TRUE-FALSE QUIZ

1. False; div $\mathbf{F}$ is a scalar field.

2. True. (See Definition 16.5.1.)

3. True, by Theorem 16.5.3 and the fact that div $\mathbf{0} = 0$.

4. True, by Theorem 16.3.2.

5. False. See Exercise 16.3.35. (But the assertion is true if D is simply-connected; see Theorem 16.3.6.)

6. False. See the discussion accompanying Figure 8 on page 1092 [ET 1068].

7. False. For example, $\operatorname{div}(y\,\mathbf{i}) = 0 = \operatorname{div}(x\,\mathbf{j})$ but $y\,\mathbf{i} \neq x\,\mathbf{j}$.

8. True. Line integrals of conservative vector fields are independent of path, and by Theorem 16.3.3, work $= \int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed path C.

9. True. See Exercise 16.5.24.

10. False. $\mathbf{F} \cdot \mathbf{G}$ is a scalar field, so $\operatorname{curl}(\mathbf{F} \cdot \mathbf{G})$ has no meaning.

11. True. Apply the Divergence Theorem and use the fact that div $\mathbf{F} = 0$.

12. False by Theorem 16.5.11, because if it were true, then div curl $\mathbf{F} = 3 \neq 0$.

EXERCISES

1. (a) Vectors starting on C point in roughly the direction opposite to C, so the tangential component $\mathbf{F} \cdot \mathbf{T}$ is negative. Thus $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$ is negative.

   (b) The vectors that end near P are shorter than the vectors that start near P, so the net flow is outward near P and div $\mathbf{F}(P)$ is positive.

2. We can parametrize C by $x = x$, $y = x^2$, $0 \le x \le 1$ so

$$\int_C x\, ds = \int_0^1 x\sqrt{1 + (2x)^2}\, dx = \tfrac{1}{12}(1 + 4x^2)^{3/2}\Big]_0^1 = \tfrac{1}{12}(5\sqrt{5} - 1).$$

3. $$\int_C yz\cos x\, ds = \int_0^\pi (3\cos t)(3\sin t)\cos t\sqrt{(1)^2 + (-3\sin t)^2 + (3\cos t)^2}\, dt = \int_0^\pi (9\cos^2 t\,\sin t)\sqrt{10}\, dt$$
$$= 9\sqrt{10}\left(-\tfrac{1}{3}\cos^3 t\right)\Big]_0^\pi = -3\sqrt{10}(-2) = 6\sqrt{10}$$

4. $x = 3\cos t \;\Rightarrow\; dx = -3\sin t\, dt$, $y = 2\sin t \;\Rightarrow\; dy = 2\cos t\, dt$, $0 \le t \le 2\pi$, so

$$\int_C y\, dx + (x + y^2)\, dy = \int_0^{2\pi} \left[(2\sin t)(-3\sin t) + (3\cos t + 4\sin^2 t)(2\cos t)\right] dt$$
$$= \int_0^{2\pi} (-6\sin^2 t + 6\cos^2 t + 8\sin^2 t\,\cos t)\, dt = \int_0^{2\pi} \left[6(\cos^2 t - \sin^2 t) + 8\sin^2 t\,\cos t\right] dt$$
$$= \int_0^{2\pi} (6\cos 2t + 8\sin^2 t\,\cos t)\, dt = 3\sin 2t + \tfrac{8}{3}\sin^3 t\Big]_0^{2\pi} = 0$$

Or: Notice that $\frac{\partial}{\partial y}(y) = 1 = \frac{\partial}{\partial x}(x + y^2)$, so $\mathbf{F}(x, y) = \langle y, x + y^2\rangle$ is a conservative vector field. Since C is a closed curve, $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C y\, dx + (x + y^2)\, dy = 0$.

