## Example: A Linear Difference Equation

Speedy One-Day Car Rental has 4 St. Louis locations: at the airport, in Clayton, in Kirkwood and in Chesterfield. Customers can rent a car at any one of these locations and drop it off at any one of the locations.

Speedy's long term records indicate (approximately) the following pattern of where cars begin and end the day.

**Cars beginning the day at**

↓

| Airport | Clayton | Kirkwood | Chesterfield | | → **end day at** |
|---|---|---|---|---|---|
| .75 | .30 | .05 | .25 | Airport | |
| .10 | .55 | .15 | .20 | Clayton | |
| .05 | .02 | .70 | .05 | Kirkwood | |
| .10 | .13 | .10 | .50 | Chesterfield | |

*These numbers can be interpreted as probabilities: for example, the probability is* $0$*.15 (15%) that a car that's at the Kirkwood location will be at Clayton branch at the end of the day.*

Suppose that at the start of a given day (call this "time $=$ 0"), the company has $c_1$ cars at the Airport office, $c_2$ at the Clayton office, $c_3$ at the Kirkwood office, and $c_4$ at the Chesterfield office. Thus the vector

$$\text{The vector } \boldsymbol{x}_0 = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} \text{cars at Airport} \\ \text{cars at Clayton office} \\ \text{cars at Kirkwood office} \\ \text{cars at Chesterfield office} \end{bmatrix}$$

gives the <u>initial state</u> ( $=$ *initial location*) of Speedy's cars at time $0$.

$$\text{The vector } \boldsymbol{x}_1 = \begin{bmatrix} .75c_1 + .30c_2 + .05c_3 + .25c_4 \\ .10c_1 + .55c_2 + .15c_3 + .20c_4 \\ .05c_1 + .02c_2 + .70c_3 + .05c_4 \\ .10c_1 + .13c_2 + .10c_3 + .50c_4 \end{bmatrix}$$

represents the new state ( $=$ *location*) of the cars after one day. It is just a linear combination

$$c_1 \begin{bmatrix} .75 \\ .10 \\ .05 \\ .10 \end{bmatrix} + c_2 \begin{bmatrix} .30 \\ .55 \\ .02 \\ .13 \end{bmatrix} + c_3 \begin{bmatrix} .05 \\ .15 \\ .70 \\ .10 \end{bmatrix} + c_4 \begin{bmatrix} .25 \\ .20 \\ .05 \\ .50 \end{bmatrix}$$

which is just a matrix-vactor product

$$\boldsymbol{x_1} = \underset{\substack{\uparrow \\ A}}{\begin{bmatrix} .75 & .30 & .05 & .25 \\ .10 & .55 & .15 & .20 \\ .05 & .02 & .70 & .05 \\ .10 & .13 & .10 & .50 \end{bmatrix}} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} .75 & .30 & .05 & .25 \\ .10 & .55 & .15 & .20 \\ .05 & .02 & .70 & .05 \\ .10 & .13 & .10 & .50 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} \boldsymbol{x}_0$$

Multiplication by this "transition" or "change-of-state" matrix $A$ shows how the state (*location*) of the cars changes from day to day: each vector $\boldsymbol{x_0}, \quad \boldsymbol{x_1} = A\boldsymbol{x_0},$ $\boldsymbol{x_2} = A\boldsymbol{x_1}, \quad \boldsymbol{x_3} = A\boldsymbol{x_2}, \quad \boldsymbol{x_4} = A\boldsymbol{x_3}$, ... gives the new location state of Speedy's cars one day later. In general,

$$\boldsymbol{x}_{k+1} = A\boldsymbol{x}_k$$

**Computation**

Suppose that $\boldsymbol{x_0} = \begin{bmatrix} 200 \\ 100 \\ 100 \\ 100 \end{bmatrix}$ is the initial state of Speedy's cars at time $0$.

Repeatedly multiplying by the transition matrix $A$ gives:

$$\boldsymbol{x_1} = A\boldsymbol{x_0} = \begin{bmatrix} .75 & .30 & .05 & .25 \\ .10 & .55 & .15 & .20 \\ .05 & .02 & .70 & .05 \\ .10 & .13 & .10 & .50 \end{bmatrix} \cdot \begin{bmatrix} 200 \\ 100 \\ 100 \\ 100 \end{bmatrix} = \begin{bmatrix} 210 \\ 110 \\ 87 \\ 93 \end{bmatrix}$$

$$\boldsymbol{x_2} = A\boldsymbol{x_1} = \begin{bmatrix} .75 & .30 & .05 & .25 \\ .10 & .55 & .15 & .20 \\ .05 & .02 & .70 & .05 \\ .10 & .13 & .10 & .50 \end{bmatrix} \cdot \begin{bmatrix} 210 \\ 110 \\ 87 \\ 93 \end{bmatrix} = \begin{bmatrix} 218 \\ 113 \\ 78 \\ 91 \end{bmatrix}$$

$$\boldsymbol{x_3} = A\boldsymbol{x_2} = \begin{bmatrix} .75 & .30 & .05 & .25 \\ .10 & .55 & .15 & .20 \\ .05 & .02 & .70 & .05 \\ .10 & .13 & .10 & .50 \end{bmatrix} \cdot \begin{bmatrix} 218 \\ 113 \\ 78 \\ 91 \end{bmatrix} = \begin{bmatrix} 224 \\ 114 \\ 72 \\ 90 \end{bmatrix}$$

$$\boldsymbol{x_4} = A\boldsymbol{x_3} = \begin{bmatrix} .75 & .30 & .05 & .25 \\ .10 & .55 & .15 & .20 \\ .05 & .02 & .70 & .05 \\ .10 & .13 & .10 & .50 \end{bmatrix} \cdot \begin{bmatrix} 224 \\ 114 \\ 72 \\ 90 \end{bmatrix} = \begin{bmatrix} 228 \\ 114 \\ 69 \\ 89 \end{bmatrix}$$

