EXTRA EXAM

Math 233 Second exam March 6, 2013

Name: Solutions
Please print above

Course: *Math 233*

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Math 233 - Exam 2 - Spring 2013

March 6, 2013

NAME:

STUDENT ID NUMBER:

General instructions: This exam has 16 questions, each worth the same amount. Check that no pages are missing and notify your proctor if you detect any problems with your copy of the exam. Mark your ID number on the six blank lines on the top of your answer card, using one line for each digit. Print your name on the top of the card. Choose the answer that is closest to the solution and mark your answer card with a PENCIL by shading in the correct box. You may use a $3\times$ 5 card with notes and any calculator that does not have graphing functions. GOOD LUCK!

1. Find the length of the curve given by

$$\mathbf{r}(t) = \langle \sqrt{2}t, e^t, e^{-t} \rangle, \ -1 \le t \le 1.$$

(A) $\sqrt{2}$

(B) $\sqrt{2}e$

(C) $2e$

(D) $e + e^{-1}$

(E) $e - e^{-1}$

(F) $e^{-1} + e$

(G) $2(e - e^{-1})$

(H) $2(e + e^{-1})$

$$r'(t) = \langle \sqrt{2}, e^t, -e^{-t} \rangle$$

$$|r'(t)| = \sqrt{2 + e^{2t} + e^{-2t}}$$

$$= \sqrt{(e^t + e^{-t})^2} = e^t + e^{-t}$$

$$L = \int_{-1}^{1} (e^t + e^{-t})\, dt = e^t - e^{-t}\Big|_{-1}^{1}$$

$$= (e - e^{-1}) - (e^{-1} - e)$$

$$= 2(e - e^{-1})$$

G

2. Find the unit tangent vector $\mathbf{T}$ to the curve $\mathbf{r}(t) = \langle cost, sint, ln(sint)\rangle$ at $t = \pi/4$. What is the first component of $\mathbf{T}(\pi/4)$?

(A) $-\sqrt{3}/4$

(B) $\sqrt{3}/4$

(C) $\sqrt{2}/2$

(D) $-\sqrt{2}/2$

(E) $1/2$

(F) $-1/2$

(G) π

(H) $-\pi$

$$T(t) = \frac{r'(t)}{|r'(t)|}$$

$$r'(t) = \langle -\sin t, \cos t, \frac{\cos t}{\sin t} \rangle$$

$$r'(\tfrac{\pi}{4}) = \langle \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1 \rangle$$

$$|r'(\tfrac{\pi}{4})| = \sqrt{\tfrac{1}{2}+\tfrac{1}{2}+1} = \sqrt{2}$$

$$\therefore \quad T(\tfrac{\pi}{4}) = \frac{\langle \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1 \rangle}{\sqrt{2}} = \langle \underline{\underline{-\tfrac{1}{2}}}, \tfrac{1}{2}, \tfrac{1}{\sqrt{2}} \rangle$$

$$\therefore \quad -\frac{1}{2}$$

F

3. Find the curvature of the ellipse

$$x(t) = 2\,cos\,t,\ y(t) = 3\,sin\,t$$

at the point $(2, 0)$.

(A) 6

(B) 1/6

(C) 2/9

(D) 4/81

(E) 81/4

(F) 81/14

(G) 4.5

(H) 3.3

$$K(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}$$

$r(t) = \langle 2\cos t, 3\sin t, 0 \rangle$

$r'(t) = \langle -2\sin t, 3\cos t, 0 \rangle$

$r''(t) = \langle -2\cos t, -3\sin t, 0 \rangle$

$(2,0) \Rightarrow t = 0$

$r'(0) = \langle 0, 3, 0 \rangle \longrightarrow |r'(0)| = 3$

$r''(0) = \langle -2, 0, 0 \rangle$

$$r'(0) \times r''(0) = \begin{vmatrix} i & j & k \\ 0 & 3 & 0 \\ -2 & 0 & 0 \end{vmatrix} = \langle 0, 0, 6 \rangle$$

$$\therefore \quad K(0) = \frac{6}{3^3} = \frac{2}{9}$$

C

4. Find the range of the function $g(x, y) = \sqrt{9 - x^2 - y^2}$.

(A) $[0, 1]$

(B) $\mathbb{R}$

(C) $[-3, 3]$

(D) $[0, 3]$

(E) $(-3, 3)$

(F) $(0, 3)$

(G) $[0, 9]$

(H) $(0, 9)$

$0 \leq g(x,y) \leq \sqrt{9} = 3$

D

5. Find the indicated limits. If a limit does not exist, write DNE.

(a) $lim_{(x,y)\to(2,1)}\,(3x^2 - xy)$

(b) $lim_{(x,y)\to(0,0)}\,\frac{x^4-24y^2}{x^2+12y^2}$

(c) $lim_{(x,y)\to(2,1)}\,\frac{xy-2y}{x^2-4}$.