5. $\int_C y^3\,dx + x^2\,dy = \int_{-1}^{1}\left[y^3(-2y) + (1-y^2)^2\right]dy = \int_{-1}^{1}(-y^4 - 2y^2 + 1)\,dy$

$= \left[-\frac{1}{5}y^5 - \frac{2}{3}y^3 + y\right]_{-1}^{1} = -\frac{1}{5} - \frac{2}{3} + 1 - \frac{1}{5} - \frac{2}{3} + 1 = \frac{4}{15}$

6. $\int_C \sqrt{xy}\,dx + e^y\,dy + xz\,dz = \int_0^1\left(\sqrt{t^4\cdot t^2}\cdot 4t^3 + e^{t^2}\cdot 2t + t^4\cdot t^3\cdot 3t^2\right)dt = \int_0^1(4t^6 + 2te^{t^2} + 3t^9)\,dt$

$= \left[\frac{4}{7}t^7 + e^{t^2} + \frac{3}{10}t^{10}\right]_0^1 = e - \frac{9}{70}$

7. C: $x = 1 + 2t \;\Rightarrow\; dx = 2\,dt$, $y = 4t \;\Rightarrow\; dy = 4\,dt$, $z = -1 + 3t \;\Rightarrow\; dz = 3\,dt$, $0 \le t \le 1$.

$$\int_C xy\,dx + y^2\,dy + yz\,dz = \int_0^1\left[(1+2t)(4t)(2) + (4t)^2(4) + (4t)(-1+3t)(3)\right]dt$$

$$= \int_0^1(116t^2 - 4t)\,dt = \left[\frac{116}{3}t^3 - 2t^2\right]_0^1 = \frac{116}{3} - 2 = \frac{110}{3}$$

8. $\mathbf{F}(\mathbf{r}(t)) = (\sin t)(1+t)\,\mathbf{i} + (\sin^2 t)\,\mathbf{j}$, $\mathbf{r}'(t) = \cos t\,\mathbf{i} + \mathbf{j}$ and

$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{\pi}((1+t)\sin t\cos t + \sin^2 t)\,dt = \int_0^{\pi}\left(\frac{1}{2}(1+t)\sin 2t + \sin^2 t\right)dt$

$= \left[\frac{1}{2}\left((1+t)\left(-\frac{1}{2}\cos 2t\right) + \frac{1}{4}\sin 2t\right) + \frac{1}{2}t - \frac{1}{4}\sin 2t\right]_0^{\pi} = \frac{\pi}{4}$

9. $\mathbf{F}(\mathbf{r}(t)) = e^{-t}\,\mathbf{i} + t^2(-t)\,\mathbf{j} + (t^2 + t^3)\,\mathbf{k}$, $\mathbf{r}'(t) = 2t\,\mathbf{i} + 3t^2\,\mathbf{j} - \mathbf{k}$ and

$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1(2te^{-t} - 3t^5 - (t^2 + t^3))\,dt = \left[-2te^{-t} - 2e^{-t} - \frac{1}{2}t^6 - \frac{1}{3}t^3 - \frac{1}{4}t^4\right]_0^1 = \frac{11}{12} - \frac{4}{e}$.

10. (a) C: $x = 3 - 3t$, $y = \frac{\pi}{2}t$, $z = 3t$, $0 \le t \le 1$. Then

$W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1\left[3t\,\mathbf{i} + (3-3t)\,\mathbf{j} + \frac{\pi}{2}t\,\mathbf{k}\right]\cdot\left[-3\,\mathbf{i} + \frac{\pi}{2}\,\mathbf{j} + 3\,\mathbf{k}\right]dt = \int_0^1\left[-9t + \frac{3\pi}{2}\right]dt = \frac{1}{2}(3\pi - 9)$.

(b) $W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{\pi/2}(3\sin t\,\mathbf{i} + 3\cos t\,\mathbf{j} + t\,\mathbf{k})\cdot(-3\sin t\,\mathbf{i} + \mathbf{j} + 3\cos t\,\mathbf{k})\,dt$

$= \int_0^{\pi/2}(-9\sin^2 t + 3\cos t + 3t\cos t)\,dt = \left[-\frac{9}{2}(t - \sin t\cos t) + 3\sin t + 3(t\sin t + \cos t)\right]_0^{\pi/2}$

