All your solutions should be written in the test booklet. No references (text, notes, etc.) are allowed during the exam.

Throughout the exam:

> $\mathbb{R}, \mathbb{R}^n$ and all their subsets (such as $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{P}$, etc.) are assumed to have the usual metric unless something different is stated.

> $X, Y, ...$ will denote topological spaces.

> (X, d), (Y, s), ... will denote <u>metric</u> spaces.

Part I Carefully state each definition or theorem :

1. Suppose $f : X \to Y$ is a function between topological spaces and that $a \in X$.

> "f is <u>continuous at a</u>" means:

> "f is a <u>homeomorphism</u>" means:

2. i) A function $f : (X, d) \to (X, d)$ is a <u>contraction mapping</u> if:

> ii) State the Contraction Mapping Theorem.

3. i) X is called a <u>Baire space</u> X if:

> ii) State the Baire Category Theorem

4. Suppose $A, B \subseteq X$.

   i) "A and B are separated" means:

   ii) X is called connected if:

5. Suppose X and Y are topological spaces. The product topology $\mathcal{T}$ on $X \times Y$ is:

6. Show the implications among these properties in a topological space: countably compact, compact, pseudocompact, sequentially compact.

Part II For each part, give an example or state briefly why no such example exists. If you give an example, you do not need to prove that it "works." However your example must be clearly and precisely.

   1) A Cauchy sequence in $\mathbb{Q}$ that has exactly two cluster points.

   2) A countable space X in which $\{x\}$ is nowhere dense for every $x \in X$.

   3) A continuous bijection f between two metric spaces where f is not a homeomorphism.

   4) A compact space that is not separable.

Part III

1. Fill the remaining cells of the table with "T" (="true") or "F" (="false"). For example, the "F" in the third column means that "it is false that a connected metric space must be separable."

		first countable	complete	separable	totally bounded
If (X, d) is Lindelöf	then (X, d) is				
If (X, d) is connected	then (X, d) is			F	
If (X, d) is Baire	then (X, d) is				
If (X, d) is sequentially compact	then (X, d) is				
If (X, d) has discrete topology $\mathcal{T}_d$	then (X, d) is				

2. Fill in the cells of the table with "T" (= "true") or "F" (= "false") to indicate whether or not the space has the stated property.

		second countable	separable	connected	Baire	metrizable
$(\mathbb{R}, \mathcal{T})$, where $\mathcal{T} = \{\emptyset, \mathbb{R}\} \cup \{(a, \infty) : a \in \mathbb{R}\}$						
$\mathbb{N}$, with topology $\mathcal{T} = \{\emptyset\} \cup \{U \subseteq \mathbb{N} : 1 \in U\}$						

Part IV   True/False Questions In each case, just answer "T" or "F"; no justification for your answer is required.

_____1. If $x \in X$, then $\{x\}$ is the intersection of a family of open sets.

_____2. If $A \subseteq X$ and D is dense in X, then $A \cap D$ is dense in A.

_____3. If $(X, \mathcal{T})$ has a finite base $\mathcal{B}$, with $|\mathcal{B}| = m \in \mathbb{N}$, then $|\mathcal{T}| \leq 2^m$

_____4. If $\mathcal{T}$ is the cofinite topology on $\mathbb{N}$, then $(\mathbb{N}, \mathcal{T})$ has the fixed point property.

_____5. Suppose $\mathbb{Q} = A \cup B$, where A and B are separated in $\mathbb{Q}$. Then $B \cap \mathrm{cl}_{\mathbb{R}} A = \emptyset$.

_____6. Suppose X has a base $\mathcal{B}$ where $|\mathcal{B}| = c$. Then every open cover $\mathcal{U}$ of X has a subcover $\mathcal{U}'$ with $|\mathcal{U}'| \leq c$.

_____7. Let $\mathcal{F}$ be the collection of closed sets in a infinite topological space $(X, \mathcal{T})$. If $\mathcal{F}$ is also a topology on X, then $\mathcal{T}$ must be the discrete topology. (***Next time add $\mathcal{T}$ nontrivial***)

_____8. If (X, d) is not connected, then its completion $(\widetilde{X}, \widetilde{d})$ is not connected.

_____9. A subspace of a pseudocompact space is pseudocompact.

_____10. Suppose $\mathcal{B}$ is a collection of subsets of X for which $\bigcup \mathcal{B} = X$ and such that if $B_1, B_2 \in \mathcal{B}$, then there is a $B_3 \in \mathcal{B}$ with $B_3 \subseteq B_1 \cap B_2$. Then $\mathcal{B}$ is a base for a topology on X.

_____11. The letter T is homeomorphic to the letter F.

_____12. Let $f : X \to \mathbb{R}$ be continuous. Suppose $E \subseteq X$ has the property that $f|E = 0$ and $f|(X - E) \geq \frac{1}{2}$. Then E must be a clopen set in X.

_____13. Let $D^2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$. If $f : D^2 \to \mathbb{Q}$ is continuous, then f must be constant.

_____14. Let C be the Cantor set and let $B = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$. Then $C \times B$ is a complete subspace of $\mathbb{R}^2$.

_____15. In a metric space (X, d) the sets A and B are separated if and only if $d(A, B) > 0$.

_____16. For a metric space (X, d), if every continuous function $f : X \to \mathbb{R}$ assumes a minimum value, then every infinite set in X has a limit point.

_____17. There are exactly c nowhere dense subsets of $\mathbb{R}$.

_____18. $A = \mathbb{R}^2 - \{(0, y) : y \in \mathbb{P}\}$ is a connected subspace of $\mathbb{R}^2$.

_____19. Suppose $f : [0, 1]^2 \to \mathbb{R}$ and $A = \{p \in (0, 1)^2 : \frac{\partial f}{\partial x} \text{ and } \frac{\partial f}{\partial y} \text{ both exist at } p\}$. Then A is totally bounded.

_____20. Suppose (X, d) is a nonempty complete metric space and that A is a totally bounded closed subspace of X. Then every sequence in A has a cluster point in A.

_____21. The closure of a discrete subspace of $\mathbb{R}$ can be uncountable.

_____22. Suppose $A, B, C \subseteq X$. If A is separated from B and B is separated from C, then A is separated from C.

_____23. If X has the cofinite topology, then every subspace is pseudocompact.

_____24. Suppose $A \subseteq \mathbb{R}^3$ is compact and that $|A| = \aleph_0$. Then A must have an isolated point.

_____25. If $f : X \to Y$ is continuous and onto, and X is metrizable, then Y is metrizable.

_____26. Let $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ and $A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1, y \geq 0\}$. There is a continuous bijection $f : A \to S^1$.

_____27. Every sequence in $[0, 1]^2$ has a Cauchy subsequence.

_____28. Suppose $A \subseteq \mathbb{R}$ and $|A| > 1$. If A is nowhere dense, then A is not connected.

_____29. Suppose $A, B \subseteq \mathbb{R}$. If A and B are nonempty and $A \cup B$ is connected, then $\mathrm{cl}_{\mathbb{R}} A \cap \mathrm{cl}_{\mathbb{R}} B \neq \emptyset$.

_____30. For $x \in \mathbb{R}$, let $\mathcal{B}_x = \{[a, b] : \ a < b \text{ and } x \in [a, b]\}$. We can define a topology on $\mathbb{R}$ by taking the collection $\mathcal{B}_x$ as a neighborhood base at x for each x in $\mathbb{R}$.