(*Note: many decimal places were carried along during those multiplications. But the* <u>*displayed*</u> *results (numbers of cars), are rounded to the nearest integer.* )

Over a longer period of time, the calculations give

$$\boldsymbol{x_{20}} = A\boldsymbol{x_{19}} = \begin{bmatrix} 236 \\ 113 \\ 62 \\ 89 \end{bmatrix}, \ldots \quad \boldsymbol{x_{40}} = A\boldsymbol{x_{39}} = \begin{bmatrix} 236 \\ 113 \\ 62 \\ 89 \end{bmatrix}, \ \ldots \ \boldsymbol{x_{100}} = A\boldsymbol{x_{99}} = \begin{bmatrix} 236 \\ 113 \\ 62 \\ 89 \end{bmatrix}, \ldots$$

*These calculations were completely painless and seemed almost instantaneous using the software package Matlab.*

If we assume that the transition matrix $A$ never changes over the time period, then it appears that the system moves toward a steady state: that is toward a value $\boldsymbol{x}$ for which $A\boldsymbol{x} = \boldsymbol{x}$. (*As a practical matter, due to round off, we will eventually see* $\boldsymbol{x_k} = \boldsymbol{x_{k+1}}$ $\boldsymbol{x_{k+2}} = \ldots$ *after* $k$ *is large enough.)*

If I restore the rounded off decimal places, the mathematical steady state vector is actually not $\begin{bmatrix} 236 \\ 113 \\ 62 \\ 89 \end{bmatrix}$ but something more like like $\boldsymbol{x_{10000}} \approx \begin{bmatrix} 236.562635457333 \\ 112.700476809723 \\ 61.768530559175 \\ 88.968357173830 \end{bmatrix}$
(even thesae values are rounded, but to 12 decimal places.

In general : suppose $\boldsymbol{x_0} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$, where $x_1, x_2, \ldots x_n$ measure different "parts" of a situation at time $0$. These "parts" are the state of the system; $\boldsymbol{x_0}$ is the initial state of the system (*time* $= 0$ ).

Suppose there is an $n \times n$ "transition" matrix $A$ such that $\boldsymbol{x_1} = A\boldsymbol{x_0}$ gives the state of the system at time 1, $\boldsymbol{x_2} = A\boldsymbol{x_1}$ gives the state at time 2, and, in general

$$\boldsymbol{x_{k+1}} = A\boldsymbol{x_k}$$
$$\uparrow$$

This is called a linear recurrence relation or linear difference equation (*it relates each state to the next in a linear way*).

One interesting question about such an relation is "what happens as $k \to \infty$" ? Does the system approach a "steady state" $\boldsymbol{x}$ for which $A\boldsymbol{x} = \boldsymbol{x}$?

To find a mathematically exact steady state (which, in this case does exist, you would want to solve $A\boldsymbol{x} = \boldsymbol{x}$. To do this, you can rearrange the equation to look like a homogeneous system $B\boldsymbol{x} = \boldsymbol{0}$ and then row reduce $B$. What is the matrix $B$ ?
Two more observations:

1) To find the steady state, try to solve $A\boldsymbol{x} = \boldsymbol{x}$, that is $(A - I)\boldsymbol{x} = \mathbf{0}$. Above this system has augmented matrix

$$\begin{bmatrix} -\frac{1}{4} & \frac{3}{10} & \frac{1}{20} & \frac{1}{4} & 0 \\ \frac{1}{10} & -\frac{9}{20} & \frac{3}{20} & \frac{1}{5} & 0 \\ \frac{1}{20} & \frac{1}{50} & \frac{-3}{10} & \frac{1}{20} & 0 \\ \frac{1}{10} & \frac{13}{100} & \frac{1}{10} & -\frac{1}{2} & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -\frac{2183}{821} & 0 \\ 0 & 1 & 0 & -\frac{1040}{821} & 0 \\ 0 & 0 & 1 & -\frac{570}{821} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ so the general solution is

$\boldsymbol{x} = x_4 \begin{bmatrix} \frac{2183}{821} \\ \frac{1040}{821} \\ \frac{570}{821} \\ 1 \end{bmatrix}$ which we can rescale and more conveniently rewrite as $\boldsymbol{x} = s \begin{bmatrix} 2183 \\ 1040 \\ 570 \\ 821 \end{bmatrix}$.

So a steady state vector (one of infinitely many, mathematically) is $\begin{bmatrix} 2183 \\ 1040 \\ 570 \\ 821 \end{bmatrix}$. BUt ti doesn't fit our problem: the entries add up to $4614$ cars; we want a steady state vector solution vector that adds up to 500 cars. To do this, we just rescale: multiply each entry by $\frac{500}{4614}$ , which gives another steady state vector $\begin{bmatrix} 2183{*}500/4614 \\ 1040{*}500/4614 \\ 570{*}500/4614 \\ 821{*}500/4614 \end{bmatrix}$

$\approx \begin{bmatrix} 236.562635457304 \\ 112.700476809710 \\ 61.768530559168 \\ 88.968357173819 \end{bmatrix}$ which is virtually the same as $x_{10000}$ computed earlier.

2) $\boldsymbol{x_k} = A\boldsymbol{x_{k-1}} = ... = A^k\boldsymbol{x_0}$

Finding a steady state is the same as asking: as $k \to \infty$, does $A^k \to$ some matrix $S$? If so, then $S\boldsymbol{x_0}$ would give the steady state.