$= -\frac{9\pi}{4} + 3 + \frac{3\pi}{2} - 3 = -\frac{3\pi}{4}$

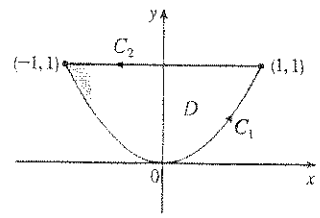
11. $\frac{\partial}{\partial y}\left[(1+xy)e^{xy}\right] = 2xe^{xy} + x^2ye^{xy} = \frac{\partial}{\partial x}\left[e^y + x^2e^{xy}\right]$ and the domain of $\mathbf{F}$ is $\mathbb{R}^2$, so $\mathbf{F}$ is conservative. Thus there exists a function f such that $\mathbf{F} = \nabla f$. Then $f_y(x,y) = e^y + x^2e^{xy}$ implies $f(x,y) = e^y + xe^{xy} + g(x)$ and then $f_x(x,y) = xye^{xy} + e^{xy} + g'(x) = (1+xy)e^{xy} + g'(x)$. But $f_x(x,y) = (1+xy)e^{xy}$, so $g'(x) = 0 \;\Rightarrow\; g(x) = K$. Thus $f(x,y) = e^y + xe^{xy} + K$ is a potential function for $\mathbf{F}$.

12. $\mathbf{F}$ is defined on all of $\mathbb{R}^3$, its components have continuous partial derivatives, and $\operatorname{curl}\mathbf{F} = (0-0)\,\mathbf{i} - (0-0)\,\mathbf{j} + (\cos y - \cos y)\,\mathbf{k} = \mathbf{0}$, so $\mathbf{F}$ is conservative by Theorem 16.5.4. Thus there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x,y,z) = \sin y$ implies $f(x,y,z) = x\sin y + g(y,z)$ and then $f_y(x,y,z) = x\cos y + g_y(y,z)$. But $f_y(x,y,z) = x\cos y$, so $g_y(y,z) = 0 \;\Rightarrow\; g(y,z) = h(z)$. Then $f(x,y,z) = x\sin y + h(z)$ implies $f_z(x,y,z) = h'(z)$. But $f_z(x,y,z) = -\sin z$, so $h(z) = \cos z + K$. Thus a potential function for $\mathbf{F}$ is $f(x,y,z) = x\sin y + \cos z + K$.

13. Since $\frac{\partial}{\partial y}(4x^3y^2 - 2xy^3) = 8x^3y - 6xy^2 = \frac{\partial}{\partial x}(2x^4y - 3x^2y^2 + 4y^3)$ and the domain of $\mathbf{F}$ is $\mathbb{R}^2$, $\mathbf{F}$ is conservative. Furthermore $f(x,y) = x^4y^2 - x^2y^3 + y^4$ is a potential function for $\mathbf{F}$. $t = 0$ corresponds to the point $(0,1)$ and $t = 1$ corresponds to $(1,1)$, so $\int_C \mathbf{F}\cdot d\mathbf{r} = f(1,1) - f(0,1) = 1 - 1 = 0$.

14. Here curl $\mathbf{F} = \mathbf{0}$, the domain of $\mathbf{F}$ is $\mathbb{R}^3$, and the components of $\mathbf{F}$ have continuous partial derivatives, so $\mathbf{F}$ is conservative. Furthermore $f(x,y,z) = xe^y + ye^z$ is a potential function for $\mathbf{F}$. Then $\int_C \mathbf{F}\cdot d\mathbf{r} = f(4,0,3) - f(0,2,0) = 4 - 2 = 2$.

15. C_1: $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j}$, $-1 \le t \le 1$;
C_2: $\mathbf{r}(t) = -t\,\mathbf{i} + \mathbf{j}$, $-1 \le t \le 1$.
Then



$$\int_C xy^2\,dx - x^2y\,dy = \int_{-1}^{1}(t^5 - 2t^5)\,dt + \int_{-1}^{1} t\,dt = \left[-\tfrac{1}{6}t^6\right]_{-1}^{1} + \left[\tfrac{1}{2}t^2\right]_{-1}^{1} = 0$$

Using Green's Theorem, we have

$$\int_C xy^2\,dx - x^2y\,dy = \iint_D \left[\frac{\partial}{\partial x}(-x^2y) - \frac{\partial}{\partial y}(xy^2)\right] dA = \iint_D (-2xy - 2xy)\,dA = \int_{-1}^{1}\int_{x^2}^{1} -4xy\,dy\,dx$$
$$= \int_{-1}^{1}\left[-2xy^2\right]_{y=x^2}^{y=1} dx = \int_{-1}^{1}(2x^5 - 2x)\,dx = \left[\tfrac{1}{3}x^6 - x^2\right]_{-1}^{1} = 0$$