(A) (DNE, DNE, DNE)

(B) $(10, DNE, DNE)$

(C) $(10, DNE, 0)$

(D) $(10, DNE, 1)$

(E) $(10, DNE, 1/4)$

(F) $(10, 1, DNE)$

(G) $(10, 0, DNE)$

(H) $(10, 1/5, DNE)$

(a) $\lim_{(x,y)\to(2,1)} \overbrace{3x^2 - xy}^{\text{continuous}} = 3\cdot 4 - 2 = 10$

(b) $(x,0) \to (0,0)$ [along x - axis] : $\lim_{x\to 0} \frac{x^4}{x^2} = \lim_{x\to 0} x^2 = 0$

$(0,y) \to (0,0)$ [along y - axis] : $\lim_{y\to 0} \frac{-24y^2}{12y^2} = \lim_{y\to 0} -2 = -2$

$0 \neq -2$ means limit DNE

(c) $\lim_{(x,y)\to(2,1)} \frac{y(x-2)}{(x-2)(x+2)} = \lim_{(x,y)\to(2,1)} \frac{y}{x+2} = \frac{1}{4}$

E

6. Given $\epsilon > 0$ and

$$f(x,y) = \frac{y}{6+4cosx}$$

find the largest $\delta > 0$ such that $|f(x,y) - 0| < \epsilon$ whenever $0 < \sqrt{x^2+y^2} < \delta$.

(A) $\delta = \epsilon/4$

(B) $\delta = \epsilon/3$

(C) $\delta = \epsilon/2$

(D) $\delta = \epsilon$

(E) $\delta = 2\epsilon$

(F) $\delta = 3\epsilon$

(G) $\delta = 4\epsilon$

(H) The limit does not exist and no such δ exists.

$$\left| \frac{y}{6+4\cos x} \right| \leq \frac{|y|}{2} \leq \frac{\sqrt{x^2+y^2}}{2} < \frac{\delta}{2}$$

since $|6+4\cos x| \geq |6-4| = 2$

So $\delta = 2\varepsilon$ works

E

7. Find $f_x(\pi/9, \pi/2, \pi/4)$ for the function $f(x, y, z) = 9x sin(y - z)$.

(A) π
(B) $\pi\sqrt{2}/2$
(C) $9\sqrt{2}/2$
(D) $9\pi/4$
(E) $\sqrt{3}/2$
(F) $1/2$
(G) $\pi/2$
(H) $\sqrt{2}/2$

$f_x = 9 \sin(y - z)$

$f_x\left(\frac{\pi}{9}, \frac{\pi}{2}, \frac{\pi}{4}\right) = 9 \sin \frac{\pi}{4} = \frac{9\sqrt{2}}{2}$

C

8. Suppose $z = f(x, y)$, where f is differentiable, and $x = g(t), y = h(t)$, where g and h differentiable. Suppose also that $g(1) = -3, g'(1) = 2, h(1) = 5, h'(1) = 4$, $f_x(1, 1) = 2\pi$, $f_x(-3, 5) = -\pi$, $f_y(1, 1) = -2\pi$, and $f_y(-3, 5) = 6\pi$. Find $\frac{dz}{dt}$ when $t = 1$.

(A) 22π

(B) 5π

(C) -4π

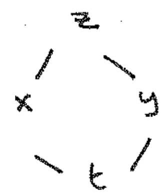
(D) -5π

(E) -8π

(F) 10

(G) -12

(H) 20



By Chain Rule:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$t = 1$:

$x = g(1) = -3$

$y = h(1) = 5$

$\frac{dx}{dt}(1) = g'(1) = 2$

$\frac{dy}{dt}(1) = h'(1) = 4$

$$\therefore \frac{dz}{dt}(1) = f_x(-3,5)\cdot 2 + f_y(-3,5)\cdot 4$$

$$= (-\pi)(2) + (6\pi)(4)$$

$$= -2\pi + 24\pi$$

$$= 22\pi$$

A

9. Let $W(s,t) = F(u(s,t), v(s,t))$, where F, u and v are differentiable. Suppose also that

$$u(-2,-3) = 1, v(-2,-3) = -7$$

$$u_s(-2,-3) = -1, v_s(-2,-3) = -101, u_t(-2,-3) = 4, v_t(-2,-3) = -5$$

$$F_u(1,-7) = 3, F_u(-2,-3) = 3, F_v(1,-7) = 2, F_v(-2,-3) = 0$$

Find $W_t(-2,-3)$.

