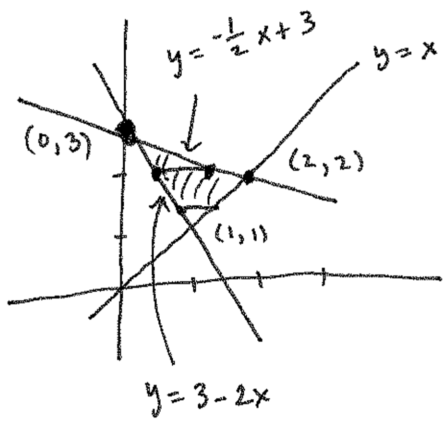
10. Suppose the volume of the solid lying below the surface $z = e^{xy}$ and above the triangle T is given by

$$\iint_T e^{xy}\,dA = \int_1^2 \int_{\frac{3-y}{2}}^{y} e^{xy}\,dx\,dy + \int_2^3 \int_{\frac{3-y}{2}}^{6-2y} e^{xy}\,dx\,dy.$$

What are the vertices of T?

(A) $(0,3), (1,1), (2,2)$
(B) $(3,0), (1,1), (2,2)$
(C) $(0,3), (1,1), (0,6)$
(D) $(3,0), (1,1), (6,0)$
(E) $(0,3), (6,6), (6,0)$
(F) $(3,0), (6,6), (0,6)$
(G) $(3/2,0), (6,6), (0,6)$
(H) $(0,3/2), (6,6), (0,6)$

Consider the lines $x = y \Updownarrow y = x$ $\quad x = \frac{3-y}{2} \Updownarrow y = 3-2x$ $\quad x = 6-2y \Updownarrow y = 3 - \frac{1}{2}x$



$\begin{cases} y = x \\ y = 3-2x \end{cases} \Rightarrow y = 3-2y \Rightarrow y = 1 \Rightarrow x = 1$
$(1,1)$

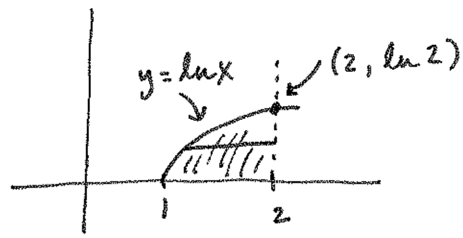
$\begin{cases} y = x \\ y = 3 - \frac{1}{2}x \end{cases} \Rightarrow y = 3 - \frac{1}{2}y \Rightarrow \frac{3}{2}y = 3 \Rightarrow y = 2 \Rightarrow x = 2$
$(2,2)$

11. Reverse the order of integration

$$\int_1^2 \int_0^{lnx} f(x,y)\, dy\, dx$$

to get an equivalent integral.

Sketch:



$$\int_0^{\ln 2} \int_{e^y}^{2} f(x,y)\, dx\, dy$$

(A) $\int_0^{lnx} \int_1^{2} f(x,y)\, dy\, dx$

(B) $\int_0^{lnx} \int_1^{2} f(x,y)\, dx\, dy$

(C) $\int_0^{ln2} \int_2^{e^y} f(x,y)\, dy\, dx$

(D) $\int_0^{ln2} \int_2^{e^y} f(x,y)\, dx\, dy$

(E) $\int_0^{ln2} \int_{e^y}^{2} f(x,y)\, dy\, dx$

(F) $\int_0^{ln2} \int_{e^y}^{2} f(x,y)\, dx\, dy$

(G) $\int_0^{ln2} \int_0^{lny} f(x,y)\, dy\, dx$

(H) $\int_1^{2} \int_0^{lny} f(x,y)\, dx\, dy$

12. Compute

$$\int_0^1 \int_x^1 \exp(\frac{x}{2y})\, dy\, dx$$

where

$$\exp(\frac{x}{2y}) = e^{\frac{x}{2y}}.$$

(A) 0.5 e

(B) e

(C) 2e

(D) 0.5(e - 1)

(E) $\sqrt{e} - 1$

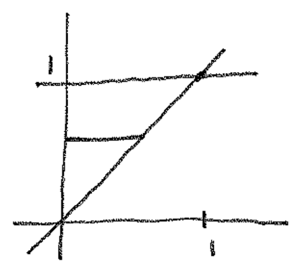
(F) 2(e - 1)

(G) e + 1

(H) 0.5(e +1)

Reverse order of integration.

Sketch:



$$\int_0^1 \left(\int_0^y e^{x/2y}\, dx \right) dy$$

$$= \int_0^1 2y\, e^{\frac{x}{2y}} \Big|_{x=0}^{x=y} dy$$

$$= \int_0^1 2y(e^{1/2} - e^0)\, dy$$

$$= y^2\Big|_0^1 (\sqrt{e} - 1)$$

$$= \sqrt{e} - 1$$

13. Compute

$$\iint\limits_D \sqrt{9 - x^2 - y^2}\, dA$$

when D is the region in the first quadrant bounded by the circle $x^2 + y^2 = 9$.