16. $\int_C \sqrt{1+x^3}\,dx + 2xy\,dy = \iint_D \left[\frac{\partial}{\partial x}(2xy) - \frac{\partial}{\partial y}\left(\sqrt{1+x^3}\right)\right] dA = \int_0^1\int_0^{3x}(2y - 0)\,dy\,dx = \int_0^1 9x^2\,dx = 3x^3\Big]_0^1 = 3$

17. $\int_C x^2y\,dx - xy^2\,dy = \iint_{x^2+y^2\le 4}\left[\frac{\partial}{\partial x}(-xy^2) - \frac{\partial}{\partial y}(x^2y)\right] dA = \iint_{x^2+y^2\le 4}(-y^2 - x^2)\,dA = -\int_0^{2\pi}\int_0^2 r^3\,dr\,d\theta = -8\pi$

18. $\operatorname{curl}\mathbf{F} = (0 - e^{-y}\cos z)\,\mathbf{i} - (e^{-z}\cos x - 0)\,\mathbf{j} + (0 - e^{-x}\cos y)\,\mathbf{k} = -e^{-y}\cos z\,\mathbf{i} - e^{-z}\cos x\,\mathbf{j} - e^{-x}\cos y\,\mathbf{k}$,
$\operatorname{div}\mathbf{F} = -e^{-x}\sin y - e^{-y}\sin z - e^{-z}\sin x$

19. If we assume there is such a vector field $\mathbf{G}$, then $\operatorname{div}(\operatorname{curl}\mathbf{G}) = 2 + 3z - 2xz$. But $\operatorname{div}(\operatorname{curl}\mathbf{F}) = 0$ for all vector fields $\mathbf{F}$. Thus such a $\mathbf{G}$ cannot exist.

20. Let $\mathbf{F} = P_1\,\mathbf{i} + Q_1\,\mathbf{j} + R_1\,\mathbf{k}$ and $\mathbf{G} = P_2\,\mathbf{i} + Q_2\,\mathbf{j} + R_2\,\mathbf{k}$ be vector fields whose first partials exist and are continuous. Then

$$\mathbf{F}\operatorname{div}\mathbf{G} - \mathbf{G}\operatorname{div}\mathbf{F} = \left[P_1\left(\frac{\partial P_2}{\partial x} + \frac{\partial Q_2}{\partial y} + \frac{\partial R_2}{\partial z}\right)\mathbf{i} + Q_1\left(\frac{\partial P_2}{\partial x} + \frac{\partial Q_2}{\partial y} + \frac{\partial R_2}{\partial z}\right)\mathbf{j} + R_1\left(\frac{\partial P_2}{\partial x} + \frac{\partial Q_2}{\partial y} + \frac{\partial R_2}{\partial z}\right)\mathbf{k}\right]$$
$$- \left[P_2\left(\frac{\partial P_1}{\partial x} + \frac{\partial Q_1}{\partial y} + \frac{\partial R_1}{\partial z}\right)\mathbf{i} + Q_2\left(\frac{\partial P_1}{\partial x} + \frac{\partial Q_1}{\partial y} + \frac{\partial R_1}{\partial z}\right)\mathbf{j}\right.$$
$$\left. + R_2\left(\frac{\partial P_1}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R_1}{\partial z}\right)\mathbf{k}\right]$$

and

$$(\mathbf{G}\cdot\nabla)\,\mathbf{F}-(\mathbf{F}\cdot\nabla)\,\mathbf{G}=\left[\left(P_2\,\frac{\partial P_1}{\partial x}+Q_2\,\frac{\partial P_1}{\partial y}+R_2\,\frac{\partial P_1}{\partial z}\right)\mathbf{i}+\left(P_2\,\frac{\partial Q_1}{\partial x}+Q_2\,\frac{\partial Q_1}{\partial y}+R_2\,\frac{\partial Q_1}{\partial z}\right)\mathbf{j}+\left(P_2\,\frac{\partial R_1}{\partial x}+Q_2\,\frac{\partial R_1}{\partial y}+R_2\,\frac{\partial R_1}{\partial z}\right)\mathbf{k}\right]$$
$$-\left[\left(P_1\,\frac{\partial P_2}{\partial x}+Q_1\,\frac{\partial P_2}{\partial y}+R_1\,\frac{\partial P_2}{\partial z}\right)\mathbf{i}+\left(P_1\,\frac{\partial Q_2}{\partial x}+Q_1\,\frac{\partial Q_2}{\partial y}+R_1\,\frac{\partial Q_2}{\partial z}\right)\mathbf{j}+\left(P_1\,\frac{\partial R_2}{\partial x}+Q_1\,\frac{\partial R_2}{\partial y}+R_1\,\frac{\partial R_2}{\partial z}\right)\mathbf{k}\right]$$