(A) -33

(B) -10

(C) 2

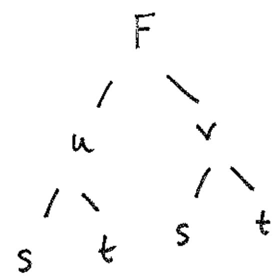
(D) 12

(E) 14

(F) 35

(G) 199

(H) 202



$(s,t) = (-2,-3)$

$\Rightarrow (u,v) = (1,-7)$

$u_t(-2,-3) = 4$

$v_t(-2,-3) = -5$

$$\frac{\partial W}{\partial t} = F_u \frac{\partial u}{\partial t} + F_v \frac{\partial v}{\partial t}$$

$$\therefore \quad \frac{\partial W}{\partial t}(-2,-3) = F_u(1,-7) \cdot u_t(-2,-3) + F_v(1,-7) \cdot v_t(-2,-3)$$

$$= (3)(4) + (2)(-5)$$

$$= 2$$

C

10. Find y' at the point $(0, 1)$ if $y\, cosx = x^2 + y^2 + xy$.

(A) -1

(B) 1

(C) -2

(D) 2

(E) -4

(F) 4

(G) -1/2

(H) 1/2

Use Implicit Differentiation with

$$\begin{cases} F(x,y) = y\cos x - x^2 - y^2 - xy \\ \dfrac{dy}{dx} = \dfrac{-F_x}{F_y} \end{cases}$$

$F_x = -y\sin x - 2x - y$

$F_y = \cos x - 2y - x$

$F_x(0,1) = -1$

$F_y(0,1) = 1 - 2 = -1$

$\therefore \quad \dfrac{dy}{dx} = -\dfrac{(-1)}{(-1)} = -1$ (A)

11. Find ∇g at the point $(1, 2)$ when $g(p, q) = p^3 - p^2q^2$.

(A) $\langle 3, 4 \rangle$
(B) $\langle 4, 3 \rangle$
(C) $\langle -3, 4 \rangle$
(D) $\langle 4, -3 \rangle$
(E) $\langle 3, -4 \rangle$
(F) $\langle -4, 3 \rangle$
(G) $\langle -4, -5 \rangle$
(H) $\langle -5, -4 \rangle$

$$\nabla g = \langle g_p, g_q \rangle = \langle 3p^2 - 2pq^2, -2p^2q \rangle$$

$$\nabla g(1,2) = \langle 3 - 8, -4 \rangle = \langle -5, -4 \rangle$$

H

12. Find the directional derivative of $f(x,y) = \sqrt{xy}$ at $P(7,7)$ in the direction of the point $Q(11,4)$.

(A) $\sqrt{2}/2$
(B) $\sqrt{2}$
(C) $\sqrt{2}/10$
(D) 1/10
(E) 1/2
(F) 5
(G) 5/2
(H) 44

$$\vec{PQ} = \langle 4, -3 \rangle$$

$$u = \frac{\vec{PQ}}{|\vec{PQ}|} = \frac{\langle 4, -3 \rangle}{\sqrt{16+9}} = \langle \tfrac{4}{5}, -\tfrac{3}{5} \rangle$$

$$f(x,y) = x^{1/2} y^{1/2}$$

$$f_x = \tfrac{1}{2} x^{-1/2} y^{1/2}$$

$$f_y = \tfrac{1}{2} x^{1/2} y^{-1/2}$$

$$\therefore \nabla f = \langle \tfrac{1}{2}\sqrt{y/x}, \tfrac{1}{2}\sqrt{x/y} \rangle$$

$$\therefore \nabla f(7,7) = \langle \tfrac{1}{2}, \tfrac{1}{2} \rangle$$

$$D_u f = \nabla f \cdot \vec{u} = \langle \tfrac{1}{2}, \tfrac{1}{2} \rangle \cdot \langle \tfrac{4}{5}, -\tfrac{3}{5} \rangle$$

$$= \frac{4-3}{10} = 1/10$$

D

13. Let $w = F(x, y, z)$ be a differentiable function and suppose the point $P(1, 2, 3)$ lies in the domain of f. Let M denote the maximum rate of change of F at P and let m denote the minimum rate of change of F at P. Which of the following are possibilities for the pair (m, M)?