(A) 3π

(B) 3.5π

(C) 4.5π

(D) 6π

(E) 12.5π

(F) 15.5π

(G) 20π

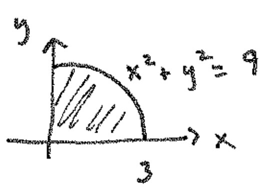
(H) 36π

$$\iint\limits_D \sqrt{9-x^2-y^2}\, dA = \frac{1}{8} \text{ volume of ball of radius } 3$$

$$= \frac{1}{8}\left(\frac{4}{3}\pi r^3\right)$$

$$= \frac{1}{2}\pi\left(\frac{3^3}{3}\right)$$

$$= \frac{9\pi}{2}$$

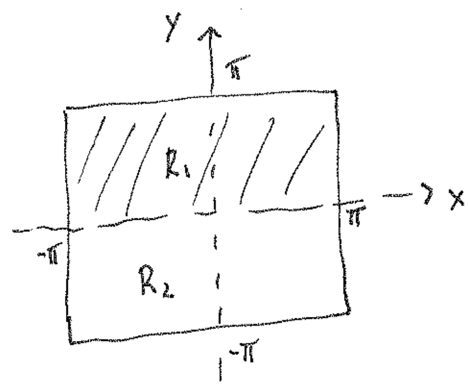


14. Compute

$$\iint_R (e^{x^2} siny + 2)\, dA$$

when $R = [-\pi, \pi] \times [-\pi, \pi]$.

(A) $8\pi^2$
(B) $e + 8\pi^2$
(C) $2e + 8\pi^2$
(D) $4e + 8\pi^2$
(E) $8e + 8\pi^2$
(F) $e^2 + 8\pi^2$
(G) $2e^2 + 8\pi^2$
(H) $4e^2 + 8\pi^2$



$$\underbrace{\iint_R e^{x^2} \sin y \, dA}_{} + \underbrace{\iint_R 2\, dA}_{2 \cdot \text{Area}(R) = 2(2\pi)(2\pi) = 8\pi^2}$$

Use symmetry:

$\sin(-y) = -\sin y$

$e^{(\pm x)^2} = e^{x^2}$

$$\therefore \iint_R e^{x^2} \sin y\, dA = \iint_{R_1} e^{x^2} \sin y\, dA + \iint_{R_2} e^{x^2} \sin y\, dA$$

$$= \iint_{R_1} e^{x^2} \sin y\, dA - \iint_{R_1} e^{x^2} \sin y\, dA$$

$$= 0$$

15. Compute

$$\iint_D \pi \cos(\frac{\pi}{2}(x^2+y^2))\,dx\,dy$$

where D is the region in the first quadrant between the circle with center the origin and radius 1 and the circle with center the origin and radius 2.

(A) $-\frac{\pi}{2}$

(B) $\frac{3\pi}{2}$

(C) $-\pi$

(D) π

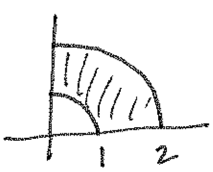
(E) $\frac{\pi^2}{2}$

(F) $\frac{\pi^4}{2}$

(G) $\frac{\pi^4}{4}$

(H) $-\frac{\pi^4}{2}$

Use polar coordinates.



$$\int_0^{\pi/2}\left(\int_1^2 \pi \cos\left(\frac{\pi}{2} r^2\right) r\,dr\right)d\theta$$

$u = \frac{\pi}{2} r^2$

$du = \pi r\,dr$

$$= \int_0^{\pi/2}\left(\int_{\frac{\pi}{2}}^{2\pi} \cos u\,du\right)d\theta$$

$$= \int_0^{\pi/2} \sin u \Big|_{\frac{\pi}{2}}^{2\pi}\, d\theta$$

$$= \int_0^{\pi/2} -\,d\theta = -\theta\Big|_0^{\pi/2} = -\frac{\pi}{2}$$

16. Compute

$$\iint_D 2\,e^{-x^2-y^2}\,dA$$

where D is the region in the first quadrant enclosed by the circle $x^2+y^2=4$, the line $y=0$, and the line $y=x$.

(A) $\frac{\pi}{e}$

(B) $\frac{\pi^2}{e^2}$

(C) $\frac{\pi^3}{e^3}$

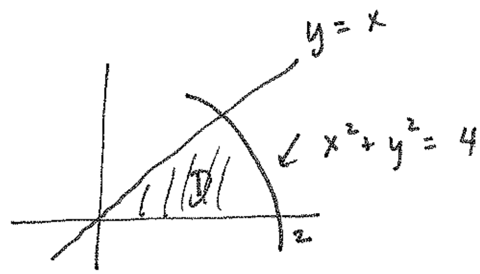
(D) $\frac{\pi^4}{e^4}$

(E) $\frac{\pi}{2}(e^2-1)$

(F) $\frac{\pi}{2e^2}(e^2-1)$

(G) $\frac{\pi}{3e^3}(e^3-1)$

(H) $\frac{\pi}{4e^4}(e^4-1)$



Use polar coordinates:

$$\int_0^{\pi/4}\left(\int_0^2 2e^{-r^2}\,r\,dr\right)d\theta$$

$u = -r^2$

$du = -2r\,dr$

$$= \int_0^{\pi/4}\left(\int_0^{-4} -e^u\,du\right)d\theta$$

$$= \int_0^{\pi/4} e^u\Big|_{-4}^{0}\,d\theta = (1-e^{-4})\cdot\frac{\pi}{4}$$

$$= \frac{\pi}{4}\left(\frac{e^4-1}{e^4}\right)$$