Hence

$$\mathbf{F}\operatorname{div}\mathbf{G}-\mathbf{G}\operatorname{div}\mathbf{F}+(\mathbf{G}\cdot\nabla)\,\mathbf{F}-(\mathbf{F}\cdot\nabla)\,\mathbf{G}$$
$$=\left[\left(P_1\,\frac{\partial Q_2}{\partial y}+Q_2\,\frac{\partial P_1}{\partial x}\right)-\left(P_2\,\frac{\partial Q_1}{\partial y}+Q_1\,\frac{\partial P_2}{\partial y}\right)-\left(P_2\,\frac{\partial R_1}{\partial z}+R_1\,\frac{\partial P_2}{\partial z}\right)+\left(P_1\,\frac{\partial R_2}{\partial z}+R_2\,\frac{\partial P_1}{\partial z}\right)\right]\mathbf{i}$$
$$+\left[\left(Q_1\,\frac{\partial R_2}{\partial z}+R_2\,\frac{\partial Q_1}{\partial z}\right)-\left(Q_2\,\frac{\partial R_1}{\partial z}+R_1\,\frac{\partial Q_2}{\partial z}\right)-\left(P_1\,\frac{\partial Q_2}{\partial x}+Q_2\,\frac{\partial P_1}{\partial x}\right)+\left(P_2\,\frac{\partial Q_1}{\partial x}+Q_1\,\frac{\partial P_2}{\partial x}\right)\right]\mathbf{j}$$
$$+\left[\left(P_2\,\frac{\partial R_1}{\partial x}+R_1\,\frac{\partial P_2}{\partial x}\right)-\left(P_1\,\frac{\partial R_2}{\partial x}+R_2\,\frac{\partial P_1}{\partial x}\right)-\left(Q_1\,\frac{\partial R_2}{\partial y}+R_2\,\frac{\partial Q_1}{\partial y}\right)+\left(Q_2\,\frac{\partial R_1}{\partial y}+R_1\,\frac{\partial Q_2}{\partial y}\right)\right]\mathbf{k}$$
$$=\left[\frac{\partial}{\partial y}\,(P_1Q_2-P_2Q_1)-\frac{\partial}{\partial z}\,(P_2R_1-P_1R_2)\right]\mathbf{i}+\left[\frac{\partial}{\partial z}\,(Q_1R_2-Q_2R_1)-\frac{\partial}{\partial x}\,(P_1Q_2-P_2Q_1)\right]\mathbf{j}+\left[\frac{\partial}{\partial x}\,(P_2R_1-P_1R_2)-\frac{\partial}{\partial y}\,(Q_1R_2-Q_2R_1)\right]\mathbf{k}$$
$$=\operatorname{curl}\,(\mathbf{F}\times\mathbf{G})$$

21. For any piecewise-smooth simple closed plane curve C bounding a region D, we can apply Green's Theorem to $\mathbf{F}(x,y)=f(x)\,\mathbf{i}+g(y)\,\mathbf{j}$ to get $\int_C f(x)\,dx+g(y)\,dy=\iint_D\left[\frac{\partial}{\partial x}\,g(y)-\frac{\partial}{\partial y}\,f(x)\right]dA=\iint_D 0\,dA=0.$

$$\partial x^2 \quad \partial y^2 + \partial z^2$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\, g + f\,\frac{\partial g}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\, g + f\,\frac{\partial g}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\, g + f\,\frac{\partial g}{\partial z}\right) \quad \text{[Product Rule]}$$

$$= \frac{\partial^2 f}{\partial x^2}\, g + 2\,\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + f\,\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\, g + 2\,\frac{\partial f}{\partial y}\frac{\partial g}{\partial y}$$