(i) $(-1/2, 1/2)$

(ii) $(0, 2)$

(iii) $(25, 4)$

(iv) $(-1/2, 1/4)$

(v) $(-1000, 10)$

(vi) $(3, 7)$

(A) all

(B) all except (i)

(C) all except (iii)

(D) (i),(ii),(iv),(v) only

(E) none

(F) (i) only

(G) (ii),(iv), (v) only

(H) (i),(iv),(v) only

$M = |\nabla F(1,2,3)|$

$m = -|\nabla F(1,2,3)|$

$\therefore \quad m = -M$

$\therefore$ only (i) is possible

F

14. Find the equation of the tangent plane to the surface $2x + y + 2z = 5e^{xyz}$ -5 at the point $(-1, 0, 1)$.

Typo:

$$2x + y + 2z = 5e^{xyz} - 5$$

(A) $x + y + 2z = 1$

(B) $x + 6y + 2z = 1$

(C) $x + 3y + z = 0$

(D) $2x + 2y - 3z = 1$

(E) $-x + z = 5$

(F) $-x + z = 0$

(G) $2x + 6y + 2z = 5$

(H) $x + y + 1.5z = 0.5$

normal of plane given by $\nabla F(-1,0,1)$

where $F(x,y,z) = 2x + y + 2z - 5e^{xyz}$

$$\nabla F = \langle F_x, F_y, F_z \rangle$$

$$= \langle 2 - 5yze^{xyz}, 1 - 5xze^{xyz}, 2 - 5xye^{xyz} \rangle$$

$\therefore$ $\nabla F(-1,0,1) = \langle 2, 1+5, 2 \rangle = \langle 2, 6, 2 \rangle$

$\therefore$ choose $\vec{n} = \langle a, b, c \rangle = \langle 1, 3, 1 \rangle$

$(x_0, y_0, z_0) = (-1, 0, 1)$

$\therefore$ eqn of plane is

$$ax + by + cz = ax_0 + by_0 + cz_0$$

$$x + 3y + z = 0$$

C

15. At what point on the surface S given by $6x = y^2 + z^2$ is the tangent plane parallel to the plane $3x + 2y + 7z = 6$?

(A) $(53/6, 2, 7)$
(B) $(53/6, -2, -7)$
(C) $(29/6, 2, 5)$
(D) $(29/6, -2, 5)$
(E) $(53/24, 1, 7/2)$
(F) $(53/24, -1, -7/2)$
(G) The tangent planes to S are never parallel to the plane $3x + 2y + 7z = 6$.
(H) All planes tangent to S are parallel to the plane $3x + 2y + 7z = 6$.

• normals of tangent plane to S given by nonzero scalar multiples of ∇F, where

$$F(x,y,z) = 6x - y^2 - z^2$$

$[\nabla F = \langle 6, -2y, -2z \rangle]$

• normals of plane $3x + 2y + 7z = 6$ given by nonzero scalar multiples of $\langle 3, 2, 7 \rangle$

$\therefore$ have $\langle 6, -2y, -2z \rangle = c \langle 3, 2, 7 \rangle$ for some $c \neq 0$

$$6 = 3c \Rightarrow c = 2$$

So $\langle 6, -2y, -2z \rangle = \langle 6, 4, 14 \rangle$

$$\Rightarrow \begin{cases} -2y = 4 \\ -2z = 14 \end{cases} \Rightarrow \begin{cases} y = -2 \\ z = -7 \end{cases}$$

(x,y,z) on S $\Rightarrow$ $x = \frac{1}{6}(y^2 + z^2) = \frac{1}{6}(4 + 49) = 53/6$

$\therefore$ $(53/6, -2, -7)$ B

16. Find symmetric equations for the normal line to the surface $xyz^2 = 6$ at the point $(3, 2, 1)$.

(A) $\frac{x-3}{2} = \frac{y-2}{7} = \frac{z-1}{14}$
(B) $\frac{x-3}{2} = \frac{y-2}{7} = \frac{z-1}{12}$
(C) $\frac{x-3}{3} = \frac{y-2}{2} = \frac{z-1}{1}$
(D) $\frac{x-3}{1} = \frac{y-2}{2} = \frac{z-1}{3}$
(E) $\frac{x-3}{2} = \frac{y-2}{3} = \frac{z-1}{14}$
(F) $\frac{x-3}{2} = \frac{y-2}{3} = \frac{z-1}{12}$
(G) $\frac{x-3}{3} = \frac{y-2}{2} = \frac{z-1}{14}$
(H) $\frac{x-3}{3} = \frac{y-2}{2} = \frac{z-1}{12}$

direction of normal line given by ∇F

where $F(x,y,z) = xyz^2$

$\nabla F = \langle yz^2, xz^2, 2xyz \rangle$

$\nabla F(3,2,1) = \langle \underset{a}{2}, \underset{b}{3}, \underset{c}{12} \rangle$

point on line : $(3,2,1)$

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

F

$$\frac{x-3}{2} = \frac{y-2}{3} = \frac{z-1}{12}$$