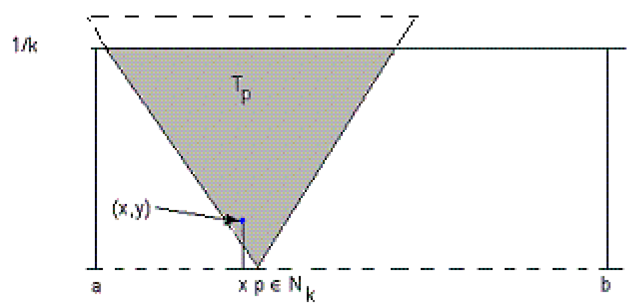
An example using the Baire Category Theorem

Suppose that for each irrational p, an equilateral triangle T_p (including its interior) is constructed in the plane with one vertex at $(p, 0)$ and its opposite side above and parallel to the x-axis.

Prove that $\bigcup\{T_p : p \in \mathbb{P}\}$ must contain a rectangle of the form $[a, b] \times (0, \frac{1}{k}]$ for some $a, b \in \mathbb{R}$ and some $k \in \mathbb{N}$.

Proof $N_k = \{p \in \mathbb{P} : T_p \textit{ has height } \geq \frac{1}{k}\}$. By hypothesis, $\mathbb{P} = \bigcup_{k=1}^{\infty} N_k$ and therefore $\mathbb{R} = \bigcup_{k=1}^{\infty} N_k \cup \bigcup_{q \in \mathbb{Q}} \{q\}$. Since $\mathbb{R}$ is second category in itself and each $\{q\}$ is nowhere dense in $\mathbb{R}$, $\exists\, k \in \mathbb{N}$ for which N_k is not nowhere dense, that is, $\text{int}_{\mathbb{R}}\, \text{cl}_{\mathbb{R}}\, N_k \neq \emptyset$. Therefore $\text{cl}_{\mathbb{R}}\, N_k$ contains some open interval (a, b).

For this k, we claim that the rectangle $[a, b] \times (0, \frac{1}{k}] \subseteq \bigcup\{T_p : p \in \mathbb{P}\}$.
If $(x, y) \in [a, b] \times (0, \frac{1}{k}]$, then $x \in [a, b]$. Since $(a, b) \subseteq \text{cl}_{\mathbb{R}}\, N_k$ there are points $p \in N_k$ arbitrarily close to x and if we choose such a p sufficiently close to x, it is geometrically clear (since $y > 0$) that $(x, y) \in T_p$:



Note: In the hypothesis, we could weaken "equilateral" to read " ... " ?