$$+ f\,\frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\, g + 2\,\frac{\partial f}{\partial z}\frac{\partial g}{\partial z} + f\,\frac{\partial^2 g}{\partial z^2} \quad \text{[Product Rule]}$$

$$= f\left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2}\right) + g\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right) + 2\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right\rangle \cdot \left\langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right\rangle$$

$$= f\nabla^2 g + g\nabla^2 f + 2\nabla f \cdot \nabla g$$

Another method: Using the rules in Exercises 14.6.37(b) and 16.5.25, we have

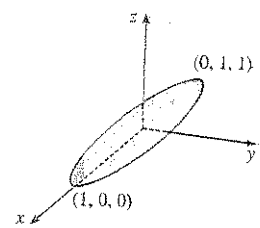
$$\nabla^2(fg) = \nabla \cdot \nabla(fg) = \nabla \cdot (g\,\nabla f + f\,\nabla g) = \nabla g \cdot \nabla f + g\nabla \cdot \nabla f + \nabla f \cdot \nabla g + f\nabla \cdot \nabla g$$
$$= g\,\nabla^2 f + f\,\nabla^2 g + 2\nabla f \cdot \nabla g$$

23. $\nabla^2 f = 0$ means that $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 0$. Now if $\mathbf{F} = f_y\,\mathbf{i} - f_x\,\mathbf{j}$ and C is any closed path in D, then applying Green's Theorem, we get

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C f_y\,dx - f_x\,dy = \iint_D \left[\tfrac{\partial}{\partial x}(-f_x) - \tfrac{\partial}{\partial y}(f_y)\right] dA$$
$$= -\iint_D (f_{xx} + f_{yy})\,dA = -\iint_D 0\,dA = 0$$

Therefore the line integral is independent of path, by Theorem 16.3.3.

24. (a) $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$, so C lies on the circular cylinder $x^2 + y^2 = 1$. But also $y = z$, so C lies on the plane $y = z$. Thus C is the intersection of the plane $y = z$ and the cylinder $x^2 + y^2 = 1$.



(b) Apply Stokes' Theorem, $\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$:

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xe^{2y} & 2x^2e^{2y} + 2y\cot z & -y^2\csc^2 z \end{vmatrix} = \langle -2y\csc^2 z - (-2y\csc^2 z), 0, 4xe^{2y} - 4xe^{2y}\rangle = \mathbf{0}$$

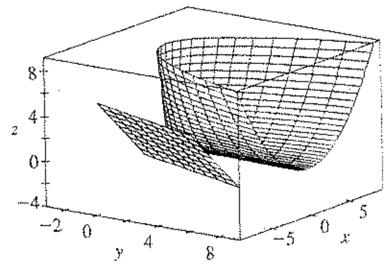
Therefore $\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \mathbf{0} \cdot d\mathbf{S} = 0$.

25. $z = f(x, y) = x^2 + 2y$ with $0 \le x \le 1$, $0 \le y \le 2x$. Thus

$$A(S) = \iint_D \sqrt{1 + 4x^2 + 4}\,dA = \int_0^1 \int_0^{2x} \sqrt{5 + 4x^2}\,dy\,dx = \int_0^1 2x\sqrt{5 + 4x^2}\,dx = \tfrac{1}{6}(5 + 4x^2)^{3/2}\Big]_0^1 = \tfrac{1}{6}\left(27 - 5\sqrt{5}\right).$$

26. (a) $\mathbf{r}_u = -v\,\mathbf{j} + 2u\,\mathbf{k}$, $\mathbf{r}_v = 2v\,\mathbf{i} - u\,\mathbf{j}$ and $\mathbf{r}_u \times \mathbf{r}_v = 2u^2\,\mathbf{i} + 4uv\,\mathbf{j} + 2v^2\,\mathbf{k}$. Since the point $(4, -2, 1)$ corresponds to $u = 1$, $v = 2$ (or $u = -1$, $v = -2$ but $\mathbf{r}_u \times \mathbf{r}_v$ is the same for both), a normal vector to the surface at $(4, -2, 1)$ is $2\,\mathbf{i} + 8\,\mathbf{j} + 8\,\mathbf{k}$ and an equation of the tangent plane is $2x + 8y + 8z = 0$ or $x + 4y + 4z = 0$.

(b)



(c) By Definition 16.6.6, the area of S is given by

$$A(S) = \int_0^3 \int_{-3}^3 \sqrt{(2u^2)^2 + (4uv)^2 + (2v^2)^2}\,dv\,du = 2\int_0^3 \int_{-3}^3 \sqrt{u^4 + 4u^2v^2 + v^4}\,dv\,du.$$

(d) By Equation 16.7.9, the surface integral is

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \int_0^3 \int_{-3}^3 \left\langle \frac{(u^2)^2}{1+(v^2)^2}, \frac{(v^2)^2}{1+(-uv)^2}, \frac{(-uv)^2}{1+(u^2)^2} \right\rangle \cdot \langle 2u^2, 4uv, 2v^2 \rangle\,dv\,du$$

$$= \int_0^3 \int_{-3}^3 \left(\frac{2u^6}{1+v^4} + \frac{4uv^5}{1+u^2v^2} + \frac{2u^2v^4}{1+u^4} \right) dv\,du \approx 1524.0190$$

27. $z = f(x, y) = x^2 + y^2$ with $0 \le x^2 + y^2 \le 4$ so $\mathbf{r}_x \times \mathbf{r}_y = -2x\,\mathbf{i} - 2y\,\mathbf{j} + \mathbf{k}$ (using upward orientation). Then

$$\iint_S z\,dS = \iint_{x^2+y^2 \le 4} (x^2 + y^2)\sqrt{4x^2 + 4y^2 + 1}\,dA$$

$$= \int_0^{2\pi} \int_0^2 r^3 \sqrt{1 + 4r^2}\,dr\,d\theta = \tfrac{1}{60}\pi\left(391\sqrt{17} + 1\right)$$

(Substitute $u = 1 + 4r^2$ and use tables.)

28. $z = f(x, y) = 4 + x + y$ with $0 \le x^2 + y^2 \le 4$ so $\mathbf{r}_x \times \mathbf{r}_y = -\mathbf{i} - \mathbf{j} + \mathbf{k}$. Then

$$\iint_S (x^2 z + y^2 z)\,dS = \iint_{x^2+y^2 \le 4} (x^2 + y^2)(4 + x + y)\sqrt{3}\,dA$$

$$= \int_0^2 \int_0^{2\pi} \sqrt{3}\,r^3 (4 + r\cos\theta + r\sin\theta)\,d\theta\,dr = \int_0^2 8\pi\sqrt{3}\,r^3\,dr = 32\pi\sqrt{3}$$

29. Since the sphere bounds a simple solid region, the Divergence Theorem applies and

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E \operatorname{div}\mathbf{F}\,dV = \iiint_E (z - 2)\,dV = \iiint_E z\,dV - 2\iiint_E dV$$

$$= 0 \left[\begin{array}{l}\text{odd function in } z \\ \text{and } E \text{ is symmetric}\end{array}\right] - 2\cdot V(E) = -2\cdot\tfrac{4}{3}\pi(2)^3 = -\tfrac{64}{3}\pi$$

Alternate solution: $\mathbf{F}(\mathbf{r}(\phi,\theta)) = 4\sin\phi\,\cos\theta\,\cos\phi\,\mathbf{i} - 4\sin\phi\,\sin\theta\,\mathbf{j} + 6\sin\phi\,\cos\theta\,\mathbf{k}$,

$\mathbf{r}_\phi \times \mathbf{r}_\theta = 4\sin^2\phi\,\cos\theta\,\mathbf{i} + 4\sin^2\phi\,\sin\theta\,\mathbf{j} + 4\sin\phi\,\cos\phi\,\mathbf{k}$, and

$\mathbf{F}\cdot(\mathbf{r}_\phi \times \mathbf{r}_\theta) = 16\sin^3\phi\,\cos^2\theta\,\cos\phi - 16\sin^3\phi\,\sin^2\theta + 24\sin^2\phi\,\cos\phi\,\cos\theta$. Then

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \int_0^{2\pi} \int_0^{\pi} (16\sin^3\phi\,\cos\phi\,\cos^2\theta - 16\sin^3\phi\,\sin^2\theta + 24\sin^2\phi\,\cos\phi\,\cos\theta)\,d\phi\,d\theta$$

$$= \int_0^{2\pi} \tfrac{4}{3}(-16\sin^2\theta)\,d\theta = -\tfrac{64}{3}\pi$$